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# On the relative equilibria of a model for the roto-orbital motion of Binary Systems

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**Abstract**

This article is a continuation of the study of the Model Full Two-Body Problem (F2BP) using nodal polar variables $(r, \theta, h, R, \Theta, H)$ and Andoyer $(\nu, \mu, \lambda, N, M, \Lambda)$. We consider a 2-DOF approximation $\mathcal{H}_{r,\nu}$ determined from the only dependent term of the variables $(r, \nu)$ of the potential by MacCullagh. In the analytical study of the relative equilibria, we recover the classical equilibria of the unperturbed model, as well as equilibria of the roto-orbital model in global variables called *inclined* equilibria, characterized by being dependent on the radius of triaxiality $\rho = J_{22}/J_2$, specifically related to the critical value $\rho = 1/3$, *perpendicular* equilibria and *special shaped* equilibria, where we exclude the equilibria defined in the limit or coplanar case. It is also shown the explicit influence of the slow nature of the rotations for the existence of these equilibria.

In addition, numerical simulations are carried out between this model and the MacCullagh model with values close to these equilibria under conditions of systems of similar mass, in particular for the Didymos-Dymorphos asteroid system and the DART asteroid-spacecraft system. From the corresponding numerical analysis we conclude that our model has better performance in the case of families of relative equilibria when $\nu = \pi/2, 3\pi/2$ with $N \neq 0$ and when the secondary field is more close to line symmetry.

**keywords:** roto-orbital dynamics, gravity gradient, relative equilibria

## 1 Introduction

In this work we analyze the roto-orbital dynamics of celestial systems based on binary sets of stars. The system called 65803 Didymos is current, formed by a main body (Didymos) and a secondary one called Dimorphos. In the [Dell'Elce et al, 2017] article, data related to physical and dynamic parameters of the roto-orbital system are shown, as well as an astrodynamic study of a restricted three-body problem. There are other models such as the Moshup and Squannit (KW4) system, studied in [Oliveira, Prado, 2020], where the main body can be considered spherical and the secondary body is triaxial. We are also interested in artificial asteroid-satellite models, so that we can determine interesting orbital zones for future space missions.

In our study, for the analysis of these systems, we consider a particular case of the F2BP, which is the result of several simplifications due to a series of physical assumptions, so that we define an approximate non-integrable model of the main problem designed by adding some terms from the gravitational gradient perturbation. Our intention through this approximate model is to know the roto-orbital dynamics of binary systems like Didymos-Dimorphos and establish a degree of validity of our approximation.

Our study is parallel to the article [Zapata and al., 2019] where the relative equilibria in slow rotation in Poisson variables of the model $H_{r,\nu}$ are analyzed, so that the equilibria in the singularities of the variables can be analyzed. We also find in [Crespo, Ferrer, 2018] a model where the value $\rho = 1/3$ connects a family of relative equilibria with unstable equilibria of the free rigid body. The relative equilibria that are analyzed in our work differ from the classic model and capture some equilibria of these works.

Other works or articles exist under this scenario of treatment of the roto-orbital dynamics, see [Kinoshita, 1972, Ferrándiz, 1979], or [Ferrer, Molero, 2014a, Ferrer, Molero, 2014b], where two roto-orbital intermediates are analyzed with an extra simplification, based on considering that the secondary body is in a relative equilibrium of the orbital dynamics, in particular describing a circular orbit around the central body. Other more current articles whose methodology we follow in this work are [Cantero, Crespo, Ferrer, 2018], [Cantero, Crespo, Ferrer, 2019] where a radial intermediary is shown.

This article is a new contribution to the previous models, which we denote by $\mathcal{H}_{r,\nu}$, which is an approximation of 2-DOF, defined with the contribution of the Hamiltonian of the orbital part, not contemplated in the [Ferrer, Molero, 2014a] model, in which the secondary body is considered to be in an orbit of constant or circular radius.

Regarding the coordinate system, we use Nodal Polar variables (orbital movement) and Andoyer variables (rotational movement), see [Soler, 2016].

The structure of the article is as follows: in Section 2 we present the general problem, in Section 3 we introduce the model under study, in Section 4 the relative equilibria are studied and in Section 5 the numerical analysis is carried out.

## 2 Approximate models for the FG2BP

In the first place, we define some hypotheses on which we will rely in order to simplify or reduce the initial problem, called the Full Two-Body Problem (F2BP).

- *H1 Relative motion coordinates: The inertial reference frame is on the main body*
- *H2 Shape and mass distribution of $B_p$: The main body $B_p$ (mass $m_p$) has spherical symmetry.*
- *H3 Size of the bodies: The dimensions of the secondary body are small compared to the distance between the centers of mass of the two bodies.*

We are interested in the roto-translatory dynamics of two bodies, under gravitational-gradient interaction, when we assume that the main body is a sphere (Hypothesis *H2*). In other words, we focus on the dynamics of the second body, being this an asteroid, satellite, etc. In addition, the distance between both bodies at all times is assumed such that the development of the potential can be truncated considering the MacCullagh approximation [MacCullangh, 1840] (Hypothesis *H3*). Then, denoting by $T_O$, $T_R$ the orbital and rotational energy and $\mathcal{P}$ the potential, the Hamiltonian function is given by

$$\begin{aligned} \mathcal{H} &= T_O + T_R + \mathcal{P} \\ &= T_O + T_R - \frac{\kappa\, m}{r} + V \\ &= \mathcal{H}_K + \mathcal{H}_R + V, \end{aligned} \tag{1}$$

where

$$eq: masareducida \kappa = \mathcal{G}(m_p + m_s), \quad m = \frac{m_p m_s}{m_p + m_s},$$

where we have that $\mathcal{G}$ is the gravitational constant, $r$ is the distance between the centers of mass of both bodies and $m$ is the reduced mass. The potential is usually divided into two parts: a term that depends only on $1/r$ and $V$, called the disturbing potential. As a result of this, we have $\mathcal{H}_K = T_O - \kappa\, m/r$ is the Keplerian part of the system and $\mathcal{H}_R = T_R$ refers to the system of the free solid.

For the orbital part of the motion we will use the nodal polar variables $(r, \theta, h, R, \Theta, H)$ [Whittaker, 1937] and the rotational part the Andoyer variables $(\nu, \mu, \lambda, N, M, \Lambda)$, see [Andoyer, 1923], also named after Serret [Tisserand, 1891, Deprit, Elipe, 1993]. so the Hamiltonian expression has the following form

$$\mathcal{H}_K = \frac{1}{2m}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\kappa m}{r}, \tag{2}$$

$$\mathcal{H}_R = \frac{q}{2}\left[\left(\frac{\sin^2\nu}{A} + \frac{\cos^2\nu}{B}\right)(M^2 - N^2) + \frac{N^2}{C}\right], \tag{3}$$

where $q = m/m_s$, and $\{A, B, C\}$ are the three main moments of inertia, defined in such a way that $A \leq B \leq C$.

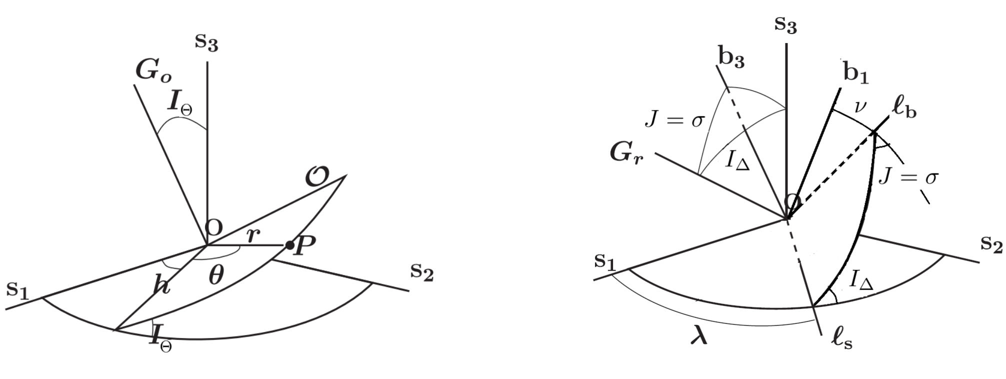



Figure 1: (a) Nodal Polar variables. (b) Andoyer Variables

The set of variables

$$(r, \theta, h, \nu, \mu, \lambda, R, \Theta, H, \Lambda, M, N)$$

is a set of symplectic variables with singularities at $I = 0, \pi$, and $J = 0, \pi$. See figure1 where the different sets of variables are represented.

