Rev. Real Academia de Ciencias. Zaragoza. **75:** 85–110, (2020). ISSN: 0370-3207

Approaching Eldan's and Lee & Vempala's bounds for the KLS conjecture in a unified method

David Alonso-Gutiérrez, Jesús Bastero

Departamento de Matemáticas - I.U.M.A.
Universidad de Zaragoza
Campus Plaza San Francisco
50009 Zaragoza, Spain
alonsod@unizar.es, bastero@unizar.es

Resumen

La principal idea de este artículo es revisar las pruebas de las mejores estimaciones conocidas para la conjetura KLS de salto espectral, demostradas por Eldan y Lee & Vempal, aplicando el esquema de localización de Eldan a dos sistemas de ecuaciones diferenciales estocásticas diferentes. Damos una prueba unificada de estas dos acotaciones obteniendo la estimación de Eldan desde el sistema de ecuaciones diferenciales estocásticas considerado por Lee & Vempala.

Abstract

The main idea of this paper is to review the proof on the best known estimates for the KLS spectral gap conjecture, given by Eldan and Lee & Vempala by applying Eldan's localization scheme to two different systems of stochastic differential equations. We give a unified proof for these two best bounds obtaining Eldan's estimate from the system of stochastic equations considered by Lee & Vempala.

1 Introduction

The Kannan-Lovász-Simonovits spectral gap conjecture (KLS) is a major problem in asymptotic geometric analysis. Its origin comes from theoretical computer sciences as a problem arising in the study of the complexity of an sampling algorithm and it is related to many other branches of mathematics like convex geometry, probability, PDE's, Riemannian geometry and information or learning theory (see [AB1], [BGVV] and the references therein (or [AB2] for a presentation of the conjecture written in Spanish)).

It concerns log-concave probabilities and can be stated in the following way:

Conjecture 1.1 (KLS spectral gap conjecture)**.** *There exists an absolute constant $C > 0$ such that, for any log-concave probability μ in $\mathbb{R}^n$*

$$\mu^+(A) \geq \frac{C}{\sqrt{\|Cov_\mu\|_{op}}} \min\{\mu(A), \mu(A^c)\}, \qquad \text{for any Borel set } A \subset \mathbb{R}^n \tag{1}$$

where

$$\mu^+(A) = \liminf_{\varepsilon \to 0} \frac{\mu(A^\varepsilon) - \mu(A)}{\varepsilon},$$

being $A^\varepsilon = \{a + x : a \in A, |x| < \varepsilon\}$, and $\|Cov_\mu\|_{op}$ is the operator norm of the covariance matrix of μ

This is a *Cheeger type isoperimetric inequality.* This conjecture was posed in [KLS], where the authors proved the Cheeger type isoperimetric inequality with constant $\frac{C}{\mathbb{E}_\mu|x|}$ (where $\mathbb{E}_\mu|x|$ denotes the expected value of the Euclidean norm with respect to the probability μ) instead of $\frac{C}{\sqrt{\|\mathrm{Cov}_\mu\|_{\mathrm{op}}}}$.

The KLS conjecture has an equivalent expression as a corresponding *Poincaré type inequality*: there exists an absolute constant $C > 0$ such that

$$\int_{\mathbb{R}^n} |f - \mathbb{E}_\mu f|^2 d\mu \leq C \|\mathrm{Cov}_\mu\|_{\mathrm{op}} \int_{\mathbb{R}^n} |\nabla f|^2 d\mu \tag{2}$$

for any log-concave probability μ in $\mathbb{R}^n$ and Lipschitz μ-integrable function f.

