

Some topics concerning the theory of singular dynamical systems

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Abstract

Some subjects related to the geometric theory of singular dynamical systems are reviewed in this paper. In particular, the following two matters are considered: the theory of canonical transformations for presymplectic Hamiltonian systems, and the Lagrangian and Hamiltonian constraint algorithms and the time-evolution operator.

Key words: *Presymplectic manifolds, singular systems, canonical transformations, Lagrangian formalism, Hamiltonian formalism.*

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1 Introduction

The aim of this paper is to carry out a brief review of several topics concerning the theory of autonomous singular dynamical systems, from a geometrical perspective. In particular, our interest will be focused on two subjects, namely: the theory of canonical transformations for singular systems, and the problem of the compatibility of the dynamical equations of Lagrangian and Hamiltonian singular systems; more precisely, the analysis of the Lagrangian and Hamiltonian constraint algorithms and their relation.

The article is based on the developments made on the references [18] and [7], for the theory of canonical transformations, and [1], [12], [13], [32], [10] and [23] for aspects related with the constraint algorithms. Thus, we will refer to these articles for more details on all these results.

All manifolds are real and C^∞ . All maps are C^∞ . Sum over crossed repeated indices is understood. Throughout this paper $i(X)\omega$ denotes the contraction between the vector field X and the differential form ω , and $L(X)\omega$ the Lie derivative of ω with respect to the vector field X .

2 Canonical transformations

2.1 Presymplectic Locally Hamiltonian Systems

Definition 1 A presymplectic locally Hamiltonian system (*p.l.h.s.*) is a triad (M, ω, α) , where (M, ω) is a presymplectic manifold, and $\alpha \in Z^1(M)$ (i.e., it is a closed 1-form), which is called a Hamiltonian form.

A p.l.h.s has associated the following equation

$$i(X)\omega = \alpha \quad , \quad X \in \mathfrak{X}(M)$$

If X exists, it is called a *presymplectic locally Hamiltonian vector field* associated to α . Nevertheless, in the best cases, this equation has consistent solutions only in a submanifold $j_C: C \hookrightarrow M$, where there exist $X \in \mathfrak{X}(M)$, tangent to C , such that

$$[i(X)\omega - \alpha]|_C = 0 \tag{1}$$

Furthermore, the solution X is not unique, in general, and this non-uniqueness is known as *gauge freedom*. In general $(C, \omega_C = j_C^*\omega)$ is a presymplectic manifold which is called the *final constraint submanifold* (f.c.s.).

The following theorem gives the local structure of p.l.h.s. (see [7]):

Theorem 1 Let (M, ω, α) be a p.l.h.s., and $j_C: C \hookrightarrow M$ the f.c.s. Then:

1. There are a symplectic manifold (P, Ω) and a coisotropic embedding $\iota_C: C \hookrightarrow P$ such that $\omega_C = \iota_C^*\Omega$.

2. For every $X \in \mathfrak{X}(M)$, tangent to C , solution to (1), there exists a family of vector fields $\mathfrak{X}(P, C) \subset \mathfrak{X}(P)$ such that, for every $X_\xi \in \mathfrak{X}(P, C)$, (a) X_ξ are tangent to C , (b) $X_\xi|_C = X|_C$, and (c) X_ξ are solutions to the equations $i(X_\xi)\Omega = \alpha_P + \xi$, where $\alpha_P \in Z^1(P)$ satisfies $\iota_C^*\alpha_P = j_C^*\alpha$, and $\xi \in Z^1(P)$ is any first-class constraint form (i.e., $\iota_C^*\xi = 0$, and the Hamiltonian vector field associated with ξ , $X_\xi \in \mathfrak{X}(P)$, is tangent to C).

3. The coisotropic embedding ι_C and the family $\mathfrak{X}(P, C)$ are unique, up to an equivalence relation of local symplectomorphisms reducing to the identity on C .