**Gravity-gradient perturbing potential. The MacCullagh approximation.**

For the purpose of formulating the perturbing potential, we assume that the dimensions of the rigid solid are small compared to with the distance to the disturbing body, which allows us to write $V$:

$$V = -\frac{\kappa m}{2m_s r^3}\,(A + B + C - 3\,\mathcal{D}) + h.o.t., \tag{4}$$

where

$$\mathcal{D} = A\,\gamma_1^2 + B\,\gamma_2^2 + C\,\gamma_3^2, \tag{5}$$

is the moment of inertia of the rigid body with respect to an axis in the direction of the line joining the center of masses with the disturber, of direction cosines $\gamma_1$, $\gamma_2$, y $\gamma_3$.

Replace (5) in (4) and taking into account that $\gamma_1^2+\gamma_2^2+\gamma_3^2 = 1$, see [Arribas, Elipe, 1993], we have to

$$V = -\frac{\mathcal{G}m}{2m_s r^3}\left[(C-B)(1-3\gamma_3^2) - (B-A)(1-3\gamma_1^2)\right], \tag{6}$$

If the orbital plane is chosen as the inertial reference frame, since the variables $h$ is not present, then the reference frame can be expressed in the associated solid by the following composition of rotations:

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = R_3(\nu)\,R_1(J)\,R_3(\mu)\,R_1(I)\,R_3(\phi)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{7}$$

where $\phi = \lambda - \vartheta$ and $\vartheta$ is the polar coordinate of orbital motion.

So that, substituting $\gamma_1$ y $\gamma_3$ given in (7), in disturbing potential (6), and after some calculations we have defined the potential of the so-called Full Model

$$V = -\frac{\kappa m}{32 m_s r^3}\left[(2C - B - A)V_1 + \frac{3}{2}(B-A)V_2\right], \tag{8}$$

where the potential $V$ is composed of $V_1$, "axial-symmetrical part", given by

$$\begin{aligned} V_1 \;=\;& (4 - 6s_J^2)\,(2 - 3s_I^2 + 3s_I^2\,C_{2,0,0}) \\ & -12 s_J c_J s_I\,[(1-c_I)\,C_{-2,1,0} + 2c_I\,C_{0,1,0} - (1+c_I)\,C_{2,1,0}] \\ & +3s_J^2\,[(1-c_I)^2\,C_{-2,2,0} + 2s_I^2\,C_{0,2,0} + (1+c_I)^2\,C_{2,2,0}]\,, \end{aligned}$$

which is independent of $\nu$, y $V_2$, "tri-axial part", given by

$$
\begin{aligned}
V_2 \;=\;& 6s_I^2 s_J^2 \,(C_{2,0,-2} + C_{2,0,2}) - 4(1-3c_I^2)s_J^2\, C_{0,0,2} \\
&+(1+c_J)^2\,[(1-c_I)^2\, C_{-2,2,2} + 2s_I^2\, C_{0,2,2} + (1+c_I)^2\, C_{2,2,2}] \\
&+(1-c_J)^2\,[(1-c_I)^2\, C_{-2,2,-2} + 2s_I^2\, C_{0,2,-2} + (1+c_I)^2\, C_{2,2,-2}] \\
&+4s_I s_J\,(1+c_J)\,[(1-c_I)\, C_{-2,1,2} + 2c_I\, C_{0,1,2} - (1+c_I)\, C_{2,1,2}] \\
&-4s_I s_J\,(1-c_J)\,[(1-c_I)\, C_{-2,1,-2} + 2c_I\, C_{0,1,-2} - (1+c_I)\, C_{2,1,-2}]\,,
\end{aligned}
$$

that carries $\nu$.

Note also that

$$C_{i,j,k} \equiv \cos(i\phi + j\mu + k\nu),$$

and that the notation has been abbreviated by writing

$$c_I \equiv \cos I, \quad s_I \equiv \sin I, \quad c_J \equiv \cos J, \quad \text{y} \quad s_J \equiv \sin J.$$

Since the variable $h$ does not appear in the Hamiltonian induces that it is an integral of the motion. The angles are given as a function of the Andoyer rotational moments, by the following expressions

$$\cos I = \Lambda/M, \qquad \cos J = N/M.$$

## 3 Approximate model $\mathcal{H}_{r,\nu}$ 2-DOF

We consider as the system defined by the following Hamiltonian function

$$\mathcal{H}_{r,\nu} = \mathcal{H}_K + \mathcal{H}_R + V_{r,\nu}(r, -, -, \nu, -, -, N, M, \Lambda),$$

where $\mathcal{H}_K$ and $\mathcal{H}_R$ are the expressions defined in 2 where the potential $V_{r,\nu}$ is a function of the variables $r, \nu$ and the three rotational moments

$$V_{r,\nu} = -\frac{\kappa m}{32 m_s r^3}\left[(2C - B - A)(4 - 6s_J^2)(2 - 3s_I^2) - 6(B - A)(1 - 3c_I^2)s_J^2 \cos 2\nu\right]. \qquad (9)$$

Before starting the analysis of the model, we scale the moments of the system per unit mass reduced $m$, and the moments of inertia per unit mass of the satellite or secondary object $m_s$. In other words, we perform the transformation

$$R' = R/m; \quad \Theta' = \Theta/m; \quad H' = H/m, \quad M' = M/m; \quad N' = N/m; \quad \Lambda' = \Lambda/m, \tag{10}$$

with

$$A' = A/m_s; \quad B' = B/m_s; \quad C' = C/m_s. \tag{11}$$

However, for the sake of simplicity for the rest of the article, we will eliminate the raw variables and continue with the initial notation, knowing that the scaling proposed has been carried out. In that case, in variables $(r, \theta, h, \nu, \mu, \lambda, R, \Theta, H, N, M, \Lambda)$, we can express the Hamiltonian per unit mass of the following form

$$\begin{aligned} \mathcal{H}_{r,\nu} &= \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\kappa}{r} \\ &\quad + \frac{q}{2}\left[\left(\frac{\sin^2\nu}{A} + \frac{\cos^2\nu}{B}\right)(M^2 - N^2) + \frac{N^2}{C}\right] \\ &\quad + \frac{\kappa}{r^3}\tau\left(\frac{2}{3} - \sin^2 I\right)\left(\frac{2}{3} - \sin^2 J(1 - \rho\cos 2\nu)\right). \end{aligned} \tag{12}$$

We will put

$$\rho = \frac{B - A}{2C - B - A} > 0, \qquad \tau = -\frac{9}{16}(2C - B - A) < 0.$$

where we have assumed that $A < B < C$.

For more information to the reader, the flattening coefficient $\rho$ is defined as $\rho = 2\dfrac{J_{22}}{J_2}$, where $J_2 = \dfrac{2C - B - A}{2\,r_s^2}$ and $J_{22} = \dfrac{B - A}{4\,r_s^2}$, where $r_s$ is the mean radius of the secondary body, and where $J_2$ and $J_{22}$ are the ellipticity coefficient and the harmonic coefficient associated.

We define the expression that is part of the disturbance as

$$\Delta^* \equiv \Delta^*(\nu, N) = \kappa\,\tau\left[\left(\frac{2}{3} - \sin^2 I\right)\left(\frac{2}{3} - \sin^2 J(1 - \rho\cos 2\nu)\right)\right], \tag{13}$$

which depends or is a function of the angle $\nu$ and the moment $N$. In the appendix we present the unperturbed roto-orbital model.

**Triaxial body orbiting a sphere. Equations of motion.**

We are going to study the case where the secondary body has triaxial geometry. In this study we will make numerical comparisons and on the other hand we will look for possible relative balances of the model in both sets of variables. We start with our initial model.