The factor $\|\mathrm{Cov}_\mu\|_{\mathrm{op}}$ appearing in both expressions (1) and (2) is just a normalization factor. Indeed, since the conjecture involves every Borel set $A \subseteq \mathbb{R}^n$, or every Lipschitz μ-integrable function in its equivalent form, making a change of variables, we can assume that μ is centered and that $\mathrm{Cov}_\mu = I_n$ (identity matrix), i.e. the new log-concave measure is isotropic and then we can reformulate both conjectures in the following way: there exists an absolute constant C such that for any isotropic log-concave probability in $\mathbb{R}^n$

$$\mu^+(A) \geq C \min\{\mu(A), \mu(A^c)\}, \qquad \text{for any Borel set } A \subset \mathbb{R}^n \tag{3}$$

or, equivalently, there exists an absolute constant C

$$\int_{\mathbb{R}^n} |f - \mathbb{E}_\mu f|^2 d\mu \leq C \int_{\mathbb{R}^n} |\nabla f|^2 d\mu \tag{4}$$

for any isotropic log-concave probability μ in $\mathbb{R}^n$ and any Lipschitz μ-integrable function f.

This conjecture remains open and the best estimates known up to now, which depend on the dimension, for the value of the constant in (1) and (2) have been obtained in two different papers by Eldan ([E1], see also [E2] for another approach) and Lee & Vempala ([LV1], see also [LV2] for a nice survey on this conjecture), respectively.

The results whose proofs we want to unify are given by the following two theorems:

Theorem 1.1 (Eldan, [E1]). *There exists an absolute constant $C > 0$ such that for any isotropic log-concave probability μ in $\mathbb{R}^n$*

$$\mu^+(A) \geq \frac{C}{\sigma_n \log n} min\{\mu(A), \mu(A^c)\} \qquad \text{for any Borel set } A \subset \mathbb{R}^n \tag{5}$$

where $\sigma_n = \sqrt{\sup \mathbb{E}_\mu \big||X| - \sqrt{n}\big|^2}$ and the sup *runs over all isotropic log-concave random vectors X in $\mathbb{R}^n$.*

Theorem 1.2 (Lee & Vempala, [LV1]). *There exists an absolute constant $C > 0$ such that for any isotropic log-concave probability μ in $\mathbb{R}^n$*

$$\mu^+(A) \geq \frac{C}{n^{1/4}} min\{\mu(A), \mu(A^c)\} \qquad \text{for any Borel set } A \subset \mathbb{R}^n. \tag{6}$$

The parameter σ_n appearing in Eldan's result is related with a different conjecture, which is the *thin shell width conjecture* proposed by Bobkov-Koldobsky ([BK]): there exists an absolute constant $C > 0$ such that for any isotropic, log-concave probability in $\mathbb{R}^n$ we have $\sigma_\mu = \sqrt{\mathbb{E}_\mu \big||x| - \sqrt{n}\big|^2} \leq C$.

If this conjecture were true it would imply that the mass in the isotropic log-concave probabilities is concentrated in a thin shell around a distance $\sqrt{n}$ from the origin, Besides, the result (5) would imply that the KLS conjecture is true up to a $\log n$ factor. As it is also very well known that the KLS conjecture is stronger than the thin shell width conjecture, the result (6) implies the best known estimate for the the last conjecture, i.e. $\sigma_n \leq Cn^{1/4}$.

Previous estimates for this parameter were found by Klartag [K] and Guedon-Milman [GM]. More information on these relations can be seen, for instance, in [BGVV] and [AB1].

The proof of both results, Theorems 1.1 and 1.2, follow the original idea developed by Eldan, the *localization scheme* introduced in [E1]: given an isotropic log-concave probability μ in $\mathbb{R}^n$, a stochastic system of differential equations originates a stochastic process of (not necessarily isotropic) log-concave probabilities $(\mu_t)_{t\geq 0}$ which are an Itô process. We can get "good" information from some μ_T and then come back to the original μ. However the two proofs propose different stochastic systems of differential equations in order to get stochastic process $(\mu_t)_{t\geq 0}$ from which we can obtain estimates.

The main purpose of this paper is to unify the two approaches and give a proof of both results together, which will follow from the same stochastic system of differential equations. Even though we are not introducing any truly new ideas in this paper, rather than carefully mixing and gluing the arguments from the aforementioned authors, it is our desire to clarify and shed light on the arguments of this beautiful and interesting theory what has moved us to write this work and bring it closer to the interested people even if they are less expert in the field.

The theorem we are going to prove in this work is the following, which collects both Theorems 1.1 and 1.2.

