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A flux-limited second order scheme for hyperbolic conservation laws with source terms

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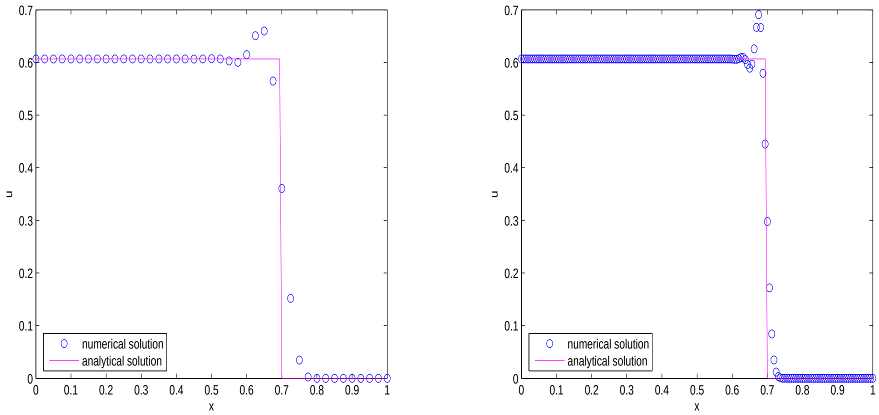
Abstract

The theoretical foundations of high-resolution TVD schemes for homogeneous scalar conservation laws and linear systems of conservation laws have been firmly established through the work of Harten [8], Sweby [13], and Roe [11]. These TVD schemes seek to prevent an increase in the total variation of the numerical solution, and are successfully implemented in the form of flux-limiters or slope limiters for scalar conservation laws and systems. However, their application to conservation laws with source terms is still not fully developed. In this work we analyze the properties of a second order, flux-limited version of the Lax-Wendroff scheme preserving steady states [5]. Our technique is based on a flux limiting procedure applied only to those terms related to the physical flow derivative.

1 Introduction

The theory of numerical schemes for homogeneous scalar conservation laws is well established. Total Variation Diminishing (TVD) schemes have proved to be particularly successful at capturing shock waves and discontinuous solutions. A problem of increasing importance in Computational Fluids Dynamics is the application of numerical methods to inhomogeneous problems such as shallow water equations. In such cases the TVD property is no longer valid. Although certain source terms may preserve the TVD property of the homogeneous part, others will actively increase the variation in the solution. An adapted one-step second-order scheme gives a very good accuracy of the solution in smooth regions although the inevitable presence of spurious overshoots in the proximity of the shock, typical of second order schemes, has been observed. As in the homogeneous case, the oscillations are not reduced if we use a fine mesh (see figure 1). This motivates the use of TVD-like schemes for inhomogeneous problems, however, although care needs to be taken in the inclusion of the source terms.



$\triangle x = 0.025$ $\triangle x = 0.00625$

Figure 1.— Second order scheme applied to $u_t + u_x = -u$.

2 Operator-Splitting

A popular method of treating inhomogeneous hyperbolic equations of the form

$$u_t + f(u)_x = s(x, u), \tag{1}$$

is to split the problem, over a time step $\triangle t$, into a homogeneous conservation part

$$u_t + f(u)_x = 0 \tag{2}$$

and an ODE part

$$u_t = s(x, u) \tag{3}$$

and to alternate between the two solutions. The numerical solution for a general scalar problem of the type (1) would be to find the numerical solution, $\bar{u}^{n+1}$, of (2) with initial data $u(x, t^n) = u^n$, a high order TVD scheme would be suitable, then use a numerical ODE solver , like various Runge-Kutta type methods, to obtain u^{n+1} from (3) with initial data $u = \bar{u}^{n+1}$.

The advantages of such an approach are clear since numerical schemes for both (2) and (3) are well developed and can be chosen to optimal effect. This is particularly true for stiff problems where much work has been undertaken using implicit ODE solvers [10]. Despite their advantages in problems with stiff source terms, the situation is by no means ideal, however. The solution of the homogeneous PDE part may cause a large departure from the true solution which will need to be recovered by the ODE solver. If the recovery is not exact, numerical errors will be introduced.

