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Orbit propagation with Lie-Deprit methods in satellite dynamics

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Abstract

In this paper we generate an analytical theory for the motion of the artificial satellite in which the tesseral model of the planetary potential is considered. We integrate the problem by applying the Lie-Deprit method with three consecutive canonical transformations: the elimination of the parallax, the Delaunay normalization and a final double normalization to eliminate the perigee and the node. We apply this theory to the 2×2, 4×4, and 5×5 tesseral models and we propagate the motion of the satellite, and also, we present the error of this analytical propagation versus the numerical propagation for a wide range of initial conditions.

1 Introduction

Analytical theories have been widely used in the study of the motion of an artificial satellite. Among the advantages of using analytical theories we may extract that they are very fast to evaluate since they consists of explicit functions of time, which results in a very useful tool in mission analysis where a fast orbit propagator is needed because it is necessary to integrate the orbit for many sets of initial conditions. Besides, analytical theories provide a good understanding of the qualitative dynamics as well as the effects of the perturbation terms [1, 2, 10, 23]. The search of frozen orbits is a good example of how the analytical theories help in the analysis of certain systems [9, 5].

On the contrary, the main difficulty of analytical theories is that it is quite complex to derive the final formulas and also to have them with the less terms as possible, although as the order of the theory increases, it is frequent to cope with millions of terms [14]. However, computers are becoming faster and cheaper, and also there are many commercial software packages that allow the construction of analytical theories of order high enough even with desk computers [4].

Due to the difficulties in building a convenient analytical theory for the satellite when tesseral harmonics are included, most of the related works have been applied to the zonal problem. The most common theories use Deprit's integration method [11] by applying a battery of canonical transformations (the elimination of the parallax [12], the elimination of the perigee [8], the Delaunay normalization [13], the Krylov-Bogoliubov method [7], etc.). In Abad *et al.* [3] an analytical tool (ATESAT) to make analytical integrations of the zonal problem of the satellite is presented. ATESAT obtains not only the analytical expressions of the transformation but the code of the software to propagate the satellite with such theory. Using ATESAT, a third order theory of the motion of an orbiter about Mars [19] was generated using a zonal model of sixth degree. The numerical results of this theory for a wide range of initial condition give an error less than 400 meters after a month.

The Coriolis term of the Hamiltonian cannot be avoided in the tesseral problem because of the appearance of the node in the expression of the potential. This, together with the size of the expression, makes very involved the analytical integration. Several attempts to handle this problem have been made. Let us mention here the works of Kaula [16], Wnuk [24, 25] based on series expansions in the eccentricity of the potential function, or the solution of Segerman and Coffey [20], that uses the relegation algorithm of Deprit *et al.* [15], or the more recent paper of Palacián [17].

In this communication we use a way presented by Serrano [21] to integrate the tesseral problem. In this method, three transformation are applied: the relegation algorithm to simplify the problem, the Delaunay normalization to eliminatete the mean anomaly and, finally, a double normalization to eliminate simultaneously the perigee and the node. We use a modern version, coded in C++ instead of C, of our old Poisson Series processor PSPC [6] to generate an analytical theory of several orders of the 2×2, 4×4 and 5×5 tesseral models. Lastly, with the C codes automatically generated by the Poisson Series Processor we propagate a big set of initial conditions during a month, and we present the errors of the analytical theory compared with the numerical integration.

2 The tesseral model of the satellite

The Hamiltonian of the tesseral $N \times N$ model of the satellite, expressed in polar-nodal variables $(r, \theta, \nu, R, \Theta, N)$, may be written as the sum of the Keplerian term, the Coriolis term, the Main Problem term and the rest of zonal and tesseral terms as

$$\mathcal{H} = \mathcal{H}_k + \mathcal{H}_c + \mathcal{H}_m + \mathcal{H}_r,$$

where we have

$$\begin{aligned} \mathcal{H}_k &= \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r}, \\ \mathcal{H}_c &= -\omega N, \\ \mathcal{H}_m &= \frac{\mu}{r}\left(\frac{\alpha}{r}\right)^2 J_2 P_2(\sin i \sin\theta), \\ \mathcal{H}_t &= \sum_{n=2}^{N} \mathcal{H}_n^t, \end{aligned} \tag{1}$$

where μ is the gravitational constant, α the equatorial Earth's radius, ω is the Earth's angular velocity, P_2 is the Legendre polynomial of order 2, and i is the inclination of the orbit.

