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An explicit formula of time as function of the true anomaly for all types of Keplerian orbits

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Abstract

This Note presents an explicit relation between time and true anomaly for Keplerian orbits, with no need to use the Kepler's equation which introduces an intermediary angle, the eccentric anomaly for elliptic motions and the hyperbolic anomaly for hyperbolic orbits, and which is the common way in computing the position on the orbit. Because of some properties of complex numbers, we prove that our formula is a real valued function, and thus it is valid for all type of Keplerian orbits, that is, the formula is valid for elliptic as well as for hyperbolic orbits, and the classical formula for parabolic ones is also obtained as the limit when the eccentricity tends to one.

Keywords: Keplerian orbits, Time law for true anomaly, Kepler's equation

1 Introduction

One of the key points in orbital dynamics is to determine the position of a spacecraft on its orbit at any instance of time, that is, $r = r(t)$. In a Kepler motion, the radial distance usually is given in terms of the true anomaly

$$r = \frac{p}{1 + e\cos f},$$

thus, it will suffice to have a "time equation" of the type $f = f(t)$ to know the sough position.

Traditionally, as can be found in every textbook of Mechanics, such time equation is given by means of another different angle, the eccentric anomaly E for elliptic orbits through Kepler's equation

$$\sqrt{\mu/a^3}(t-t_0) = \frac{2\pi}{T}(t-t_0) = E - e\sin E,$$

resulting that the radial distance is $r = a(1 - e\cos E)$.

For hyperbolic orbits, a new similar anomaly is introduced, the hyperbolic eccentric anomaly, F, in such a way that $\sqrt{\mu/a^3}(t-t_0) = e\sinh F - F$, and $r = a(e\cosh F - 1)$.

The only case in which we find in textbooks an explicit relation between the true anomaly and time is for the parabolic case. Indeed, the relation is

$$\frac{1}{3}\tan^3\frac{f}{2} + \tan\frac{f}{2} = \sqrt{\frac{\mu}{2q^3}}(t-t_0), \tag{1}$$

where $2q$ is the *semilatus rectum* of the parabola, that is the limit of p when $e \to 1$ and $a \to \infty$.

Thus, it seems that nobody dared to try to obtain an equation time relating directly t and f, which could be dirctly obtained by integrating the differential equation $r^2\,df = Gdt$. We found in the textbook of Schaub and Junkins [4, p. 405] a justification to do not proceed in such way, with the sentence *"analytically solving it involves finding a solution to a nonstandard elliptic integral, clearly not a very attractive proposition."* But this statement is not true, because there is no square root involved, and hence no possibility of having elliptic functions. This motivated us to compute from the *aerial relation*, $r^2\,df = G\,dt$, the quadrature

$$\frac{G}{p^2}(t-t_0) = \frac{\mu^2}{G^3}(t-t_0) = \int_0^f \frac{1}{(1+e\cos s)^2}ds \equiv \Phi(f;e), \tag{2}$$

where we assumed that $f = 0$ at $t = t_0$, and used the well known relation $p = G^2/\mu$, with p the semi *latus rectum* of the conic, G the constant norm of the angular momentum ($G = \|\mathbf{G}\| = \|\mathbf{r} \times \dot{\mathbf{r}}\|$), μ the Keplerian parameter, and e is the eccentricity of the conic, that will be considered as a parameter, while f is the true anomaly.

Let us make some remarks about the function $\Phi(f;e)$.

1. For all $e \geq 0$, the function $\Phi(f;e)$ is an strict monotonically increasing function of f

2. For $e \in [0, 1)$, the function $\Phi(f; e)$ is well defined for all $f \geq 0$.

3. For $e = 1$, the integral (2) is singular for $f = \pi$; besides, $\Phi(\pi; 1) = +\infty$. Therefore, $\Phi(\pi; 1)$ is only defined for $f \in [0, \pi)$ and when $f < \pi$, the $\lim_{f \to \pi} \Phi(f; 1) = +\infty$.

4. For $e > 1$, the function $\Phi(f; e)$ is only defined in the interval $[0, f_e]$, such that $\cos f_e = 1/e$ and besides, when $f < f_e$, the $\lim_{f \to f_e} \Phi(f; e) = +\infty$.

In the Note, first, we managed to solve the quadrature (2) straightforward, by hand. After that, we obtained another equivalent solution by using an algebraic manipulator. The result is a not very apealing equation (3), but at least we have a *ley horaria*. In the obtaining of this function, we did not any assumption on the value eccentricity, then this formula is valid for whatever value of the eccentricity, no matter its value. To have a unique time law formula for both elliptical and hyperbolic motions is quite relevant in orbital problems where the osculating eccentricity may pass from values lesser than one to others greater than the unit. In those cases the so-called *universal variables* based on Stumpff's functions are usually employed, see e.g. Abad's textbook [1, Chap. 10]. With our equation, we may skip universal variables and use the more natural formulation in terms of the true anomaly and classical orbital elements.