There are some other potential pitfalls in using a fractional-step method to handle source terms. This approach performs very poorly in those situations where u_t is small relative to the other two terms, in particular when steady or quasi-steady solutions are being sought. For such solutions, highly accurate numerical simulations can only be obtained from numerical methods that respect the balance that occurs between the flux gradient and the source term when u_t is small, and it is known ([9]) that this balance is not likely to be respected when using a fractional step approach.

3 An adapted second-order method

Many numerical methods, like fractional step methods, have difficulties preserving steady states and cannot accurately calculate small perturbations of such states, as we have observed in the previous section.

The source term has to be incorporated into the algorithm, avoiding fractional steps. In general, the source term can be approximated in two ways: A pointwise approach, where the source term approximation is calculated at the nodal points, and an upwind characteristic based approach, where the source term is approximated in a more physical way. Roe [12] put forward the idea of upwinding the source terms in inhomogeneous conservation laws, in a manner similar to that for constructing numerical fluxes for solving homogeneous conservation laws. Further work in this direction was carried out by Bermúdez and Vázquez-Cendón [1], who started by considering the problem

$$u_t + au_x = s(x, u). \tag{4}$$

The solution of this inhomogeneous linear equation with nonlinear source, considering a constant ($a > 0$), at time $t = (n + 1)\triangle t$ can be calculated by integrating along the characteristic through (x_i, t_{n+1}) between t_n and t_{n+1} to give

$$u(x_i, t_{n+1}) = u(x_i - a\triangle t, t_n) - \int_{t_n}^{t_{n+1}} b(x_i - a(t_{n+1} - \xi), u(x_i - a(t_{n+1} - \xi), \xi)d\xi. \tag{5}$$

In the above integral b is clearly dependent on data in the upwind domain, indicating a need for an upwind treatment of source term.

In this sense, Gascón and Corberán in [5] presented an extension of the one-step Lax-Wendroff scheme for inhomogeneous conservation laws by rewriting (1) as

$$u_t + g_x = 0 \quad \text{where} \quad g = f(u) - \int_0^x s(\xi, u(\xi, t))d\xi, \tag{6}$$

and g is considered as a function of x and u. However, if we consider $g = g(x, t)$, then a second order method is obtained by the scheme

$$U_i^{n+1} = U_i^n - \frac{\triangle t}{\triangle x}(g_{i+\frac{1}{2}}^{n+\frac{1}{2}} - g_{i-\frac{1}{2}}^{n+\frac{1}{2}}), \tag{7}$$

where the estimation of the new flux, g, at the point mid-way between grid points is obtained by an expression based on Taylor's expansion

$$g_{i+\frac{1}{2}}^{n+\frac{1}{2}} = g_{i+\frac{1}{2}}^n + \frac{\triangle t}{2}\frac{\partial g}{\partial t}\bigg|_{i+\frac{1}{2}}^n. \tag{8}$$

By introducing the following notation:

$$\alpha^n_{i+\frac{1}{2}} = \frac{\triangle t}{\triangle x}\frac{\partial f}{\partial u}\bigg|^n_{i+\frac{1}{2}}, \quad \beta^n_{i+\frac{1}{2}} = \frac{\triangle t}{2}\frac{\partial s}{\partial u}\bigg|^n_{i+\frac{1}{2}}, \quad b^n_{ik} \approx \int_{x_i}^{x_k} -s(\xi, u(\xi, t_n))d\xi^1,$$

and using simple algebraic manipulations, the scheme admits the expression

$$\begin{aligned} U_i^{n+1} &= U_i^n - \frac{\triangle t}{\triangle x}(f^{LW}_{i+\frac{1}{2}} - f^{LW}_{i-\frac{1}{2}}) - \frac{\triangle t}{\triangle x}(b^n_{i-\frac{1}{2}i} + b^n_{ii+\frac{1}{2}}) \\ &- \frac{\triangle t}{2\triangle x}(\beta^n_{i+\frac{1}{2}}(f^n_{i+1} - f^n_i + b^n_{ii+1}) + \beta^n_{i-\frac{1}{2}}(f^n_i - f^n_{i-1} + b^n_{i-1i})) \end{aligned} \quad (9)$$

with

$$f^{LW}_{i+\frac{1}{2}} = \frac{1}{2}(f^n_{i+1} + f^n_i - b^n_{ii+\frac{1}{2}} + b^n_{i+\frac{1}{2}i+1} - \alpha^n_{i+\frac{1}{2}}(f^n_{i+1} - f^n_i + b^n_{ii+1})). \quad (10)$$