(P, C, Ω) is the coisotropic canonical system associated to (M, ω, α) .

2.2 Canonical Transformations for p.l.h.s.

Let $\mathfrak{X}_{lh}(C)$ be the set of locally Hamiltonian vector fields in (C, ω_C) ; that is, $\mathfrak{X}_{lh}(C) = \{X_C \in \mathfrak{X}(C) \mid L(X_C)\omega_C = 0\}$.

Definition 2 Let $(M_i, \omega_i, \alpha_i)$ ($i = 1, 2$) be a p.l.h.s., with f.c.s. $j_{C_i}: C_i \rightarrow M_i$, and $\omega_{C_i} = j_{C_i}^*\omega_i$, such that $\dim M_1 = \dim M_2$, $\dim C_1 = \dim C_2$, and $\text{rank } \omega_{C_1} = \text{rank } \omega_{C_2}$.

A canonical transformation *between these systems* is a pair (Φ, ϕ) , with $\Phi \in \text{Diff}(M_1, M_2)$ and $\phi \in \text{Diff}(C_1, C_2)$, such that:

1. $\Phi \circ J_{C_1} = J_{C_2} \circ \phi$; that is, we have the commutative diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{\Phi} & M_2 \\ \uparrow J_{C_1} & & \uparrow J_{C_2} \\ C_1 & \xrightarrow{\phi} & C_2 \end{array}$$

2. $\phi_*(\mathfrak{X}_{lh}(C_1)) \subset \mathfrak{X}_{lh}(C_2)$.

The generalization of Lee Hwa Chung's theorem to presymplectic manifolds allows us to prove that (see [18]):

Proposition 1 *Condition 2 is equivalent to saying that there exists $c \in \mathbb{R}$ such that $J_{C_1}^*(\Phi^*\omega_2 - c\omega_1) = \phi^*\omega_{C_2} - c\omega_{C_1} = 0$.*

c is called the *valence* of the canonical transformation. So, *univalent canonical transformations* are the *presymplectomorphisms* between C_1 and C_2 .

Let (P_i, C_i, Ω_i) be the coisotropic canonical systems associated with the p.l.h.s $(M_i, \omega_i, \alpha_i)$, $i = 1, 2$. The class of ϕ is defined by $\{\phi\} = \{\Psi \in \text{Diff}(P_1, P_2) \mid \Psi \circ \iota_{C_1} = \iota_{C_2} \circ \phi\}$. So we have the diagram

$$\begin{array}{ccccc} & & \Psi & & \\ & & \xrightarrow{\hspace{2cm}} & & \\ P_1 & & & & P_2 \\ & \nearrow & & & \nwarrow \\ & M_1 & \xrightarrow{\Phi} & M_2 & \\ & \uparrow J_{C_1} & & \uparrow J_{C_2} & \\ & C_1 & \xrightarrow{\phi} & C_2 & \end{array}$$

And therefore we have (see [7]):

Theorem 2 *There exists $\psi \in \{\phi\}$ which is a symplectomorphism between the symplectic manifolds (P_1, Ω_1) and (P_2, Ω_2) .*

As a particular situation, we can analyze the canonical transformations in a p.l.h.s. Thus, let (M, ω, α) be a p.l.h.s., with f.c.s. $J_C: C \hookrightarrow M$. Consider the involutive distribution $\ker \omega_C$ in C , and assume that the quotient space $\hat{C} = C / \ker \omega_C$ is a manifold with natural projection $\hat{\pi}: C \rightarrow \hat{C}$ (it is called the *reduced phase space* associated to the p.l.h.s.). Then the form ω_C is $\hat{\pi}$ -projectable; hence there exists $\hat{\Omega} \in \Omega^2(\hat{C})$ such that $\omega_C = \hat{\pi}^*\hat{\Omega}$. Furthermore, $(\hat{C}, \hat{\Omega})$ is a symplectic manifold, and we have the following result (see [7]):

Proposition 2 *Every canonical transformation (Φ, ϕ) in (M, ω, α) leaves the distribution $\ker \omega_C$ invariant. As a consequence, there exists a unique $\hat{\phi} \in \text{Diff}(\hat{C})$ such that:*

1. $\hat{\phi} \circ \hat{\pi} = \hat{\pi} \circ \phi$.
2. $\hat{\phi}$ is a symplectomorphism.