The system of differential equations associated with the Hamiltonian (12) is given by

$$\begin{aligned}
\dot{r} &= R,\\
\dot{R} &= \frac{1}{r^4}\left(r\Theta^2 - \kappa r^2 + 3\Delta^*\right),\\
\dot{\nu} &= N\left[\frac{q}{D}(1-\chi\cos 2\nu) + \frac{\kappa}{r^3}\frac{2\tau}{M^2}\left(\frac{2}{3}-\sin^2 I\right)(1-\rho\cos 2\nu)\right],\\
\dot{N} &= (A-B)\sin^2 J\left[\frac{q}{2AB}M^2 + \frac{9\kappa}{8r^3}\left(\frac{2}{3}-\sin^2 I\right)\right]\sin 2\nu,
\end{aligned} \tag{14}$$

where we have used for abbreviation the constants defined by Andoyer and Kinoshita

$$\frac{1}{D} = \frac{1}{C} - \frac{1}{2}\left(\frac{1}{A}+\frac{1}{B}\right), \qquad \chi = \frac{C(B-A)}{C(A+B)-2AB}.$$

In addition, we have the following equations that may be solved by quadratures.

$$\begin{aligned}
\dot{\theta} &= \frac{\Theta}{r^2},\\
\dot{\mu} &= \frac{qM}{2}\left(\frac{1}{A}+\frac{1}{B}\right)\left[1-\frac{B-A}{A+B}\cos 2\nu\right] - \frac{\kappa}{r^3}\frac{2\tau}{M}\left\{\cos^2 I\left(\cos^2 J - \frac{1}{3}\right)\right.\\
&\quad \left. + \cos^2 J\left(\cos^2 I - \frac{1}{3}\right) + \rho\left[\cos^2 I(1-\cos^2 J) - \left(\cos^2 I - \frac{1}{3}\right)\cos^2 J\right]\cos 2\nu\right\},\\
\dot{\lambda} &= \frac{\kappa}{r^3}\frac{2\tau}{M}\cos I\left[\left(\frac{2}{3}-\sin^2 J\right) + \rho\sin^2 J\cos 2\nu\right],
\end{aligned} \tag{15}$$

and the integrals of the motion

$$\dot{h} = \dot{H} = \dot{\Theta} = \dot{M} = \dot{\Lambda} = 0. \tag{16}$$

The non-dependence of the Hamiltonian on the moment $H$ induces that the variable $\dot{h}$ =0, so this variable is constant, and we can consider any inclination of the orbital plane.

Looking at these equations, note that this first Hamiltonian we just made is not completely separable, as indicated by the subsystem $(\dot{\nu}, \dot{N})$ of (14), which is not decoupled since it contains the radial variable $r$, and the rotational variable $N$ is not part of the set of integrals of (16). In other words, the first model we have found is of two degrees of freedom (2-DOF), or what is the same, the most basic non-integrable approximate intermediate that we can find from roto-orbital motion. However, by basic we do not intend to say trivial, since we are faced with a model that presents 5 integrals (parameters distinguished) and 3 physical parameters (the moments of inertia). In short, a system that depends on 8 fixed quantities requires a deep analysis and thorough to know all the subcases of its dynamics.

## 4 Relative equilibria

In this section we are going to determine the families of relative equilibria of the differential system of motion (14) and (15). We start by recovering the classic equilibriums of the roto-orbital problem. The reader can see Appendix A for the unperturbed roto-orbital model and its associated relative equilibria.

### *4.1 Scenario 1. Extension of classical relative equilibria.*

Consider the values for the variables $\nu$ and their conjugate moment $N$, $\nu = 0, \pi$, $N = 0$ or the values $\nu = \pi/2, 3\pi/2$, in the equation of $\dot{\lambda} = 0$ of the differential system (15) so we have that

$$\dot{\lambda} = \frac{\kappa}{r^3}\frac{2\tau}{M}\left(\pm\rho - \frac{1}{3}\right) = 0, \tag{17}$$

so we have the options

$$\cos I = 0 \Rightarrow \Lambda = 0, \tag{18}$$

$$\rho = \frac{1}{3}, \quad \text{only if} \quad \nu = 0, \pi, \tag{19}$$

therefore we have that in the perturbed model there are relative equilibria for $\Lambda = 0$ and there is a family of rigid bodies represented by the condition (19) for any arbitrary value of $M$, see figure (2).

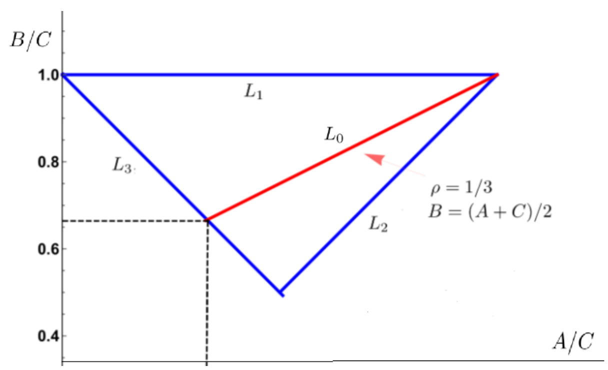



Figure 2: Family of solids represented in the parameter space as the red line $L_0$ that represents the equation $\rho = 1/3$. $L_1$ is the line of solids of type *prolate*. $L_2$ is the solid line of type *flat*. $L_3$ is the solid line of type *oblate*.

The modules of the radius vectors are given by the expression

$$r = \frac{\Theta^2 + \sqrt{\Theta^4 + 12\kappa\Delta^*}}{2\kappa} > r_p, \tag{20}$$

where

$$\Delta^* = \frac{-\kappa\,\tau}{3}\left(\frac{-1}{3} \pm \rho\right), \tag{21}$$

where we take the minus when $\nu = 0, \pi$ and the plus if $\nu = \pi/2, 3\pi/2$. So we can establish the following result:

**Proposition 4.1.** *Let the differential system of motion given by (14) and (15), there are relative equilibria for the classical values $N = 0, \nu = 0, \pi$ or $N = 0, \nu = \pi/2$, $\nu = 3\pi/2$, where the conditions (18) and (19) hold, and where the radius vectors have as modulus the expression (20).*

These equilibria correspond to the equilibria of [Crespo, Ferrer, 2018], which are called Perpendicular Equilibria ($\Lambda = 0$), and Special Shape Equilibria ($\rho = 1/3$).

These equilibria are also determined in the article by [Kinoshita, 1970], where the so-called *spoke* equilibria are determined, which are equilibria with the secondary body aligned with the radius vector, but for axialsymmetric solids.

### *4.2 Scenario 2. Relative equilibria with $N = 0$ and $\Lambda \neq 0$.*

Next, we begin the study of general relative equilibria in the perturbed model. In the same way as before, we are going to start by analyzing when the differential equation for $\dot\lambda$ vanishes.

**Estudio de la ecuación $\dot\lambda = 0$.** We are interested in studying the possible values when $\dot\lambda = 0$. In the first place, for any triaxial body, since $\sin 2I = 0$, $\lambda$ remains fixed for $I = k\,\pi/2$, so the possible values of the moment are $\Lambda = 0$ and the limiting case $M = |\Lambda|$, which is analyzed in [Crespo, Ferrer, 2018].

Another case for $\lambda$ to be fixed, and solving for the equation $\dot\lambda = 0$ is the expression

$$\cos 2\nu = \frac{3\sin^2 J - 2}{3\rho \sin^2 J}, \tag{22}$$

where we consider $N \neq M$. We start from the assumption that $N \neq 0$. If we substitute this value in $\dot\nu$ we get an expression of the form

$$\dot\nu = \frac{6Mqr^3\chi + 4\mathrm{D}\kappa\,\rho\tau(1 - 3\cos^2 I) + 9Mqr^3(\rho - \chi)\sin^2 J}{9\sin^2 J\,\mathrm{D}\,M^2\,\rho\, r^3}\,N,$$

so if we want $\dot\nu = 0$ then

$$6M^2qr^3\chi + 4\mathrm{D}\kappa\rho\tau(1 - 3\cos^2 I) + 9M^2\,q\,r^3(\rho - \chi)\sin^2 J = 0,$$

and

$$r_\nu^3 = \frac{-4\kappa\, D\rho\,\tau(1-3\cos^2 I)}{M^2 q(6\chi + 9(\rho-\chi)\sin^2 J)},$$

in moments of inertia

$$r_\nu^3 = \frac{3\kappa ABC(A+B-2C)(1-3\cos^2 I)}{4M^2 q\big((A+B-2C)C + 3(A-C)(B-C)\sin^2 J)\big)}. \tag{23}$$

This expression must verify the condition of $r_\nu > 0$, which is verified in a certain region. You should also check that $r_\nu > r_p$. Next, we will compare this expression with the one that appears when studying the zeros of $\dot{N} = 0$.