The expression of the tesseral part can be decomposed into

$$\begin{aligned} \mathcal{H}_2^t &= -\frac{\mu}{r}\left(\frac{\alpha}{r}\right)^2 \sum_{j=1}^{2} \left(C_{2,j}\cos j\lambda + S_{2,j}\sin j\lambda\right) P_{2,j}(\sin i \sin\theta), \\ \mathcal{H}_n^t &= -\frac{\mu}{r}\left(\frac{\alpha}{r}\right)^n \sum_{j=0}^{n} \left(C_{n,j}\cos j\lambda + S_{n,j}\sin j\lambda\right) P_{n,j}(\sin i \sin\theta), \end{aligned} \tag{2}$$

with $P_{n,j}$ the associated Legendre polynomials, and $C_{n,j}$ and $S_{n,j}$ are the tesseral coefficients of the planet gravitational potential. For details, the reader is addressed to [16].

The relative order of the terms of the Hamiltonian has been extensively studied by Serrano [22]. Taking into consideration the relative value of the mean motion of the satellite with respect to the rotation angular velocity of the planet, Serrano [22] suggests to scale the Hamiltonian in the following way

$$\mathcal{H} = \mathcal{H}_0 + \epsilon\mathcal{H}_1 + \frac{\epsilon^2}{2!}\mathcal{H}_2 + \frac{\epsilon^4}{4!}\mathcal{H}_4,$$

where

$$\mathcal{H}_0 = \mathcal{H}_k,$$

$$\mathcal{H}_1 = \frac{\mathcal{H}_k}{\epsilon} = -\omega^* N,$$

(3)
$$\mathcal{H}_2 = \frac{2!\,\mathcal{H}_m}{\epsilon^2} = \frac{\mu}{r}\left(\frac{\alpha}{r}\right)^2 J_2^* P_2(\sin i \sin\theta),$$

$$\begin{aligned}\mathcal{H}_4 = \frac{4!\,\mathcal{H}_r}{\epsilon^4} &= -\frac{\mu}{r}\left(\frac{\alpha}{r}\right)^2 \sum_{j=1}^{2}\left(C_{2,j}^*\cos j\lambda + S_{2,j}^*\sin j\lambda\right) P_{2,j}(\sin i\sin\theta)\\ &-\frac{\mu}{r}\sum_{n=3}^{6}\left(\frac{\alpha}{r}\right)^n \sum_{j=0}^{n}\left(C_{n,j}^*\cos j\lambda + S_{n,j}^*\sin j\lambda\right) P_{n,j}(\sin i\sin\theta).\end{aligned}$$

Note that we introduce a fictitious small parameter ϵ as the order of the Coriolis term with respect to the Keplerian Hamiltonian,

(4)
$$\epsilon = \frac{|\mathcal{H}_c|}{|\mathcal{H}_k|} = \frac{\omega\sqrt{\mu a(1-e^2)}\cos i}{\mu/2a}$$

and we rewrite the constants in the form

$$\omega \;=\; \epsilon\,\omega^*, \quad J_2 \;=\; \frac{\epsilon^2}{2!}J_2^*, \quad C_{i,j} \;=\; \frac{\epsilon^4}{4!}C_{i,j}^*.$$

Figure 1 represents the level curves of the function $\epsilon = \epsilon(a, i)$ given by Eq. (4). The horizontal axis represents the semi major-axis (a), with a range between 1 and the semi major axis of a geostationary orbit (we take the equatorial radius as unit of distance), whereas the vertical axis is for the inclination (i) with a range between 0º and 180º. Out of the yellow area the value of ϵ increases and consequently, the proposed scaling is not adequate for these regions.