4 A Flux limiter scheme

The motivation for this work is to analyze the properties of a second order, flux-limited version of the Lax-Wendroff scheme which preserves the TVD property, in the sense that it avoids oscillations around discontinuities, while preserving steady states([5]).

We consider the Lax-Wendroff method (9) adapted to a balance law (1), this is a second order method that generates spurious oscillations near discontinuities (see figure 1). In order to construct a "flux-limiting" method, we consider the numerical flux in (9) of the form

$$F^n_{i+\frac{1}{2}} = F^{LO}_{i+\frac{1}{2}} + \phi_{i+\frac{1}{2}}(F^{HI}_{i+\frac{1}{2}} - F^{LO}_{i+\frac{1}{2}}), \quad (11)$$

using (10) as a high order numerical flux($F^{HI}_{i+\frac{1}{2}}$). As a low order numerical flux($F^{LO}_{i+\frac{1}{2}}$), our choice is

$$F^{LO}_{i+\frac{1}{2}} = \frac{1}{2}(f^n_{i+1} + f^n_i - b^n_{ii+\frac{1}{2}} + b^n_{i+\frac{1}{2}i+1} - sign(\alpha^n_{i+\frac{1}{2}})(f^n_{i+1} - f^n_i + b^n_{ii+1})). \quad (12)$$

We called this method the **TVDB** scheme, and we can notice that the numerical flux incorporates information on the source term in its definition, for this reason, we define the variable $r_{i+\frac{1}{2}}$, that is always the ratio of the upwind change to the local change, as:

$$r_{i+\frac{1}{2}} = \begin{cases} \dfrac{f_i - f_{i-1} + b_{i-1i}}{f_{i+1} - f_i + b_{ii+1}}, & sign(\alpha_{i+\frac{1}{2}}) > 0; \\ \\ \dfrac{f_{i+2} - f_{i+1} + b_{i+1i+2}}{f_{i+1} - f_i + b_{ii+1}}, & sign(\alpha_{i+\frac{1}{2}}) < 0. \end{cases} \quad (13)$$

As a flux limiter function $\phi_{i+\frac{1}{2}} = \phi(r_{i+\frac{1}{2}})$ we use the minmod limiter,

$$\phi_{i+\frac{1}{2}} = max(0, min(r_{i+\frac{1}{2}}, 1)) \quad (14)$$

[1]We make sure that b^n_{ik} approximation guarantee that the scheme satisfy the exact C-property, i.e., it is exact when applied to the stationary case.

In Fig 2-left, we display the numerical results obtained after applying this scheme to $u_t + u_x = -u$. A slight oscillation can be observed, whose amplitude decreases with the mesh width, as shown in figure 2right.

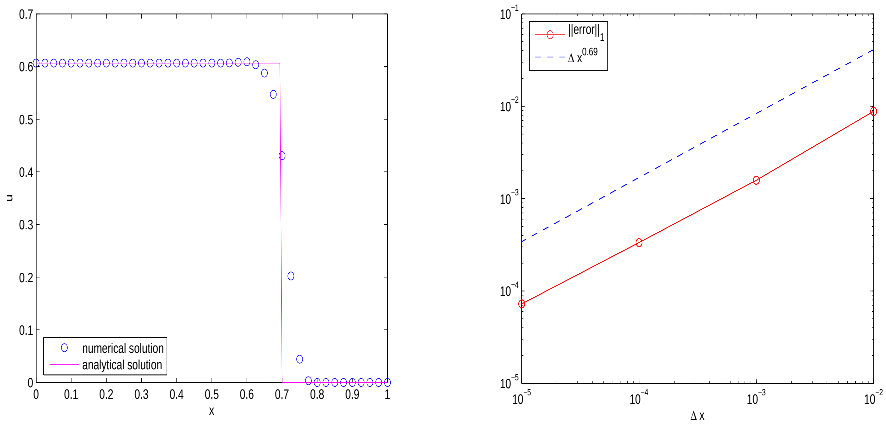


Figure 2.— $u_t + u_x = -u$. Left: TVB method. Right Error for the TVDB method.