Some applications of the geometric theory of p.l.h.s. and their canonical transformations are the following: the study of canonical transformations for regular autonomous systems (it includes a new geometrical description of these kinds of systems based on the *coisotropic embedding theorem*) [8], the analysis of the geometric structure and construction of canonical transformations for the free relativistic massive particle [8], the discussion about time scaling transformations in Hamiltonian dynamics and their application to celestial mechanics [9], and the construction of realizations of symmetry groups for singular systems and, as an example, the Poincaré realizations for the free relativistic particle [35].

3 Constraints and the Evolution Operator

3.1 Lagrangian Dynamical Systems

(See [3] for details).

Let Q be a n -dimensional differential manifold which constitutes the *configuration space* of a dynamical system. Its tangent and cotangent bundles, $\tau_Q: TQ \rightarrow Q$ and $\pi_Q: T^*Q \rightarrow Q$, are the (*phase spaces of velocities and momenta* of the system).

A *Lagrangian dynamical system* is a couple (TQ, \mathcal{L}) , where $\mathcal{L} \in C^\infty(TQ)$ is the *Lagrangian function* of the system. Using the canonical elements of TQ , the vertical endomorphism $\mathcal{S} \in \mathfrak{X}_1^1(TQ)$ and the Liouville vector field $\Delta \in \mathfrak{X}(TQ)$, we can define the *Lagrangian 2-form* $\omega_{\mathcal{L}} = -d(d\mathcal{L} \circ \mathcal{S})$, and the *Lagrangian energy function* $E_{\mathcal{L}} = \Delta(\mathcal{L}) - \mathcal{L}$. Moreover, we define the *Legendre transformation* associated with \mathcal{L} , $\mathcal{FL}: TQ \rightarrow T^*Q$, as the fiber derivative of the Lagrangian.

\mathcal{L} is a *singular Lagrangian* if $\omega_{\mathcal{L}}$ is a presymplectic form (which is assumed to have constant rank) or, what is equivalent, \mathcal{FL} is no longer a local diffeomorphism. In particular, \mathcal{L} is an *almost-regular Lagrangian* if: (i) $M_0 = \mathcal{FL}(TQ)$ is a closed submanifold of T^*Q , (ii) \mathcal{FL} is a submersion onto M_0 , and (iii) for every $p \in TQ$, the fibres $\mathcal{FL}^{-1}(\mathcal{FL}(p))$ are connected submanifolds of TQ . Then (TQ, \mathcal{L}) is an *almost-regular Lagrangian system*.

For almost-regular Lagrangian systems, $(TQ, \Omega_{\mathcal{L}}, dE_{\mathcal{L}})$ is a p.l.h.s., and we have the so-called *Lagrangian dynamical equation*

$$i(\Gamma_{\mathcal{L}})\omega_{\mathcal{L}} = dE_{\mathcal{L}} \quad (2)$$

Variational considerations lead us to impose that, solutions $\Gamma_{\mathcal{L}} \in \mathfrak{X}(TQ)$ to (2) must be *second-order differential equations* (SODE); that is, holonomic vector fields in TQ .

Geometrically this means that

$$\mathcal{S}(\Gamma_{\mathcal{L}}) = \Delta \quad (3)$$

For singular Lagrangians this does not hold in general, and (3) must be imposed as an additional condition (*SODE-condition*). Integral curves of vector fields satisfying (2) and (3) are solutions to the Euler-Lagrange equations.