3 Lie-Deprit method

The well known Lie-Deprit method [11] consists of finding a canonical Lie transformation of generator

$$W(\boldsymbol{x}, \boldsymbol{X}; \epsilon) = \sum_{n\geq 0}\frac{\epsilon^n}{n!}\,W_n(\boldsymbol{x}, \boldsymbol{X}),$$

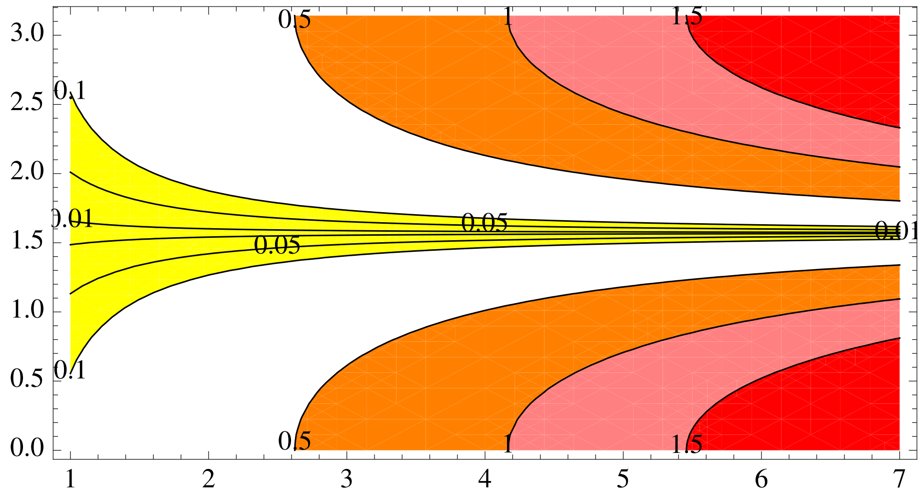

Figure 1: Level curves of the function $\epsilon = \epsilon(a, i)$. Yellow area corresponds to the smaller values of ϵ.

that transforms the original Hamiltonian

$$\mathcal{H}(\boldsymbol{x}, \boldsymbol{X}; \epsilon) = \sum_{n \geq 0} \frac{\epsilon^n}{n!} \mathcal{H}_n(\boldsymbol{x}, \boldsymbol{X}),$$

of a given dynamical system, into a new Hamiltonian

$$\mathcal{K}(\boldsymbol{y}, \boldsymbol{Y}; \epsilon) = \sum_{n \geq 0} \frac{\epsilon^n}{n!} \mathcal{K}_n(\boldsymbol{y}, \boldsymbol{Y}),$$

enjoying certain properties.

By changing the notation as $\mathcal{H}_{n,0} = \mathcal{H}_n, \quad \mathcal{H}_{0,n} = \mathcal{K}_n$, we may write the relation among the old and new Hamiltonians and the generator of the transformation by means of the so-called Lie triangle algorithm

$$\mathcal{H}_{i,j} = \mathcal{H}_{i+1,j-1} + \sum_{k=0}^{i} \binom{i}{k} (\mathcal{H}_{i-k,j-1}, W_{k+1}) \,. \tag{5}$$

where (f, g) stands for the Poisson bracket, defined as

$$(f, g) = \sum_{i}^{3} \left(\frac{\partial f}{\partial X_i} \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial X_i} \right).$$

To compute the transformed Hamiltonian we apply an iterative process showed in the following scheme (for details, see the original paper of Deprit [11]):

$$
\begin{array}{ccccccccccccc}
 & & & & & & \mathcal{H}_{0,0} & & & & & & \\
 & & & & & \swarrow & & & & & & & \\
 & & & & \mathcal{H}_{1,0} & & \longrightarrow & & \mathcal{H}_{0,1} & & & & \\
 & & & \swarrow & & & & \swarrow & & & & & \\
 & & \mathcal{H}_{2,0} & & \longrightarrow & & \mathcal{H}_{1,1} & & \longrightarrow & & \mathcal{H}_{0,2} & & \\
 & \swarrow & & & & \swarrow & & & & \swarrow & & & \\
\mathcal{H}_{3,0} & & \longrightarrow & & \mathcal{H}_{2,1} & & \longrightarrow & & \mathcal{H}_{1,2} & & \longrightarrow & & \mathcal{H}_{0,3} \\
\swarrow & & & \swarrow & & & & \swarrow & & & & \swarrow &
\end{array}
$$