- **Study of equations $\dot{r} = \dot{R} = \dot{\nu} = \dot{N} = 0$.** Apart from the fact that $R = 0$ is immediate, the structure of the system defined for the other three equations suggests that their study goes from the bottom up.

Considering $\dot{N} = 0$. has four factors. In the following it is assumed that $A \neq B$. So, there is only one type of possible solution under study associated with

$$\frac{q}{2AB}M^2 + \frac{9\kappa}{8r^3}\left(\frac{2}{3} - \sin^2 I\right) = 0.$$

So we have to

$$r_N^3 = \frac{1}{\epsilon_A}\frac{1-3\cos^2 I}{M^2} > 0 \tag{24}$$

where we have introduced $\epsilon_A$ as a function of the physical parameters

$$\epsilon_A = \frac{4}{3}\frac{q}{\kappa}\frac{1}{AB},$$

and $1 - 3\cos^2 I > 0$, this means that we have the condition

$$0 < |\Lambda| < \sqrt{1/3}\, M. \tag{25}$$

We equate the expressions (23) and (24). By clearing and simplifying we are left with $\sin^2 J = 0$, which tells us that $N = M$, therefore it is a borderline case, since it is not in the domain of Andoyer variables and is postponed for a later work. Therefore the only possible case for $\dot{\nu} = 0$ to be checked is for $N = 0$.

Now, if we substitute the condition $N = 0$ into (22), then we have

$$\cos 2\nu = \frac{1}{3\rho} < 1. \tag{26}$$

Therefore, there is a region in the parameter space, which represents a family of solids, where we have relative equilibriums for different values of the angle $\nu$. This expression is decisive for all our subsequent analysis. These equilibria correspond to tilted equilibria type 2 of [Crespo, Ferrer, 2018].

Following our analysis, from (24), $r_N$ is a function only of the rotational moment. Therefore, since we also have the physical constraint imposed by the radius of the main body $r_p$ we must also take into account the restriction for rotational integrals. $\Lambda$ and $M$:

$$\epsilon_A \, r_p^3 < \frac{1 - 3\cos^2 I}{M^2}. \tag{27}$$

Note also that from Eq. (27) the following astrodynamic constraint between $r_N$ and $M$:

$$M < \frac{1}{\sqrt{\epsilon_A} \, r_p^{3/2}}, \tag{28}$$

Also, substituting the above expressions in Eq. $\dot{\nu} = 0$ we find that it is satisfied when $N = 0$, when the plane of the body is inclined $\pi/2$ with respect to the plane of angular momentum and therefore $\cos J = 0$.

Finally, considering Eq. $\dot{R} = 0$ we get the equation

$$r\Theta^2 - \kappa r^2 + 3\Delta^* = 0,$$

where the equilibrium radius $r = r_N$ where is a function only of the parameters. So considering the expression for $\Delta^*$ when $N = 0$, and also $\cos 2\nu = \dfrac{1}{3\rho}$ we get that

$$\Theta^2 = \kappa r_N, \tag{29}$$

where $r_N$ is the expression (24), so that

$$\Theta^2 = \frac{\kappa}{(\epsilon_A)^{1/3}} \frac{\left(1 - 3\cos^2 I\right)^{1/3}}{M^{2/3}}, \tag{30}$$

so that the orbital moment is a function of $\Theta = \Theta(\Lambda, M)$, and of the moments of inertia, where $r_N > r_0$, where $r_0$ is the radius of the main body, and where $M$ checks the condition (28) and $\Lambda$ checks the condition (25). This last expression expresses the coupling of the dynamic system.

**Theorem 4.2.** *Let the differential equations of motion be given by the expressions (14) and (15), there are general relative equilibria for the case $\Lambda \neq 0$ when the condition for the rotational angular momentum given by (28), and the condition for the orbital angular momentum given by (30), and also the value of the angle $\nu$ is given by the expression (26).*

The condition (26) tells us the following about the equilibria.

From that equation it follows that the line of equation $y = \dfrac{1}{3}x + \dfrac{2}{3}$ follows, where $y = B/C$ and $x = A/C$, that is, the normalized moments of inertia. It is also equivalent to $B = (C + A)/2$, (see Fig.3).

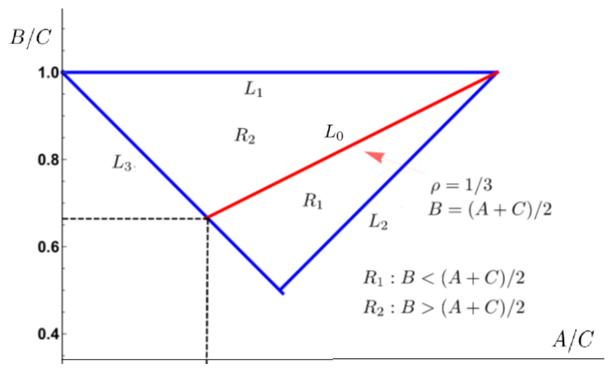



Figure 3: Region of existence $R_2$ in the parameter space, for relative equilibria so that $\dot{\lambda} = 0$ for case (a) (N=0). The red line $L_0$ represents the equation $\rho = 1/3$.

Since the relative equilibria in the unperturbed model are verified for the values of $N = 0$ and $\nu = \pi/2$, we have that the angle $\nu$ fluctuates between $0 < \nu < 35.2644°$ in the plane $N = 0$, therefore the relative equilibria in unperturbed model is bifurcated.

- In the equation of $\dot{\mu}$, we can introduce the equilibrium conditions so that $\nu$ fixes ($N = 0$) and $\lambda$ fixes $\cos 2\nu = 1/3\rho$ , so we are left with

$$\dot{\mu} = \frac{2(A + B) - C}{3AB} M\, q, \tag{31}$$

It is trivial to check that $\mu$ behaves as in the classical case.

### *4.3 Scenario 3. Relative equilibria with $N, \Lambda \neq 0$.*

Let us now see the case when $\sin 2\nu = 0$ is verified in the equation of $\dot{N} = 0$. We then have that it can be $\nu = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$, but now we suppose that $N \neq 0$ .

#### 4.3.1 Subcase $\nu = \frac{\pi}{2}, \frac{3\pi}{2}$

• **Study of equation** $\dot{\lambda} = 0$. When $\dot{\lambda} = 0$ we get the equation

$$2 - 3\sin^2 J - 3\rho \sin^2 J = 0, \tag{32}$$

so that

$$\sin^2 J = \frac{2}{3(\rho + 1)} < 1, \tag{33}$$

which is verified in the entire domain of $\rho$, and from the point of view of conjugate moments

$$\sqrt{\frac{2}{3}}M > N > \sqrt{\frac{1}{3}}M.$$

This last expression induces that the relative equilibria appear in two sections of a parallel phase flow, so they are tilted type 3 equilibria of [Crespo, Ferrer, 2018].