5 Burgers' equation with source terms

In a variety of physical problems one encounters source terms that are balanced by internal forces and this balance supports multiple steady state solutions that are stable. Typical of these are gravity-driven flows such as those described by the shallow water equations over a nonuniform ocean bottom. In this section we show a scalar 1-D approximation of balance laws of this kind.

5.1 The Embid problem

This problem was presented in [3] as a simple scalar approximation to the 1-D equations that model the flow of a gas through a duct of variable cross-section.

$$\begin{cases} u_t + (\frac{u^2}{2})_x = (6x-3)u, & 0 < x < 1 \\ u(0,t) = 1, u(1,t) = -0.1. \end{cases} \tag{15}$$

There are two entropy satisfying steady solutions for the Embid problem. One is stable in time with a standing shock at $x_1 = 0.18$ and the other with an unstable standing shock at $x_2 = 0.82$. The steady solutions for the Embid problem are

$$u(x) = \begin{cases} 1 + 3x^2 - 3x, & x < x_i; \\ -0.1 + 3x^2 - 3x, & x > x_i. \end{cases} \tag{16}$$

for $i = 1, 2$. We computed the steady profiles by taking initial data with a jump at the stable location, using a CFL number equal to 0.8 and by marching in time until convergence criterion

$$\sum_i |u_i^{n+1} - u_i^n| \leq 10^{-10}$$

was satisfied (figure 3left).

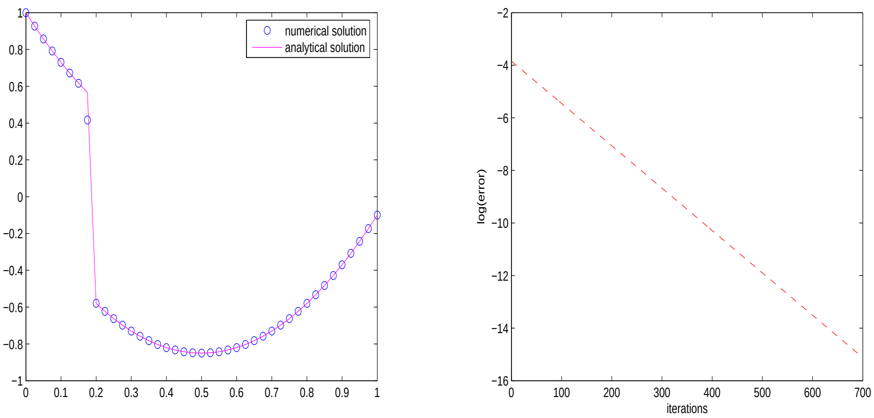


Figure 3.— Embid problem. Left: TVB scheme. Right: Convergence history

The TVDB numerical solution reproduces the exact steady solution except for one internal shock point. The scheme requires 383 iterations to reach the stationary solution with a residual less than 10^{-10} and using the minmod limiter. Figure 3right shows the logarithm of the residual errors with respect to the number of iterations for both schemes.

5.2 Greenberg et al. tests

In order to test the methods described above, we show the numerical result following the tests in [7]. Let us consider the equation

$$u_t + (\frac{u^2}{2})_x + a_x(x)u = 0. \tag{17}$$

where

$$a(x) = 0.9 \begin{cases} 0, & x < 0; \\ (\cos(\pi \frac{x-1}{2}))^{30}, & 0 \leq x \leq 2; \\ 0, & 2 < x. \end{cases} \tag{18}$$

Figure 4left is the numerical solution of (17) with the initial data $u + a = 1$ at time 1 (Experiment 1). The l_1-error is $6.9044 \cdot 10^{-17}$ for the TVDB scheme, thus the C-property is ensured.