The *Lagrangian problem* consists in finding a submanifold $J_{S_f} : S_f \hookrightarrow TQ$, and $\Gamma_{\mathcal{L}} \in \mathfrak{X}(TQ)$, tangent to S_f , such that

$$[i(\Gamma_{\mathcal{L}})\omega_{\mathcal{L}} - dE_{\mathcal{L}}]|_{S_f} = 0 \quad , \quad [\mathcal{S}(\Gamma_{\mathcal{L}}) - \Delta]|_{S_f} = 0$$

Now, if $\mathcal{FL}_0 : TQ \rightarrow M_0$ is the restriction of \mathcal{FL} to M_0 , we have that $\omega_{\mathcal{L}}$ and $E_{\mathcal{L}}$ are \mathcal{FL}_0 -projectable: there exist $\omega_0 \in \Omega^2(M_0)$, and $h_0 \in C^\infty(M_0)$ such that $\omega_{\mathcal{L}} = \mathcal{FL}_0^*\omega_0$, $E_{\mathcal{L}} = \mathcal{FL}_0^*h_0$. Then, (M_0, ω_0, dh_0) is a p.l.h.s. which is called the *canonical Hamiltonian system* associated with the Lagrangian system $(TQ, \omega_{\mathcal{L}}, dE_{\mathcal{L}})$. So we have the *Hamiltonian dynamical equation*

$$i(X_0)\omega_0 = dh_0 \quad ; \quad X_0 \in \mathfrak{X}(M_0)$$

and the *Hamiltonian problem* consists in finding a submanifold $J_{M_f} : M_f \hookrightarrow M_0$ and $X_0 \in \mathfrak{X}(M_0)$ tangent to M_f such that

$$[i(X_0)\omega_0 - dh_0]|_{M_f} = 0$$

3.2 Constraint Algorithms

In order to solve the Hamiltonian problem stated for an almost-regular system different kinds of Hamiltonian constraint algorithms were developed. The first was the local-coordinate *Dirac constraint algorithm* [16], but there were also geometric algorithms: the *Presymplectic Constraint Algorithm* (PCA) of *Gotay, Nester, Hinds* [21], and others by *Marmo, Tulczyjew* et al. [30], [31], etc. All of them give a sequence of submanifolds which, in the best cases, stabilizes giving the f.c.s.: $T^*Q \leftarrow M_0 \leftarrow M_1 \leftarrow \dots \leftarrow M_f$.

For the Lagrangian problem, the first attempt was not to consider the SODE-problem (3), and then develop a Lagrangian constraint algorithm by simply applying the P.C.A. to the Lagrangian dynamical equation (2), obtaining a sequence of submanifolds $TQ \leftarrow P_1 \leftarrow \dots \leftarrow P_f$ [19]. The SODE problem is studied later in [20], obtaining a submanifold S_f which solves the lagrangian problem, but which is not defined by constraints and is not maximal. Later, *Kamimura* [28] and *Batlle, Gomis, Pons, Román-Roy* [1] developed local-coordinate Lagrangian constraint algorithms in which the Lagrangian dynamical equation (2) and the SODE-condition (3) were both considered at the same time. The f.c.s. S_f obtained at the end of the corresponding sequence, $T^*Q \leftarrow S_1 \leftarrow \dots \leftarrow S_f$, is

maximal and is defined by constraints. The relation between the aforementioned sequences of submanifolds is explained in the following diagram

$$\begin{array}{ccccccc}
 \text{TQ} & \leftrightarrow & P_1 & \leftrightarrow & \dots & \leftrightarrow & P_f \\
 & & & \swarrow & & \swarrow & \\
 & & & & S_1 & \leftrightarrow & \dots & \leftrightarrow & S_f
 \end{array} \tag{4}$$

In particular, the submanifolds P_i , $i = 1 \dots f$, are defined by constraints that can be expressed as \mathcal{FL} -projectable functions which give all the Hamiltonian constraints, and are related with the *first-class Hamiltonian constraints*. Furthermore, the submanifolds S_i , $i = 1 \dots f$, are defined by adding constraints that are not \mathcal{FL} -projectable, and they are related with the *second-class Hamiltonian constraints*.