The equation of $\dot{\nu} = 0$ is of the form

$$\frac{q}{D}(1 + \chi) + \frac{\kappa}{r^3}\frac{2\tau}{M^2}\left(\frac{2}{3} - \sin^2 I\right)(1 + \rho) = 0,$$

so the radius of the orbit at equilibrium is the expression

$$r_\nu^3 = \frac{3\kappa AC}{4qM^2}\left(1 - 3\cos^2 I\right) > 0, \tag{34}$$

$r_\nu$ is just a function of rotational momentum. Therefore, we have the physical constraint imposed by the radius of the main body $r_p$. We must also take into account the restriction for rotational integrals $\Lambda$ y $M$:

$$\epsilon_B r_p^3 < \frac{\left(1 - 3\cos^2 I\right),}{M^2} \tag{35}$$

where

$$\epsilon_B = \frac{4q}{3\kappa AC},$$

and also $\cos^2 I < 1/3$, that is, $54.7356° < I < 90°$. Then

$$M < \frac{1}{\sqrt{\epsilon_B}}\frac{1}{r_p^{3/2}}, \tag{36}$$

therefore we obtain a bounded condition of the rotational angular momentum. If we isolate the orbital angular momentum from the equation of $\dot{R}$, the expression remains

$$\Theta^2 = \kappa r_\nu - 3\frac{\kappa\,\tau}{r_\nu}\left(1 - 3\cos^2 I\right)\left(\frac{2}{3} - \sin^2 J(1 + \rho)\right).$$

which tells us the dependency with $r_\nu$, and by (33) we get

$$\Theta^2 = \kappa r_\nu,$$

and thus

$$\Theta^2 = \frac{\kappa}{(\epsilon_B)^{1/3}} \frac{(1-3\cos^2 I)^{1/3}}{M^{2/3}}. \tag{37}$$

• **Study of the equation** $\dot{\mu} = 0$. We substitute the values of $\nu = \pi/2, 3\pi/2$ and the expression (33) so we have

$$\dot{\mu} = \frac{(-2A+B+C)q - 3C}{3AC} M, \tag{38}$$

so that $\dot{\mu} \neq 0$.

4.3.2 Subcase $\nu = 0, \pi$.

• **Study of the equation** $\dot{\lambda} = 0$. For this case, if we want $\dot{\lambda} = 0$, it must be

$$2 - 3\sin^2 J + 3\rho \sin^2 J = 0, \tag{39}$$

so that

$$\sin^2 J = \frac{2}{3(1-\rho)} < 1, \tag{40}$$

therefore we have the condition of $\rho$ and the conjugate moments of the form

$$\rho < \frac{1}{3}, \quad N < \frac{M}{\sqrt{3}}.$$

This last expression induces that the relative equilibria bifurcate in two sections of a parallel phase flow, as described in the article by [Crespo, Ferrer, 2018], Inclined Equilibriums of type 1.

The equation of $\dot{\nu} = 0$ is of the form

$$\frac{q}{D}(1-\chi) + \frac{\kappa}{r^3}\frac{2\tau}{M^2}\left(\frac{2}{3} - \sin^2 I\right)(1-\rho) = 0,$$

so the radius of the orbit at equilibrium is the expression

$$r_{\nu 2}^3 = \frac{3\kappa BC}{4qM^2}\left(1 - 3\cos^2 I\right) > 0, \tag{41}$$

$r_{\nu 2}$ it is just a function of the rotational moment. Therefore, we have the physical constraint imposed by the radius of the main body $r_p$.

We must also take into account the restriction for rotational integrals $\Lambda$ and $M$:

$$\epsilon_C r_p^3 < \frac{(1 - 3\cos^2 I)\,,}{M^2} \tag{42}$$

where

$$\epsilon_C = \frac{4q}{3\kappa BC},$$

and also $\cos^2 I < 1/3$, that is, $54.7356° < I < 90°$. Then

$$M < \frac{1}{\sqrt{\epsilon_C}} \frac{1}{r_p^{3/2}}, \tag{43}$$

therefore we obtain a bounded condition of the rotational angular momentum. If we isolate the orbital angular momentum from the equation of $\dot{R}$, the expression remains

$$\Theta^2 = \kappa r_{\nu 2} - 3\frac{\kappa\,\tau}{r_{\nu 2}}\left(1 - 3\cos^2 I\right)\left(\frac{2}{3} - \sin^2 J(1-\rho)\right).$$

which shows the de- pendency with $r_\nu$, and by (40) we get

$$\Theta^2 = \kappa r_{\nu 2},$$

from the way

$$\Theta^2 = \frac{\kappa}{(\epsilon_C)^{1/3}} \frac{(1 - 3\cos^2 I)^{1/3}}{M^{2/3}}. \tag{44}$$

• **Study of the equation** $\dot{\mu} = 0$. We substitute the values of $\nu = 0, \pi$ and the expression (33) so we have

$$\dot{\mu} = \frac{(A - 2B + C)q - 3C}{3BC} M, \tag{45}$$

so that $\dot{\mu} \neq 0$.

**Theorem 4.3.** *Let the differential equations of motion be given by the expressions (14) and (15), there are general relative equilibria for the case of $N \neq 0$ and $\Lambda \neq 0$ for the values of the angle $\nu = \pi/2, 3\pi/2$ or $\nu = 0, \pi$ and when the condition for the rotational angular momentum given by ( 36) and (43), and the condition for the orbital angular momentum given by (37) and (44) respectively.*

## 5 Numerical simulation

In this section we are going to show through numerical integrations the differences between the original problem (Full Model-FM) and the model $H_{r,\nu}$. We want to check the effects on the dynamics for various values of dynamic constants of movement (integrals), specifically the rotational angular momentum and physical parameters (moments of inertia), that is, triaxiality.

We are going to analyze two binary systems for the numerical simulations, one formed by the asteroid11997AE12 and the DART spacecraft and the other by the Didymos-Dimorphos asteroid system. The equilibria under study are those previously analyzed analytically, which were defined as Scenario 2 and Scenario 3 with $\nu = \pi/2$.

### *5.1 Systems data*

**Case 1: System 11997 AE12 and spaceship DART**

The numerical simulation is first established with a system formed by an asteroid where the main body is spherical with mass $m_p = 3.23492 \cdot 10^{12}$ kg and mean radius $r_p = 0.420$ km and the secondary body is an spaceship called DART with mass $m_s = 624$ Kg and mean radius $r_s = 0.0196$ km and moments of inertia $a = 0.00194\, \mathrm{m_s}\, \mathrm{R_p^2}$, $b = 0.01196\, \mathrm{m_s}\, \mathrm{R_p^2}$, $c = 0.01234\, \mathrm{m_s}\, \mathrm{R_p^2}$.

The gravitation constant takes the value

$$G = 6.67384 \cdot 10^{-11} \mathrm{N\, m^2/Kg^2} = 9.29 \cdot 10^{-28}\, r_p^3/\mathrm{min}^2\, \mathrm{Kg},$$

and the value of $\kappa = 0.003792\, r_p^3/\mathrm{min}^2$ y $q = 0.999582$.

**Case 2: Binary System Didymos-Dimorphos** In this case we have a primary body that has an approximate diameter of 780 m and the secondary body called Dymorphos at a distance of 1.2 km from the main body. The physical parameters for the main body are $m_p$=$2,253 \cdot 10^{12}$ kg, with an average radius of $r_p$=766 km, and for the secondary body we have the dimensions $x$=285.5 m, y=231, 5 m, and z=174.5 m, with $m_s$=$2,488 \cdot 10^{11}$ kg, so $a = 0.0132463\, \mathrm{m_s}\, \mathrm{r_p^2}$, $b = 0.0187063\, \mathrm{m_s}\, \mathrm{r_p^2}$ y $c = 0.0210626\, \mathrm{m_s}\, \mathrm{r_p^2}$.

Besides, the simulations have been carried out for five orbital periods. The computations have been developed with a package written in [Wolfram Mathematica 12.0].