On the other hand, the initial condition used to generate figure 4right at time 1.5 (Experiment 2) is

$$u + a = \begin{cases} 1.3, & x < 0.2; \\ 1, & 0 < x. \end{cases} \tag{19}$$

In this case, we cannot observe any spurious oscillations in the numerical solution.

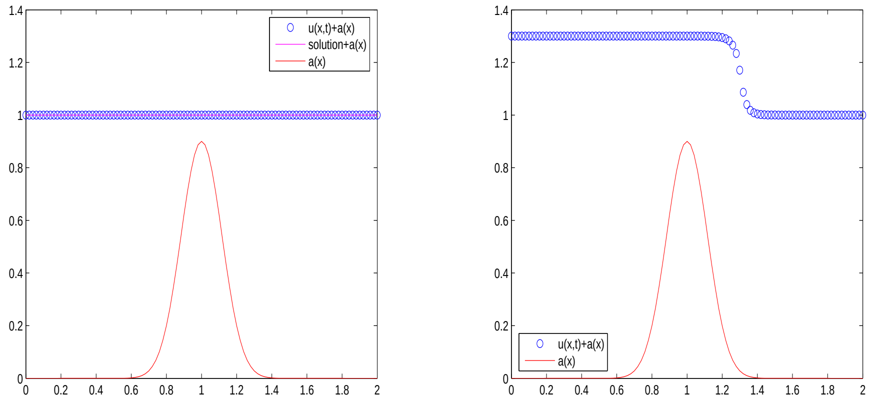


Figure 4.— Left: Experiment 1. Right: Experiment 2.

6 The shallow water equations

The shallow water equations form a hyperbolic system of conservation laws that approximately describes various geophysical flows. We consider source terms due to topography, but we do not consider wind effects and Coriolis force. In this case, the one dimensional shallow water equations are as follows

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{s}(x, \mathbf{u}), \tag{20}$$

where

$$\mathbf{u} = \begin{pmatrix} h \\ q \end{pmatrix}, \quad \mathbf{f}(\mathbf{u}) = \begin{pmatrix} q \\ \frac{q^2}{h} + \frac{g}{2}h^2 \end{pmatrix}, \quad \text{and} \quad \mathbf{s}(x, \mathbf{u}) = \begin{pmatrix} 0 \\ -ghz_x \end{pmatrix}.$$

We propose an extension of the TVDB scheme to this system via the so-called characteristic-based approach, see [4], [2]. This technique makes use of the fact that all the eigenvalues of the Jacobian matrix of the convective flux vector are real, and the matrix is diagonalizable, i.e., there is a complete set of N linearly independent eigenvectors.

The following series of numerical experiments are standard in the literature. Firstly, and in order to give a numerical validation of the C-property, we consider the following steady flow cases.

Steady flow

Following [14], let consider a channel with a length of 20 m defined as

$$z(x) = 0.2e^{-\frac{2}{5}(x-10)^2}, \tag{21}$$

with the quiescent state $h + z = 2$m ($q = 0$) as initial condition. As expected, the TVDB scheme exactly preserves the steady state (see figure 5), because the $\mathcal{L}^1$ error of the numerical solution at time 50 s is $3.3267 \cdot 10^{-14}$, which is roundoff error.

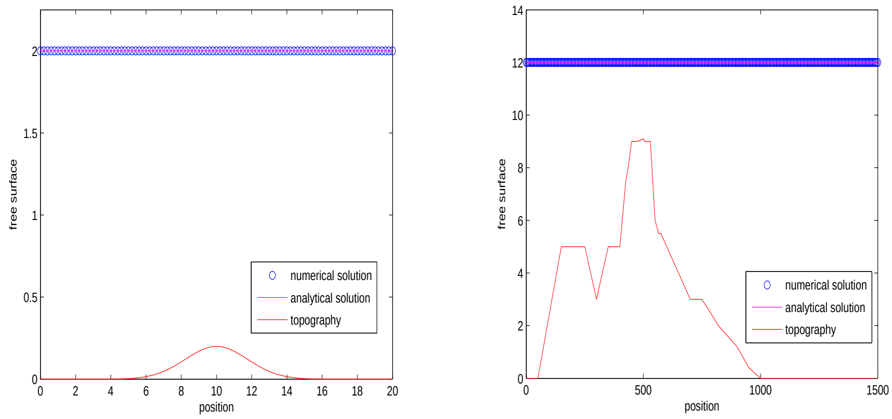


Figure 5.— Flow at rest. Left: Smooth topography. Right: Complex topography.