At order n (from $n=1$), we compute the line $\mathcal{H}_{i,j}$, with $i+j=n$, and $i=n-1,\ldots,0$, from the element $\mathcal{H}_{n,0}$, the terms $\mathcal{H}_{i,j}$, of the previous lines, and the elements W_i, $(i \le n)$. Hence

$$
\begin{array}{llll}
\mathcal{H}_{n-1,1} & = & \mathcal{H}_{n,0} & +\ldots+k_{12}(\mathcal{H}_{20},W_{n-2})+k_{11}(\mathcal{H}_{10},W_{n-1})+(\mathcal{H}_{00},W_n) \\
\mathcal{H}_{n-2,2} & = & \mathcal{H}_{n-1,1} & +\ldots+k_{22}(\mathcal{H}_{11},W_{n-2})+\quad(\mathcal{H}_{01},W_{n-1}) \\
\mathcal{H}_{n-3,3} & = & \mathcal{H}_{n-2,2} & +\ldots+\quad(\mathcal{H}_{02},W_{n-2}) \\
\ldots & = & \ldots & \\
\mathcal{H}_{0,n} & = & \ldots &
\end{array} \tag{6}
$$

where the k_{ij} coefficients are

$$k_{12}=\binom{n-1}{n-3}=\frac{1}{2}(n-1)(n-2), \quad k_{11}=\binom{n-1}{n-2}=(n-1), \quad k_{22}=\binom{n-2}{n-3}=(n-2).$$

From these relations we may define the tilde elements by the expression

$$\mathcal{H}_{i,j}=\widetilde{\mathcal{H}}_{i,j}+(\mathcal{H}_{00},W_n)=\widetilde{\mathcal{H}}_{i,j}-\mathcal{L}_0 W_n, \quad i<n.$$

where the so-called Lie operator $\mathcal{L}_0$ is defined as the Poisson bracket $\mathcal{L}_0(F)=(H_{0,0},F)$.

Deprit's method tries to compute, iteratively, line by line, the elements W_n. Let us suppose we have completed the $n-1$ first lines of the Lie triangle and the elements $W_1,\ldots W_{n-1}$ are known; then we may compute the tilde elements in line n by the formulas

$$
\begin{array}{lllllll}
\widetilde{\mathcal{H}}_{n,0} & = & \mathcal{H}_{n,0} & & & & \\
\widetilde{\mathcal{H}}_{n-1,1} & = & \widetilde{\mathcal{H}}_{n,0} & +\ldots+ & k_{12}(\mathcal{H}_{20},W_{n-2}) & + & k_{11}(\mathcal{H}_{10},W_{n-1}), \\
\widetilde{\mathcal{H}}_{n-2,2} & = & \widetilde{\mathcal{H}}_{n-1,1} & +\ldots+ & k_{22}(\widetilde{\mathcal{H}}_{11},W_{n-2}) & + & (\widetilde{\mathcal{H}}_{01},W_{n-1}), \\
\widetilde{\mathcal{H}}_{n-3,3} & = & \widetilde{\mathcal{H}}_{n-2,2} & +\ldots+ & (\widetilde{\mathcal{H}}_{02},W_{n-2}) & , & \\
\ldots & = & \ldots & & & & \\
\widetilde{\mathcal{H}}_{0,n} & = & \ldots & & & &
\end{array}
$$

and, finally, we obtain the element W_n by solving the homological equation

$$\widetilde{\mathcal{H}}_{0,n} = \mathcal{H}_{0,n} + \mathcal{L}_0 W_n. \tag{7}$$

The usual way to apply the Deprit's method is to split the tilde element $\widetilde{\mathcal{H}}_{0,n}$ into two parts

$$\widetilde{\mathcal{H}}_{0,n} = \mathcal{H}_{0,n}^{(1)} + \mathcal{H}_{0,n}^{(2)},$$

in such a way that we easily find a first integral of the partial differential equation

$$\mathcal{L}_0 W_n = \mathcal{H}_{0,n}^{(2)},$$

and we take $\mathcal{H}_{0,n}^{(1)}$ as the order n of the new Hamiltonian

$$\mathcal{H}_{0,n} = \mathcal{H}_{0,n}^{(1)}.$$

4 Solving the tesseral problem of the satellite

An analytical theory for the motion of the artificial satellite, using the Lie-Deprit method, consists of applying several Lie canonical transformations until we are able to integrate the problem. As it is easy to imagine, there are several choices. We can integrate the problem by applying only one transformation, the *Delaunay normalization* [13], which requires to have the Hamiltonian of the problems expressed in terms of the Delaunay variables, and then, we need to replace the powers $(1/r)^n$ into power series of the eccentricity, which reduces the valididy of the theory to only quite small eccentricities. In order to avoid such series expansions of the eccentricity, more than one transformation must be applied.