Just in the last stage of writing the manuscript, we discovered in Geyling and Westerman's monograph [3, Chapter 2] (a text mostly unknown in universities libraries) a formula equivalent to ours. However, they did not perform any quadrature, but they used classical formulas relating true and eccentric anomalies to convert Kepler's equation into a function depending on the true anomaly. They did the same for the hyperbolic case ending up with another function for hyperbolic orbits. In so doing, they did not realize the general use of the sough formula.

2 The obtaining of the time equation

The formal integration of formula (2), after some cumbersome process gives

$$\Phi(f; e) = \frac{2}{(1 - e^2)^{3/2}} \arctan\left(\sqrt{\frac{1 - e}{1 + e}} \tan \frac{f}{2}\right) - \frac{e \, \sin f}{(1 - e^2)(1 + e \, \cos f)}.$$

By using a symbolic processor like Mathematica, the result is

$$\Phi(f; e) = \frac{-2}{(e^2 - 1)^{3/2}} \operatorname{arctanh}\left(\frac{e - 1}{\sqrt{e^2 - 1}} \tan \frac{f}{2}\right) + \frac{e \, \sin f}{(e^2 - 1)(1 + e \, \cos f)}.$$

It can be numerically proved that both equations coincide. Hence, for the equation of time we choose a mix of both formulas, namely

$$\begin{aligned}\frac{\mu^2}{G^3}\,(t-t_0) = \Phi(f;e) = {} & \frac{2}{(1-e^2)^{3/2}}\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{f}{2}\right) + \\ & \frac{e\ \sin f}{(e^2-1)(1+e\ \cos f)}.\end{aligned} \tag{3}$$

Obviously, this formula is singular at $e=1$ (parabolic motion). However we may avoid this singularity by computing the limit when e tends to one. Indeed,

$$\lim_{e\to 1}\Phi(f) = \frac{1}{12}\sec\frac{f}{2}\left(3\sin\frac{f}{2}+\sin\frac{3f}{2}\right) = \frac{1}{2}\left(\frac{1}{3}\tan^3\frac{f}{2}+\tan\frac{f}{2}\right),$$

which is the well known Baker's time equation for parabolic motion (1).

Equation (3) has two drawbacks. The first one, is that there is a discontinuity at $f=\pi$, because $\lim_{x\to\pi^-}\tan x/2 = +\infty$ and $\lim_{x\to\pi^+}\tan x/2 = -\infty$, hence

$$\lim_{f\to\pi^-}\Phi(f;e) = 2(1-e^2)^{-3/2}\pi/2, \qquad \lim_{f\to\pi^-}\Phi(f;e) = -2(1-e^2)^{-3/2}\pi/2.$$

Thus, the graphic has "jumps" at $f=\pi, 3\pi, \ldots$. But this difficulty can be avoided by using instead of the function $\arctan(y/x)$ the equivalent one for numerical computations `atan2(y,x)`, quite common in programming languages like C^{++} or even in *Mathematica* (in this case, the function is `ArcTan[x,y]`). With this change, the function $\Phi(f)$ is an increasing and continuous function in the domain $[-2\pi, 2\pi]$. But with this trick we moved the singularity from $f=\pi$ to $f=2\pi, 4\pi, \ldots$. One possible way to circumvent this obstacle is by defining a new function as

$$\widetilde{\Phi}(f;e) = \Phi(f;e) + \left\lfloor\left(\frac{f+\pi}{2\pi}\right)\right\rfloor 2(1-e^2)^{-3/2},$$

where $\lfloor\Box\rfloor$ denotes the floor function of expression $\Box$. With this, the function $\widetilde{\Phi}(f;e)$ still is not defined for $f=2\pi, 4\pi, \ldots$, but its letft- and right-hand limits are equal and it is possible to define a continuous extension.

The second challenge, and the most important, is that we claim that Eq. (3) is valid for both elliptic and hyperbolic motions. However, in the formula there is the expression $\sqrt{1-e}$. How can it be possible that the formula be still valid when $e>1$?

In formula (3) there are two places where a pure imaginary number appears when $e > 1$. First, in factor $(1-e^2)^{-3/2}$. The other place is due to the expression $\sqrt{(1-e)/(1+e)}$, but

$$(1-e^2)^{-3/2} = \mathbf{i}\,(|1-e^2|)^{-3/2}, \qquad \sqrt{\frac{1-e}{1+e}} = \mathbf{i}\sqrt{\frac{|1-e|}{1+e}},$$

with $\mathbf{i} = \sqrt{-1}$.