Usually, the bottom topography is not smooth. With the aim of evaluating the performance of a numerical scheme in the presence of complex and possibly non-smooth geometry, the following experiment was proposed in the workshop on dam-break wave simulation [6]. The initial data is the water at rest at a level of 12m. Numerical results obtained after a simulation of 200s are displayed in figure (5), we can observed that also in this test the $\mathcal{L}^1$ error, which is $1.4989 \cdot 10^{-15}$ is roundoff error.

The interest of the next three tests (extracted from [6]) is to study the convergence of this scheme towards a steady state. In these simulations a bottom topography of 25m length is defined as:

$$z(x) = \begin{cases} 0.2 - 0.05(x-10)^2, & 8\text{m} < x < 12\text{m} \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

In all cases, the initial data are $h + z =$constant and $q = 0$. The analytical solution is computed with the Bernoulli equation

$$\frac{q^2}{2gh^2} + h + z = H_a,$$

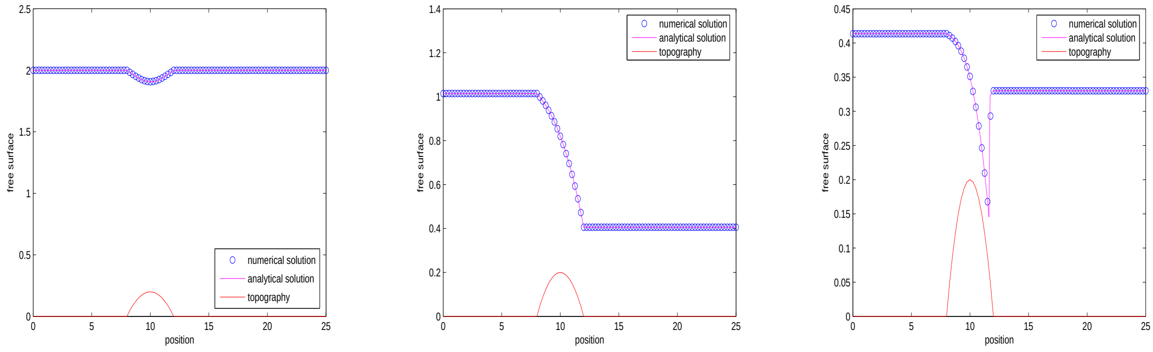
where H_a is the upstream head, q is the steady discharge and h the water level.

For the initial conditions are $h + z = 2$m, and $q = 0$ and boundary conditions

- downstream: $h = 2$ m
- upstream: $q = 4.42$ m$^2/s$.

The resulting flow is a subcritical flow (6-(a)). If we use as boundary conditions

- downstream: $h = 0.66$ m only when $F_r < 1$
- upstream: $q = 1.53$ m^2/s.



(a) (b) (c)

Figure 6.— Steady flow over a hump. (a) Subcritical flow. (b) Transcritical flow without shock. (c) Transcritical flow with shock.

where $F_r = u/\sqrt{gh}$ is the Froude number, and $h+z = 0.66$ m and $q = 0$ as initial condition, then a transcritical flow without shocks is obtained, see figure 6-(b). Transcritical flow, with a shock (see figure 6-(c)), is obtained if $h + z = 0.33$ m and $q = 0$ is used as initial data, and in this case the boundary conditions are

- downstream: $h = 0.33$ m
- upstream: $q = 0.18$ m^2/s.