To integrate the zonal problem of the satellite [3, 7, 18] we apply first the *elimination of the parallax* [12]. The elimination of the parallax is a classical Lie canonical transformation that reduces the complexity of the Hamiltonian, expressed in polar-nodal variables, by reducing the terms proportional to $1/r^n$, $(n \geq 2)$ into terms only proportional to $1/r^2$. Once we apply the elimination of the parallax transformation, we obtain a new Hamiltonian depending on the mean anomaly ℓ and the perigee g. The *elimination of the perigee* transformation [8] eliminates the perigee angle of the expression of the Hamiltonian. Finally, a Delaunay normalization eliminates the mean anomaly and makes the final Hamiltonian independent of the coordinates, then, the problem becomes trivially integrable.

The previous method cannot be applied to the tesseral problem because the node angle ν explicitly appears in the Hamiltonian. A new method to integrate this problem has been proposed by Serrano [21]. This method is based on three Lie canonical transformations, namely, a modified version of the elimination of the parallax that includes the node; a Delaunay normalization to eliminate the mean anomaly and, finally, a double normalization to eliminate simultaneously the perigee and the node.

4.1 Elimination of the parallax

The original elimination of the parallax algorithm cannot be directly applied to the tesseral problem since the variable ν appears in the expressions; however we may change slightly the algorithm in order to simplify the Hamiltonian.

The elimination of the parallax is based on the expression of the operator $\mathcal{L}_0$ in polar–nodal variables

$$\mathcal{L}_0 = R\frac{\partial}{\partial r} - \left(\frac{\mu}{r^2} - \frac{\Theta^2}{r^3}\right)\frac{\partial}{\partial R} + \frac{\Theta}{r^2}\frac{\partial}{\partial \theta}. \tag{8}$$

Let us suppose we are dealing with expressions of the form

$$\mathcal{F}^0_m = \Gamma^c_k \cos(j\nu) + \Gamma^s_k \sin(j\nu), \quad \mathcal{F}^\theta_m = \Gamma^c_k \cos(i\theta + j\nu) + \Gamma^s_k \sin(i\theta + j\nu), \tag{9}$$

where the script m in $\mathcal{F}$ represents a triad of elements i, j, k, and Γ^c_k, Γ^s_k belong to the kernel of $\mathcal{L}_0$, i.e. $\mathcal{L}_0\Gamma^c_k = 0$, $\mathcal{L}_0\Gamma^s_k = 0$.

For $\mathcal{F}^\theta_m$ we may write

$$\mathcal{L}_0\mathcal{W}^\theta_m = \frac{\Theta}{r^2}\mathcal{F}^\theta_m, \quad \text{with} \quad \mathcal{W}^\theta_m = \frac{\Gamma^c_k}{i}\sin(i\theta + j\nu) + \frac{-\Gamma^s_k}{i}\cos(i\theta + j\nu). \tag{10}$$

In order to apply the Lie-Deprit method let us suppose that we know the expression of $\widetilde{\mathcal{H}}_{0,n}$, in which two kind of terms appear:

- terms $\widetilde{\mathcal{H}}^r_{0,n}$ with the factor $1/r^n$, $n \geq 2$,
- terms $\widetilde{\mathcal{H}}^*_{0,n}$ in which the variable r does not appear. These terms come from the computation of the Poisson bracket $(\mathcal{H}_{0,1}, W_n)$ because r does not appear neither in W_n (see the next step) nor in $\mathcal{H}_{0,1} = \mathcal{H}_{1,0}$.

Taking into account the relations

$$\frac{1}{r} = \frac{\mu}{\Theta^2}\left(1 + C\cos\theta + S\sin\theta\right), \quad C = e\cos g, \quad S = e\sin g,$$

where C and S are the *state functions* that belong to the kernel, we may reduce the exponent of r in $\widetilde{\mathcal{H}}^r_{0,n}$ changing $1/r^n$ by

$$\frac{1}{r^n} = \frac{1}{r^2}\left(\frac{\mu}{\Theta^2}\left(1 + C\cos\theta + S\sin\theta\right)\right)^{n-2}.$$