The remaining question was how to describe geometrically the submanifolds S_i and their properties. First, this problem was solved in [12], [13] for S_1 , i.e.; the submanifold of compatibility conditions for (2) and (3), obtaining as the main result that:

Theorem 3 *For every SODE $\Gamma \in \mathfrak{X}(\text{TQ})$, we have:*

$$S_1 = \{p \in \text{TQ} \mid [i(Z)(i(\Gamma)\omega_{\mathcal{L}} - dE_{\mathcal{L}})](p) = 0, \forall Z \in \mathcal{M}\}$$

where $\mathcal{M} = \{Z \in \mathfrak{X}(\text{TQ}) \mid \mathcal{S}(Z) \in \ker \mathcal{FL}_* = \ker \omega_{\mathcal{L}} \cap \mathfrak{X}^{\text{V}(\tau_Q)}(\text{TQ})\}$.

In particular, for every $Z \in \ker \omega_{\mathcal{L}} \subset \mathcal{M}$, $i(Z)(i(\Gamma)\omega_{\mathcal{L}} - dE_{\mathcal{L}}) = i(Z)(dE_{\mathcal{L}})$ define the submanifold P_1 where (2) is compatible. They are called dynamical constraints, and are related to the existence of primary first-class Hamiltonian constraints.

For every $Z \notin \ker \omega_{\mathcal{L}}$, $Z \in \mathcal{M}$, these functions are called SODE-constraints, and are related to the existence of primary second-class Hamiltonian constraints.

For S_i , $i = 2 \dots f$, the problem was studied in [32], by imposing tangency conditions for solutions to (2) and (3). The main results are the following: there are two kinds of Lagrangian constraints defining every S_i , $i = 1, \dots, f$:

- *Dynamical constraints*, which define the submanifolds P_i , $i = 1, \dots, f$, in the sequence (4). They are related with the solutions to the eq. (2), and all of them can be expressed as \mathcal{FL} -projectable functions which give all the Hamiltonian constraints.

- *SODE (non-dynamical) constraints*, coming from the SODE-condition (3). They are not \mathcal{FL} -projectable.

Hence, the relation among the sequences of submanifolds in the Lagrangian and Hamiltonian formalisms is given in the following diagram

$$\begin{array}{ccccccc}
 & & \text{TQ} & \leftrightarrow & P_1 & \leftrightarrow & \dots & \leftrightarrow & P_f \\
 & \swarrow & & & & \swarrow & & & \swarrow \\
 & & & & & & S_1 & \leftrightarrow & \dots & \leftrightarrow & S_f \\
 & \swarrow & & & & \swarrow & & & \swarrow & & \\
 \mathcal{FL} & & \downarrow & \mathcal{FL}_0 & & \downarrow & & \downarrow & & \downarrow & \\
 \text{T}^*Q & \leftrightarrow & M_0 & \leftrightarrow & M_1 & \leftrightarrow & \dots & \leftrightarrow & M_f
 \end{array}$$

Furthermore, the tangency conditions for dynamical constraints give dynamical and SODE constraints, while the tangency conditions for SODE-constraints remove gauge degrees of freedom and do not give new constraints.