Let us now recall the relation [2, formula (4.6.16)], $\arctan(z) = \mathbf{i}\,\mathrm{arctanh}\, z$, hence, since both factors are pure imaginary numbers, its product is a real number, and consequently, the function $\Phi(f)$ is a real valued function. In Figure 1 we show several examples of the function $\Phi(f)$ for several eccentricities ($e = 0.4$, 0.7. 0.9 and 1.4). Note in the plots that for the elliptic orbits the t -period is different in each case. The reason is that we keep G constant, hence, since $a(1-e^2) = G^2/\mu$, an increase of the eccentricity also implies an increase of the semimajor axis and therefore of the period.

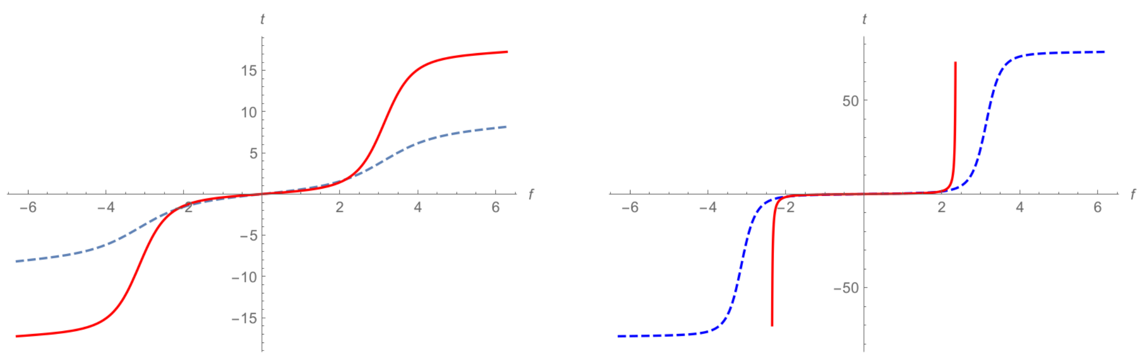

Figure 1: The function $t = t(f)$ in the interval $(-2\pi, 2\pi]$ for different eccentricities: Left) $e = 0.4$ (dashed line), $e = 0.7$ (continuous line); and Right) $e = 0.9$ (dashed line) and $e = 1.4$ (continuous line). In this last case, f is limited by the asymptote of the hyperbola.

The most complicate aspect of this formulation, is the obtaining of the inverse function $f = \Phi^{-1}(t; e)$ with respect to f; the eccentricity e plays the role of parameter. It has to be solved numerically. It is not our goal in this Note to give efficient methods to find the inverse function $\Phi^{-1}(t; e)$. We show in Figure 2 two samples for eccentricities $e = 0.8$ and $e = 1.8$, obtained with Mathematica's function `FindRoot`. Note that for the hyperbolic case, the true anomaly tends asymptotically to the asymptote of the hyperbola, while for the elliptic case, the true anomaly increases continuously.

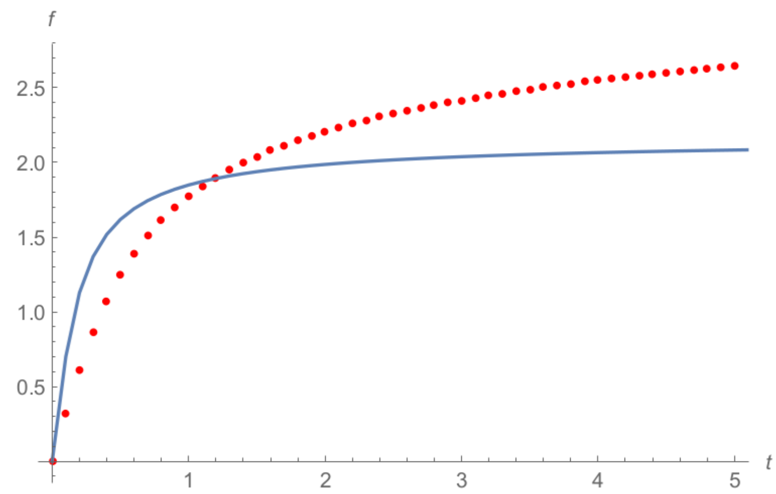

Figure 2: The function $f = \Phi^{-1}(t; e)$ for two eccentricities: elliptic case $e = 0.8$ (dotted line) and hyperbolic case $e = 1.8$ (continuous line). Note that in this last case the graphic asymptotically tends towards the asymptote of the hyperbola.

3 Conclusions

We propose in this Note a direct formula $t = \Phi(f)$ for Kepler motion, with no need to use the eccentric anomaly as intermediate step. This formula is a real valued function whatever the value of the eccentricity, which could be of interest in computing ephemeris. Thus, there is no need to have a different formulation for each type of orbit, and neither is needed the use of *universal variables* or Stumpff's functions.

Acknowledgments

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