Then, expanding the expression of $\widetilde{\mathcal{H}}^r_{0,n}$ we obtain an expression of the form

$$\widetilde{\mathcal{H}}^r_{0,n} = \widetilde{\mathcal{H}}^0_{0,n} + \widetilde{\mathcal{H}}^\theta_{0,n}, \quad \widetilde{\mathcal{H}}^0_{0,n} = \sum_m \frac{\Theta}{r^2}\mathcal{F}^0_m, \quad \widetilde{\mathcal{H}}^\theta_{0,n} = \sum_m \frac{\Theta}{r^2}\mathcal{F}^\theta_m. \tag{11}$$

Finally we may choose the expression of the new Hamiltonian

$$\mathcal{H}_{0,n} = \widetilde{\mathcal{H}}^*_{0,n} + \widetilde{\mathcal{H}}^0_{0,n}, \tag{12}$$

and $\mathcal{L}_0 W_n$ (needful to refresh the diagonal) will be

$$\mathcal{L}_0 W_n = \widetilde{\mathcal{H}}_{0,n} - \mathcal{H}_{0,n} = \widetilde{\mathcal{H}}^\theta_{0,n}. \tag{13}$$

Finally, since $\mathcal{L}_0 W_n = \widetilde{\mathcal{H}}^\theta_{0,n}$ has the form $(\Theta/r^2)\sum_m \mathcal{F}^\theta_m$, then, taking into account (10) the generator, W_n can be obtained by the expression

$$W_n = \sum_m \mathcal{W}^\theta_m. \tag{14}$$

4.2 Delaunay Normalization

Since Delaunay's normalization consists of eliminating the mean anomaly ℓ from the Hamiltonian, first, let us change the Hamiltonian from Polar–Nodal to Delaunay variables, then the Keplerian Hamiltonian becomes

$$\mathcal{H}_K = -\frac{\mu^2}{2L^2},$$

and the Lie operator $\mathcal{L}_0$ is

$$\mathcal{L}_0 = \frac{\mu^2}{L^3}\frac{\partial}{\partial \ell} \tag{15}$$

To obtain the n-th order of the transformation we start with the value of $\widetilde{\mathcal{H}}_{0,n}$ previously computed at the order $(n-1)$-th.

Then, choose as the new Hamiltonian the average over ℓ of the term $\widetilde{\mathcal{H}}_{0,n}$

$$\mathcal{H}_{0,n} = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{\mathcal{H}}_{0,n}\, d\ell, \tag{16}$$

and the n-th term of the generating function will be obtained, from the homological equation, by the quadrature

$$W_n = \frac{L^3}{\mu^2}\int \left(\widetilde{\mathcal{H}}_{0,n} - \mathcal{H}_{0,n}\right)\, d\ell, \tag{17}$$

4.3 Double normalization

After Delaunay normalization the first three orders of the Hamiltonian

$$\mathcal{H}_{0,0} = \mathcal{H}_{0,0}(L), \quad \mathcal{H}_{1,0} = \mathcal{H}_{1,0}(H), \quad \mathcal{H}_{2,0} = \mathcal{H}_{2,0}(L,G,H),$$

depend, respectively, on the moments L, H and (L,G,H), whereas the rest of terms are of the form $\mathcal{H}_{n,0} = \mathcal{H}_{20}(\_, g, h, L, G, H)$. From these expressions we deduce the following Lie-derivatives

$$\mathcal{L}_0 = \frac{\mu^2}{L^3}\frac{\partial}{\partial \ell}, \quad \mathcal{L}_1 = -\omega\frac{\partial}{\partial h}, \quad \mathcal{L}_2 = L^*\frac{\partial}{\partial \ell} + G^*\frac{\partial}{\partial g} + H^*\frac{\partial}{\partial h},$$

where L^*, G^*, H^* are functions of the momenta L, G, H.

With this, we decompose the generator W_n in three parts

$$W_n = W_n^{\ell}(\ell, g, h, L, G, H) + W_n^{h}(\_, g, h, L, G, H) + W_n^{g}(\_, g, \_, L, G, H),$$

then, we have

$$\mathcal{L}_0 W_n = \mathcal{L}_0 W_n^{\ell}, \quad \mathcal{L}_1 W_n = \mathcal{L}_1 W_n^{h}, \quad \mathcal{L}_2 W_n = \mathcal{L}_2 W_n^{g}.$$

The objective of the *Double normalization* transformation is to eliminate simultaneously both variables g and h. Since these variables do not appear in the first three orders of the Hamiltonian, then we choose

$$\mathcal{H}_{01} = \mathcal{H}_{10}, \quad \mathcal{H}_{02} = \mathcal{H}_{20}.$$

With these requirements, the equation (6) leads to a value of $W_1^{(h)} = 0$ whereas $W_1^{(g)}$ remains undetermined.