Quasi stationary flow

This last test were proposed in [9] by LeVeque in order to evaluate the capability of the scheme to accurately compute small perturbations of the water surface over a variable topography, in this case is given as

$$z(x) = \begin{cases} 0.25\left(\cos\left(\pi\dfrac{x-0.5}{0.1}\right)+1\right), & \text{if } |x-0.5| < 0.1 \\ 0, & \text{otherwise,} \end{cases} \tag{23}$$

on $0 < x < 1$and with $g = 1$. The initial condition is

$$\begin{aligned} h + z &= 1 + 0.001 \text{ for } 0.1 < x < 0.2 \\ q &= 0, \end{aligned}$$

which represents a small hump perturbation of the quiescent state $(h, q) = (1 - z, 0)$. LeVeque uses this test to show the disadvantages of schemes that do not preserve steady states. In figure 7, we show the numerical result at time 0.7, using different limiters. We could observe in this example the main features of both limiters..

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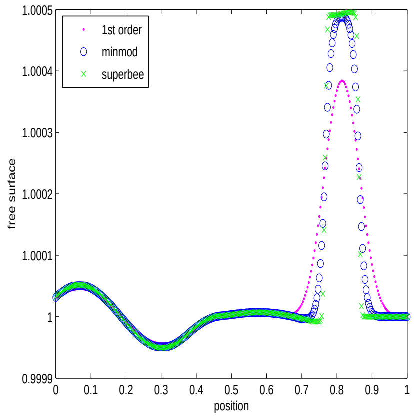

Figure 7.— Numerical solution for the quasi stationary flow, $t = 0.7$s, for the first order and second order using two different limiters.

References

[1] Bermúdez, A., Vázquez, M.: Upwind methods for hyperbolic conservation laws with source terms. Computers & Fluids **23**, 1049–1071 (1994)

[2] Donat, R., Mulet, P.: The two-jacobian scheme for systems of conservation laws. In: Analysis and Simulation of Fluid Dynamics, pp. 89–108. Birkhäuser (2007)

[3] Embid, P., Goodman, J., Majda, A.: Multiple steady states for 1-d transonic flow. SIAM Journal on Scientific and Statistical Computing **5**, 21–41 (1984)

[4] Fedkiw, R.P., Merriman, B., Donat, R., Osher, S.: The penultimate scheme for systems of conservation laws: finite difference ENO with Marquina's flux splitting. In: Innovative methods for numerical solutions of partial differential equations (Arcachon, 1998), pp. 49–85. World Sci. Publ., River Edge, NJ (2002)

[5] Gascón, L., Corberán, J.M.: Construction of second-order tvd schemes for nonhomogeneous hyperbolic conservation laws. J. Comput. Phys. **172**(1), 261–297 (2001)

[6] Goutal, N., Maurel, F. (eds.): Proceedings of the 2nd Workshop on Dam Break Wave Simulation. EDF-DER Report HE-43/97/016/B, 42,45 (1997)

[7] Greenberg, J.M., Leroux, A.Y.: A well-balanced scheme for the numerical processing of source terms in hyperbolic equations. SIAM J. Numer. Anal. **33**(1), 1–16 (1996)

[8] Harten, A.: High resolution schemes for hyperbolic conservation laws. Journal of Computational Physics **135(2)**, 260–278 (1997)

[9] LeVeque, R.J.: Balancing source terms and flux gradients in high-resolution Godunov methods: the quasi-steady wave-propagation algorithm. J. Comput. Phys. **146**(1), 346–365 (1998)

[10] LeVeque, R.J., Yee, H.C.: A study of numerical methods for hyperbolic conservation laws with stiff source terms. J. Comput. Phys. **86**(1), 187–210 (1990)

[11] Roe, P.: Generalized formulation of tvd lax-wendroff schemes. ICASE **84**, 53 (1984)

[12] Roe, P.: Upwind differencing schemes for hyperbolic conservation laws with source terms. In: in proceedings of Nonlinear Hyperbolic problems,edited by C. Carasso, P.Raviart, and D. Serre, Lecture Notes in Mathematics (Springer-Verlag), vol. 1270, p. 41 (1986)

[13] Sweby, P.K.: High resolution schemes using flux limiters for hyperbolic conservation laws. SIAM Journal on Numerical Analysis **21**(5), 995–1011 (1984)

[14] Vukovic, S., Sopta, L.: ENO and WENO schemes with the exact conservation property for one-dimensional shallow water equations. J. Comput. Phys. **179**(2), 593–621 (2002)