3.3 The Time Evolution Operator K

As a final remark, the complete relation between Hamiltonian and Lagrangian constraints is given by the so-called *time evolution operator* K , which relates Lagrangian and Hamiltonian constraints, and also solutions to the Lagrangian and Hamiltonian problems. (It is also known by the name of the *relative Hamiltonian vector field* [33]). It was introduced and studied for the first time by *J. Gomis* et al. [1], developing some previous ideas of *K. Kamimura* [28]. They give the local-coordinate definition of this operator, whose expression in coordinates is

$$K = v^A \left(\frac{\partial}{\partial q^A} \circ \mathcal{FL} \right) + \frac{\partial \mathcal{L}}{\partial q^A} \left(\frac{\partial}{\partial p^A} \circ \mathcal{FL} \right)$$

The intrinsic definition and the geometric study of its properties was carried out independently in [10] and [23]. In the first work, K was defined using the *Skinner-Rusk unified formalism* in $\mathrm{T}Q \oplus \mathrm{T}^*Q$. In the second article, the concept of *section along a map* plays the crucial role, and so we have:

Definition 3 *Let $(\mathrm{T}Q, \omega_{\mathcal{L}}, E_{\mathcal{L}})$ be a Lagrangian system, and $\Omega \in \Omega^2(\mathrm{T}^*Q)$ the canonical form. The time-evolution operator K associated with $(\mathrm{T}Q, \omega_{\mathcal{L}}, E_{\mathcal{L}})$ is a map $K: \mathrm{T}Q \longrightarrow \mathrm{TT}^*Q$ verifying the following conditions:*

1. (Structural condition): K is vector field along \mathcal{FL} , $\pi_{\mathrm{T}^*Q} \circ K = \mathcal{FL}$.
2. (Dynamical condition): $\mathcal{FL}^*[i(K)(\Omega \circ \mathcal{FL})] = dE_{\mathcal{L}}$.
3. (SODE condition): $\mathrm{T}\tau_Q \circ K = \mathrm{Id}_{\mathrm{T}Q}$.

$$\begin{array}{ccc} \mathrm{T}Q & \xleftarrow{\mathrm{T}\tau_Q} & \mathrm{TT}^*Q \\ \mathrm{Id}_{\mathrm{T}Q} \downarrow & \nearrow K & \downarrow \pi_{\mathrm{T}^*Q} \\ \mathrm{T}Q & \xrightarrow{\mathcal{FL}} & \mathrm{T}^*Q \end{array}$$

Then, the relation between Lagrangian and Hamiltonian constraints is established as follows (see [1], [10], [23]):

Proposition 3 *If $\xi \in C^\infty(\mathrm{T}^*Q)$ is a i th-generation Hamiltonian constraint, then $L(K)\xi$ is a $(i + 1)$ th-generation Lagrangian constraint.*

In particular, if ξ is a first-class constraint (resp. a second-class constraint) for M_f , then $L(K)\xi$ is a dynamical constraint (resp. a SODE constraint).

The time-evolution operator has also been used for studying different kinds of problems concerning singular systems. For instance, the operator K has been extended for analyzing higher-order singular dynamical systems [2], [11], [26], [27]. It is also used for treating constrained systems in general (*linearly singular systems*) [24]. It has been defined and its properties studied for non-autonomous dynamical systems [6]. Finally, K has been applied for analyzing gauge symmetries and other structures for singular systems [22], [25]. Furthermore, sections along maps in general are analyzed and used in different kinds of physical and geometrical problems in [4], [14], [15].

4 Discussion and outlook

Some of the previous problems have been studied in the sphere of first-order classical field theories, specially their *multisymplectic formalism* [5]. So, a geometric constraint algorithm has recently been completed for Lagrangian and Hamiltonian singular field theories [29], and the definition and properties of the operator K have been carried out for field theories [17], [34].

Other potentially interesting topics could be the generalization of some of the above results; such as: to study the local structure of pre-multisymplectic Hamiltonian field theories (previous generalization of the *coisotropic embedding theorem* for premultisymplectic manifolds); the study of canonical transformations for Hamiltonian field theories (*multisymplectomorphisms* and *pre-multisymplectomorphisms*), and the application of the operator K to analyze the relation between Lagrangian and Hamiltonian constraints of singular field theories (which could require prior development of the non-covariant formulation, i.e., *space-time splitting*, of field theories).

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