The homological equation, in this case, can be expressed in the form

$$\widetilde{\mathcal{H}}_{n0} = \mathcal{H}_{0n} + \frac{1}{2}n(n-1)\mathcal{L}_2 W_{n-2}^{(g)} + n\mathcal{L}_1 W_{n-1}^{(h)}.$$

To solve the order $n \geq 3$ we split $\widetilde{\mathcal{H}}_{n,0}$ in three terms

$$\widetilde{\mathcal{H}}_{n,0} = \widetilde{\mathcal{H}}^{\emptyset}_{n,0}(\_,\_,\_,L,G,H) + \widetilde{\mathcal{H}}^{g}_{n,0}(\_,g,\_,L,G,H) + \widetilde{\mathcal{H}}^{h}_{n,0}(\_,g,h,L,G,H),$$

where

- $\widetilde{\mathcal{H}}^{\emptyset}_{n,0} \in \ker(\mathcal{L}_1) \cap \ker(\mathcal{L}_2)$, i.e. the terms does not depend of g and h.
- $\widetilde{\mathcal{H}}^{g}_{n,0} \in \ker(\mathcal{L}_1)$, i.e. terms that depend only on g, but not on h.
- $\widetilde{\mathcal{H}}^{h}_{n,0}$ are the terms depending on h. The variable g may appear on these terms, but it is not mandatory.

With these assumptions we write

$$\mathcal{H}_{0n} = \widetilde{\mathcal{H}}^{\emptyset}_{n,0},$$

$$\tfrac{1}{2}n(n-1)\mathcal{L}_2 W_{n-2}^{(g)} = \widetilde{\mathcal{H}}^{g}_{n0}, \quad \Longrightarrow \quad W_{n-2}^{(g)} = \frac{2}{n(n-1)G^*}\int \widetilde{\mathcal{H}}^{g}_{n0}\,dg, \tag{18}$$

$$n\mathcal{L}_1 W_{n-1}^{(h)} = \widetilde{\mathcal{H}}^{h}_{n0}, \quad \Longrightarrow \quad W_{n-1}^{(h)} = -\frac{1}{n\,\omega}\int \widetilde{\mathcal{H}}^{h}_{n0}\,dh.$$

and in this way, we complete the generator at order $n-2$ and besides we compute one term of the generator at order $n-1$.

5 Application of the theory to the orbit propagation

To check the use of this analytical integration of the artificial satellite problem as an orbit propagator, the first task is to construct the analytical theory by using a Poisson Series processor. We do that with a modern version, coded in C++ instead of C, of our old Poisson Series processor PSPC [6].

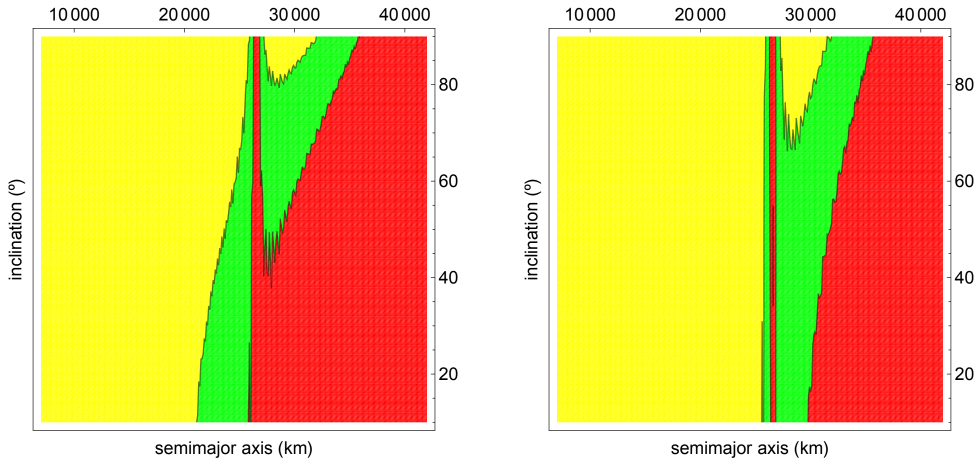

Figure 2: Differences between analytical and numerical integrations for the (2×2) -tesseral model. Left) results for a 6th-order theory. Right) same model and a 7th-order theory. Horizontal axis is the semi-major axis $a \in [7000, 42000]$ km (from low orbits till geostationary orbits), and vertical axis represents the inclination $i \in [0^o, 90^o]$. Initial conditions (a, i) inside the yellow region provide an error less than one km after one month of propagating the orbit. Green area give errors between one and two km. Red area represents errors greater than two km after the propagation of the orbit for one month.