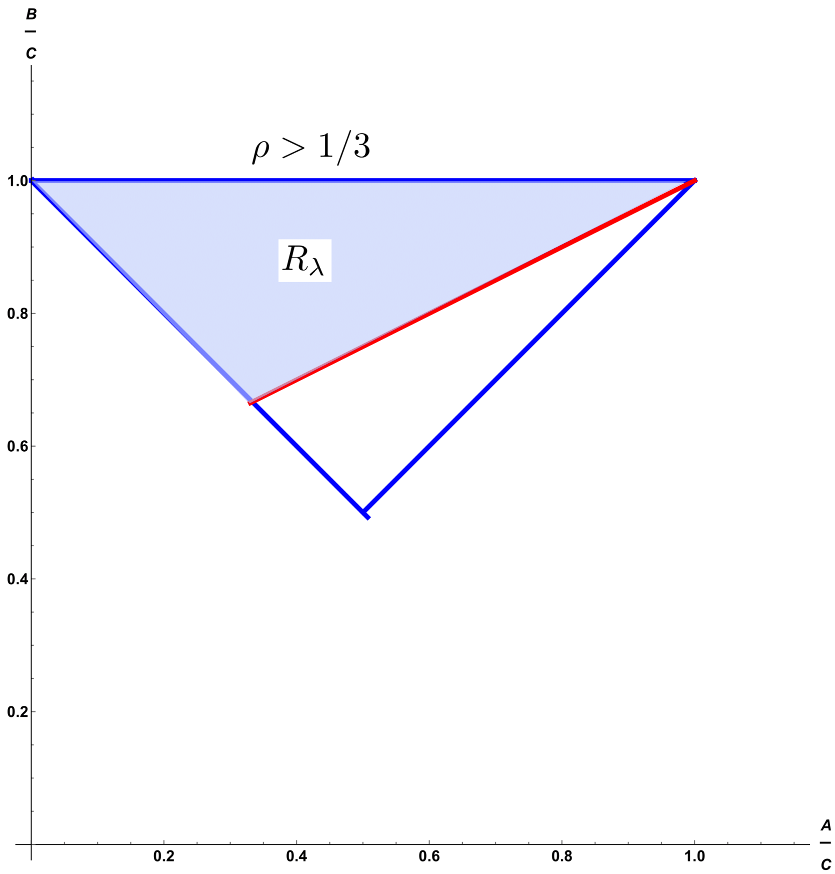



Figure 4: Representation of the secondary bodies (DART and DIMORPHOS) on the parameter space of the moments of inertia. The red line represents the equation $\rho = 1/3$. Note that the closeness of the function line $\rho = 1/3$ indicates a greater triaxiality. We are interested in how this difference in triaxiality affects numerical simulations.

### *5.2 Data of the variables in the equilibria*

- **(Scenario 2)**

**Case 1: System 11997 AE12-DART** For this equilibrium we take a value for the rotational angular momentum $M_0$ close to the value of the relative equilibrium determined by the expression (28), so we take the value $M_0 = 0.000034\, r_p^2/\text{min}$. The initial values of the angles of the system are

$$\theta_0 = 0,\ \nu_0 = 34.49^\circ,\ \mu_0 = 0,\ \lambda_0 = \frac{\pi}{2}\,.$$

The value of the angles that define the inclination of the planes of the body and of Andoyer are due to the condition in equilibrium takes the value $J = 90°$, and the angle $I$ is chosen, so we have that

$$I = 70\,\frac{\pi}{180}\,rad,\;\; J = 90\,\frac{\pi}{180}\,rad,$$

therefore the conjugate moments have initial values equal to

$$R_0 = 0,\; N_0 = M_0 \cdot \cos J = 0,\; \Lambda_0 = M_0 \cdot \cos I = 0.00034\, r_p^2/\mathrm{min}$$

In this way we have by the expressions (24) and (29) we have the values $r_{equi} = 2.75\, R_p$, and $\Theta_{equi} = 0.077\, r_p^2/\mathrm{min}$.

Carrying out a study of the integration times and the time spent on the variable $\mu$ around an orbital period, we obtain a rotation period of $T_r$ = 647 min=10.78 h and an orbital period of $T_o = 613$ min =10.21 h. Then 1:1 resonance is observed.

**Caso 2: System Didymos-Dimorphos**

In this case we obtain the values of $M_{equi} = 0.0001\, r_p^2/\mathrm{min}$, $r_{equi} = 2.93776\, r_p$ and $\Theta_{equi} = 0.08\, r_p^2/\mathrm{min}$. The values of the initial angles, plane angles, and moments are the same as in the previous case.

For this case we have a rotation period of $T_r$ = 16.72 h and an orbital period of $T_o$ = 11.28 h. We have in this case a 3:2 resonance.

- **Scenario 3 with** $\nu = \pi/2$

**Case 1: System 11997 AE12-DART**

In the first place, the value of the rotational angular momentum will be a value close to equilibrium determined by the condition (43), so we take the initial value $M_0$ equal to $0.00034\, r_p^2/\mathrm{min}$.

The initial values of the angles of the system are

$$\theta_0 = 0\,,\; \nu_0 = \pi/2\,,\; \mu_0 = 0\,,\; \lambda_0 = \frac{\pi}{2}\,.$$

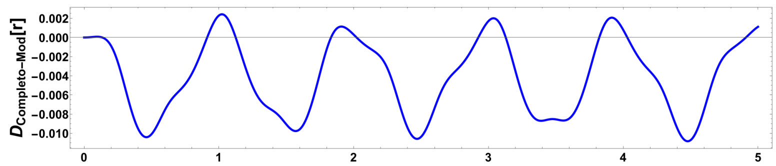
The value of the angles that define the inclination of the planes of the body and Andoyer are for the equilibrium condition with $J = 36°$, and $I = 75°$ chosen, therefore the conjugate moments have initial values equal to

$$R_0 = 0,\; N_0 = M_0 \cos J = 0,\; \Lambda_0 = M_0 \cos I = 0.0018\, r_p^2/\mathrm{min}$$

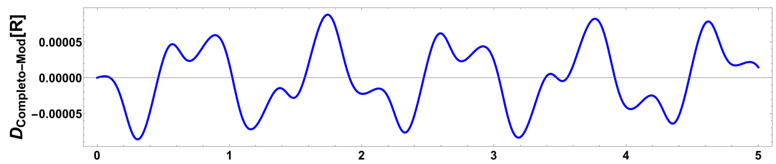
In this way we have by the expressions (41) and (44) that $r_{equi} = 2.95\, r_p$ and $\Theta_{equi} = 0.08\, r_p^2/\text{min}$. The numerical simulation shows for this case a rotation period of $T_r = 553$ min=9.21 h and an orbital period $T_o = 681.6 = 11.36$ h.

**Case 2: System Didymos-Dimorphos**

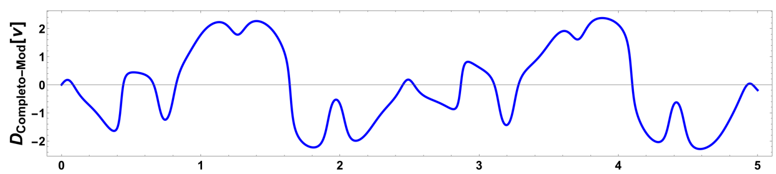
In this case and following the same previous expressions, we have that the rotational moment takes the value $M_{equi} = 0.0001\, r_p^2/\text{min}$, and $r_{equi} = 2.93776\, R_p$ and $\Theta_{equ} = 0.08\, r_p^2/\text{min}$. The angles of the planes are $J = 41.2°$, $I = 75°$ (chosen). The values of the initial angles and moments are the same as in the previous case. Carrying out a study in the numerical simulation we obtain a rotation period of $T_r = 12.77$ h and an orbital period of $T_o = 12.52$ h. Is observed 1:1 resonance.