We build up the theory for the models (2×2), (4×4) and (5×5) of the potential. In the two first cases we obtain a 6-th and 7-th order theory. In the model (5×5) we obtain only the 6-th order theory because of an *out of memory* error when we try to make the 7-th order with a computer of 32 GB of total memory.

The `C` code to propagate the orbit has been automatically written by the Poisson series processor from the direct and inverse expressions of the three transformations.

To check this program we compare it with a propagation of the same model obtained by using `dopri853`, a Runge-Kutta method of order 8(5,3) due to E. Hairer (see `http://www.unige.ch/ hairer/software.html`).

We begin, in both cases, from a set of initial conditions that cover a wide range of semimajor-axis (from low to geostationary orbits) and inclinations (from 0^o to 90^o). We propagate the orbit during a month and, finally, we compare the position obtained with the numerical integration with the one obtained with the analytical integration.

Figures 2 and 3 represent the results of these comparisons. In these figures the horizontal axis stands for the semi major axis a, from 7000 to 42000 km. The vertical axis represents the inclination i, in degrees.

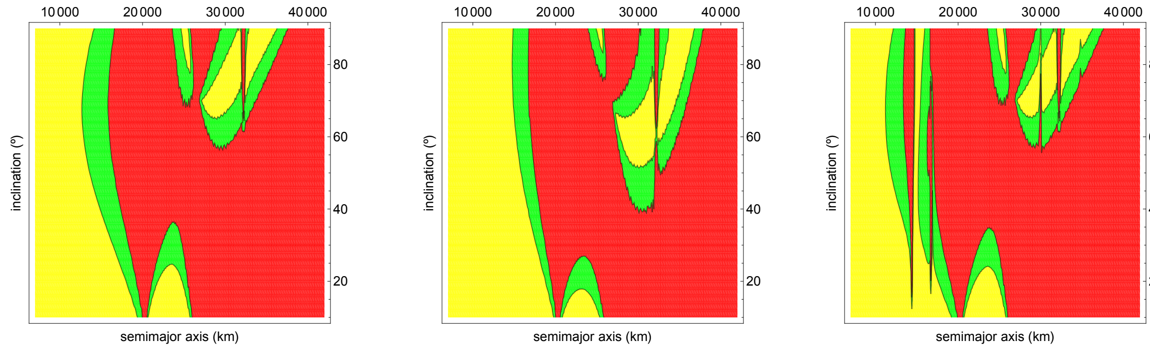

Figure 3: Differences between analytical and numerical integrations for the (4×4)-model: Left) a 6th order theory; Center) a 7th order theory. Right) The same analysis, but now for the (5×5)-model and a 6th order theory. Horizontal axis is the semi-major axis $a \in [7000, 42000]$ km (from low orbits till geostationary orbits), and vertical axis represents the inclination $i \in [0^o, 90^o]$. Initial conditions (a, i) inside the yellow region provide an error less than one km after one month of propagating the orbit. Green area give errors between one and two km. Red area represents errors greater than two km after the propagation of the orbit for one month.

The left part of the Figure 2 represents the comparison between a 6-th order theory of the model 2×2 with respect to the numerical integration of the same mode. The right part is the comparison of the 7-th order theory of the same model with respect to the numerical integration of this model.

Figure 3 contains the comparison of the 6-th and 7-order of the (4×4)-model (left and middle plots respectively) and the 6-th order of the (5×5)-model (right plot). The yellow area of these figures contains initial conditions (a, i) which give an error less than one km after propagating the orbit during one month. Green area represents error between one and two km. Red area corresponds to initial conditions giving errors greater than two km after one month. We can see that the analytical propagation gives good results for small values of the semi major axis, which agree with the relation of the small parameter of the theory and the semi major axis.

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