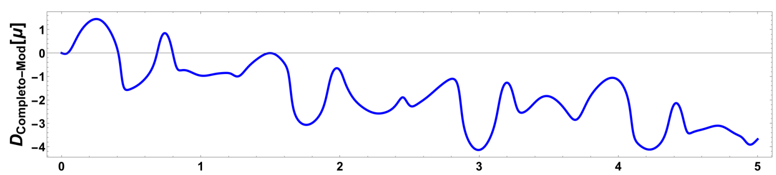

(a) $\Delta r = 0.002$ Rp



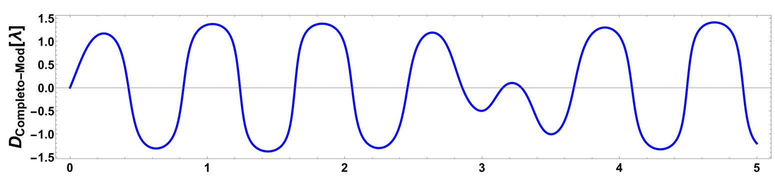

(b) $\Delta R = 0.00005$ Rp/min




(c) $\Delta \nu = 2$ rad

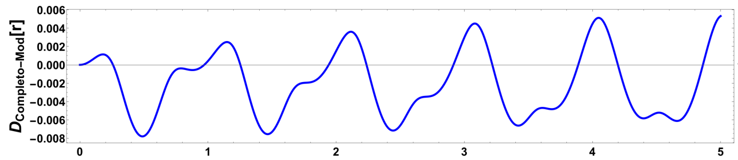



(d) $\Delta \mu = 4$ rad

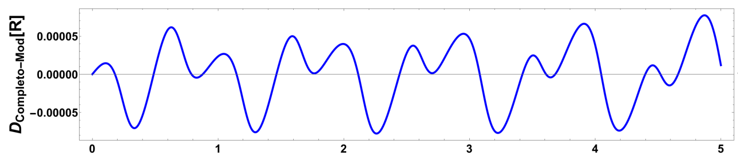



(e) $\Delta \lambda = 4$ rad

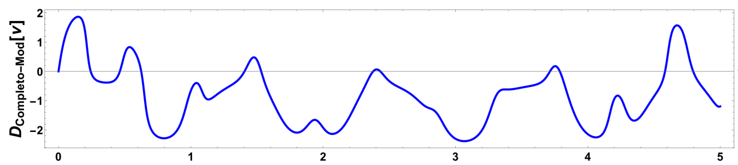
Figure 5: FM vs. Model $H_{r,\nu}$ (Scenario 2) DART $\nu = 34.49°$. $M = 0.00034\, r_p^2/\text{min}$, $I = 70°$. We can observe in this figure the differences in equilibria (A) in the orbital variables in the first 2 graphs. The difference in the variable $r$ can be seen from time to time, as well as at the moment $R$. Taking into account that each orbital period is $T_o = 613$ min=10.21 h, then a difference of $0.002\, Rp$ is observed in 5 orbital periods. For rotational variables, we can establish that for the variable $\nu$ we have a maximum difference of 2 rad, in a semiperiodic movement, for the variable $\mu$ there is a maximum difference in the 5 orbital periods of 4 rad, and a descending drift is observed and for the variable $\lambda$ it is 1.5 rad, and the differences maintain a quasi-periodic movement.



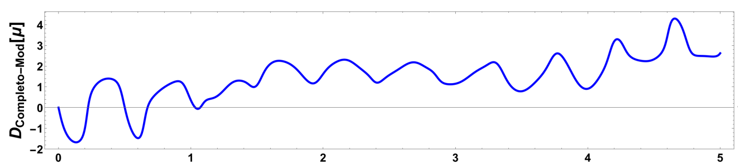

(a) $\Delta r = 0.006$ Rp



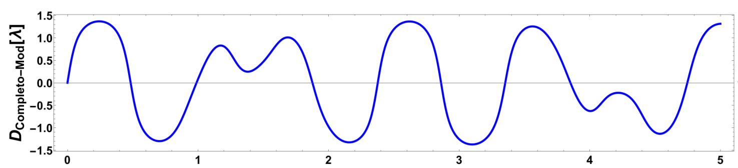

(b) $\Delta R = 0.00005$ Rp/min




(c) $\Delta\nu = 2$ rad

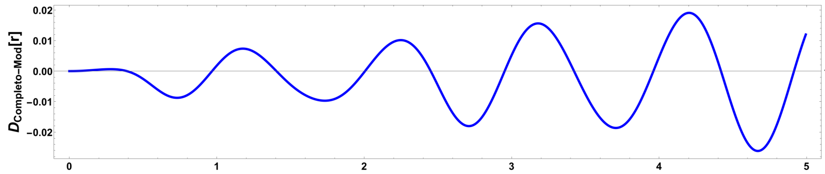



(d) $\Delta\mu = 4$ rad

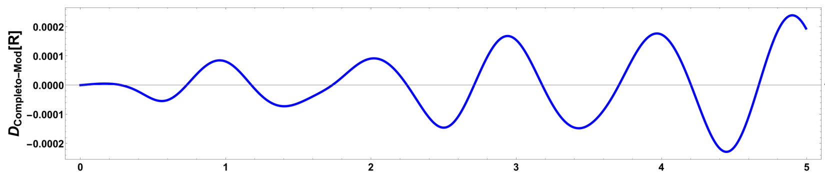



(e) $\Delta\lambda = 1.5$ rad

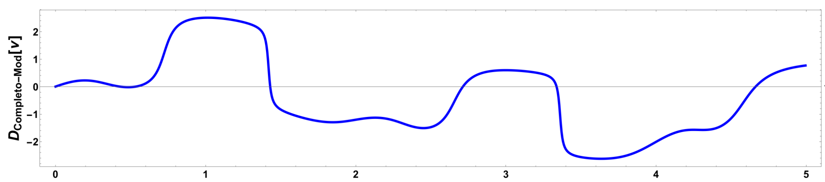
Figure 6: FM vs. model $H_{r,\nu}$. (Scenario 3) DART $\nu = \pi/2$. $M = 0.000034\, r_p^2$/min. $I = 75°$. $J = 36°$., variables $\nu$, $\mu$ and $\lambda$. We can observe in this figure the differences in equilibria (B). In this case, since the orbital period is $T_o = 681.6{=}11.36$ h, the difference in the variable $r$ the differences are periodic in all orbital periods. The maximum difference in 5 periods is 0.006 Rp. In the rotational variables, which we can see in the same figure, we can establish in the variable $\nu$ a maximum difference of 2 rad in a semiperiodic movement or a certain periodicity. Regarding $\mu$ the difference is 4 rad. The variable $\lambda$ reaches a maximum difference of 1.5 rad and there are differences that change from 1 rad in one orbital period to change to a difference of $-1$ rad that is maintained in one orbital period. A 1:1 resonance is shown in this relative equilibria.



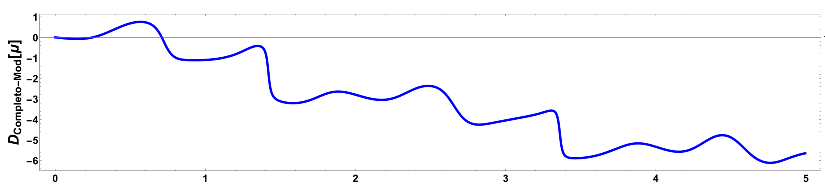

(a) $\Delta r = 0.02$ Rp



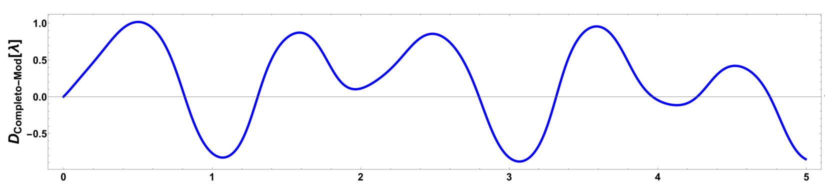

(b) $\Delta R = 0.002$ Rp/min




(c) $\Delta\nu = 2$ rad

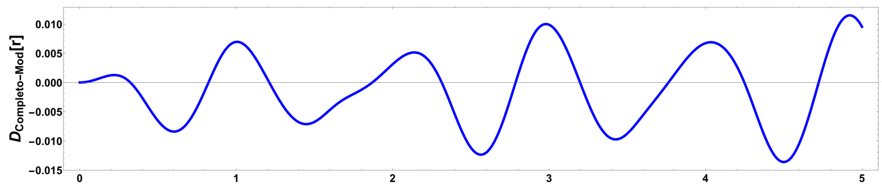



(d) $\Delta\mu = 6$ rad

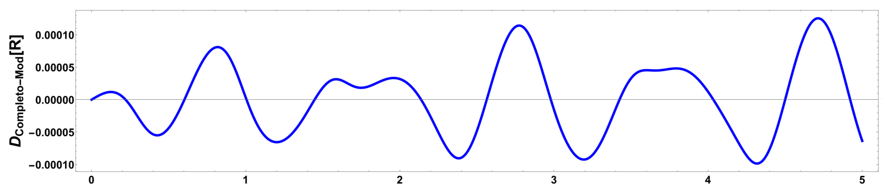



(e) $\Delta\lambda = 1$ rad

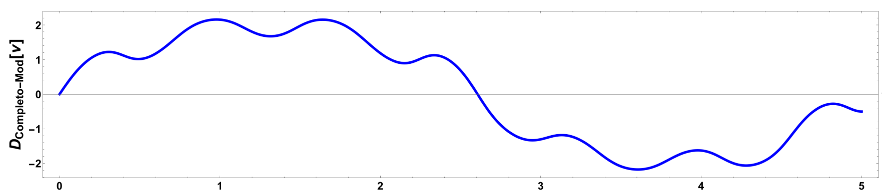
Figure 7: FM vs. Model $H_{r,\nu}$ (Scenario 2) Didymos-Dimorphos system. $\nu = 25.08°$. $M = 0.0001\, r_p^2$/min, $N = 0$. $I = 75°$. We can see in this figure the comparisons in two orbital variables and the three rotational ones. We observe that in the case of equilibrium (A), we can observe how the differences in the orbital variables increase. We have a periodic movement in the variables $\nu$ pro with descending drift. In the variable $\mu$ a descending drift of up to 6 rad difference and in the variable $\lambda$ if a periodicity of the differences with a maximum difference of 1 rad is observed.



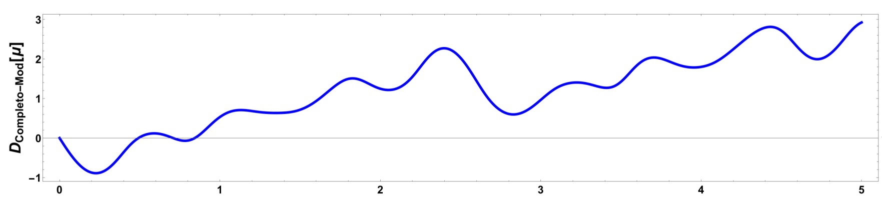

(a) $\Delta r = 0.01$ Rp



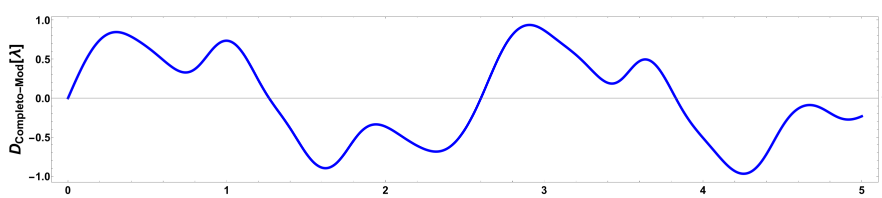

(b) $\Delta R = 0.0001$ Rp/min




(c) $\Delta\nu = 2$ rad




(d) $\Delta\mu = 3$ rad




(e) $\Delta\lambda = 1$ rad

Figure 8: FM vs. Model $H_{r,\nu}$. (Scenario 3) Didymos-Dimorphos case $\nu = \pi/2$. $M = 0.00005\, r_p^2/\text{min}$. $I = 25°$. $J = 38°$. We can see in this figure the comparisons in two orbital variables and the three rotational ones. We observe that in the equilibrium case (B), we can observe how the differences in the orbital variables are periodic, contrary to the previous case. We have a periodic movement in the variables $\nu$ with a maximum difference of 2 rad. In the variable $\mu$ an ascending drift of up to 3 rad difference and in the variable $\lambda$ if a periodicity of the differences with a maximum difference of 1 rad is observed.

## 6 Conclusions and future work.

This work is a continuation of other studies already started in [Ferrer, Molero, 2014a] and [Crespo, Ferrer, 2018], where an intermediary model is presented. Our study takes up the concept of an approximate model, introducing a 2-GDL approximation, which unlike previous models, establishes the relative equilibria in the Polar Nodal and Andoyer variables. This makes the study simpler than in other works that define the model with Variables Total Angular Momentum.

We find three relative equilibrium scenarios associated with this model, which are determined with an upper bound condition for the rotational angular momentum and an equation that establishes the coupling of the system. It was to be expected that the Coplanar equilibria in [Crespo, Ferrer, 2018] are outside our domain in Andoyer variables. Numerical analysis close to the relative equilibriums show better global results in the case of Scenario 3, which is shown in the study of both the Didymos-Dimorphos system and the 11997 AE12-DART system. The differences observed in the numerical comparisons determine that we have to add the resonant term. In relation to triaxiality, we see that the model behaves better for a more axial-symmetric body such as DART than for a more triaxial body such as Dimorphos.

On the other hand, unlike the Global or Poisson variables, we cannot establish equilibria in the regions with singularities associated with the variables. Still, this roto-orbital model captures the classical equilibria and the classical dynamics of the free rigid solid is not maintained in the determined relative equilibria, since there is a bifurcation of them. This type of degeneration is not the one observed in previous works, so we consider as future work the analysis of additional perturbations with terms of type $\cos 4\nu$, with the intention of trying to determine completely. We also plan to implement applications to cases of artificial satellites, comparing simulations with real observations.

## 7 Acknowledgments

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# A Appendix

### A.1 Unperturbed roto-orbital model.

The Hamiltonian of the unperturbed system is given by $\mathcal{H} = \mathcal{H}_K + \mathcal{H}_R$, (2) and (3),

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\kappa}{r} \\ &+\frac{q}{2}\left[\left(\frac{\sin^2\nu}{A} + \frac{\cos^2\nu}{B}\right)(M^2 - N^2) + \frac{N^2}{C}\right], \end{aligned} \tag{46}$$

where $q = m/m_s$. The system of differential equations associated with the Hamiltonian (46) is given by

$$\begin{aligned} \dot{r} &= R, & \dot{\nu} &= q\,N\left(\frac{\sin^2\nu}{A} + \frac{\cos^2\nu}{B}\right) + \frac{q\,N}{C}, \\ \dot{R} &= \frac{\Theta^2}{r^3} - \frac{\kappa}{r^2}, & \dot{N} &= \frac{q(A-B)}{2AB}(M^2 - N^2)\sin 2\nu, \end{aligned} \tag{47}$$

and the squares

$$\dot{\theta} = \frac{\Theta}{r^2}, \quad \dot{\mu} = q\left(\frac{\sin^2\nu}{A} + \frac{\cos^2\nu}{B}\right) M, \quad \dot{\lambda} = 0, \tag{48}$$

$$\dot{\theta} = \frac{\Theta}{r^2} \tag{49}$$

with

$$\dot{h} = \dot{H} = \dot{M} = \dot{\Lambda} = 0.$$

The first integrals $\lambda$, $\Lambda$ and $M$ reflect what we already know: that they are associated to the angular momentum vector which is an integral vector of the model. In other words, they tell us that we have an invariant rotational plane.

### A.2 Relative equilibria

Relative balances orbital part. The orbital part is Kepler's problem.

We have the following equation when $\dot{R} = 0$ so

$$\frac{\Theta^2}{r^3} - \frac{\kappa}{r^2} = 0, \tag{50}$$

so we have to

$$r = \frac{\Theta^2}{\kappa}, \qquad R = 0, \tag{51}$$

which are orbits of constant radius.

Relative equilibria rotational part. The rotational part is the Rigid body in free rotation.

In the generic case ($A \neq B$ are different and $M \neq 0$), we have relative equilibria when $N = 0$ and $\nu = \frac{\pi}{2}k$, ($k = 0, 1, 2, 3$). In other words, we have special periodic orbits $P_i(\mu, \nu, \Lambda, M, N)(t)$ given by:

$$P_1(t) = \left(\frac{qM}{B}\,t, 0, \Lambda, M, 0\right),\ P_2(t) = \left(\frac{qM}{B}\,t, \pi, \Lambda, M, 0\right),$$

$$P_3(t) = \left(\frac{qM}{A}\,t, \pi/2, \Lambda, M, 0\right),\ P_4(t) = \left(\frac{qM}{A}t, 3\pi/2, \Lambda, M, 0\right).$$

Note that for all $\Lambda$ the value of $\lambda$ is constant. Let us now analyze the energy integral. Substituting the values in the equilibrium we have the following expression

$$\mathcal{H} = -\frac{1}{2}\frac{\kappa}{\Theta^2} + \frac{q}{2B}M^2, \tag{52}$$

which depends on the orbital moment $\Theta$ and the rotational moment $M$.