Theorem 1.3. *There exists an absolute constant $C > 0$ such that for any isotropic log-concave probability μ in $\mathbb{R}^n$ the following isoperimetric inequality holds*

$$\mu^+(A) \geq \frac{C}{\min\{\sigma_n \log n, n^{1/4}\}} \min\{\mu(A), \mu(A^c)\}$$

for any Borel set $A \subseteq \mathbb{R}^n$.

The paper is organized in the following way. In Section 2 we will introduce notation, some definitions and some previous results we are going to use in order to develop our proof. In Section 3 we will introduce Eldan's localization scheme, presenting the system of stochastic differential equations we will consider in this work, which will define the aforementioned stochastic process of log-concave probabilities $(\mu_t)_{t\geq 0}$. In Section 4 we will give an overview of the strategy we follow in order to stress out the event whose probability is needed so that the estimates for the KLS constant can be obtained. The trace of the covariance matrix of the probabilities $(\mu_t)_{t\geq 0}$ will be needed to bound the probability of such event from below. They will be proved to be small enough with some probability in Section 5. Finally, in Section 6 we will put all the inequalities together to complete the proof of Theorem 1.3.

2 Notation and definitions

In this section we will introduce some notation and definitions which are common in this framework. Some well-known results will also be explained either by giving their proof or a reference to it.

We will denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^n$ and also the absolute value on $\mathbb{R}$. S^{n-1} will denote the Euclidean unit sphere. A probability measure μ on $\mathbb{R}^n$ is called log-concave if for any compact subsets $A, B \subseteq \mathbb{R}^n$ and for any $0 \le \lambda \le 1$

$$\mu((1-\lambda)A + \lambda B) \ge \mu(A)^{1-\lambda}\mu(B)^{\lambda}.$$

The following theorem by Borell [B], characterizes this kind of probabilities: Let μ be a non degenerate log-concave probability measure on $\mathbb{R}^n$, (i.e. not concentrated in any hyperplane). Then, μ is log-concave if and only if μ is absolutely continuous with respect to the Lebesgue measure and its density f is log-concave, i.e. $d\mu(x) = f(x)dx = e^{-V(x)}dx$, where the function $V : \mathbb{R}^n \to (-\infty, \infty]$ is convex.

In the sequel we will use the probabilistic notation $\mathbb{E}_\mu g := \int_{\mathbb{R}^n} g(x)d\mu(x)$ and $\mathrm{Var}_\mu g := \mathbb{E}_\mu(g - \mathbb{E}_\mu g)^2$ for any μ-integrable function g.

First reductions:

We say that μ is *isotropic* if its barycenter $b_\mu := \mathbb{E}_\mu x = 0$ and its covariance matrix

$$\mathrm{Cov}_\mu = A_\mu := \mathbb{E}_\mu(x - b_\mu) \otimes (x - b_\mu) = I_n,$$

where I_n is the identity matrix. Every non degenerate log-concave probability $d\mu(x) = f(x)dx$ admits an affine transformation such that $d\nu(y) = |\det(A_\mu)|^{1/2} f(b_\mu + A_\mu^{1/2} y)dy$ is an isotropic log-concave probability. In particular with this change of variables, it is easy to prove that if an isotropic probability μ satisfies Poincaré's inequality (2) with some constant C, then for any non-degenerate linear map T the log-concave probability measure

$\mu \circ T$, given by $(\mu \circ T)(A) = \mu(T(A))$ for any Borel set A, satisfies (2) with the same constant C. Therefore, if there exists a constant C_n such that every isotropic log-concave probability in $\mathbb{R}^n$ satisfies (2) with constant C_n, then every log-concave probability in $\mathbb{R}^n$ satisfies (2) with the same constant C_n and if there exists a constant $\tilde{C}_n$ such that every isotropic log-concave probability in $\mathbb{R}^n$ with compact support satisfies (2) with constant $\tilde{C}_n$, then every log-concave probability in $\mathbb{R}^n$ with compact support satisfies (2) with the same constant $\tilde{C}_n$.

Furthermore, if any log-concave probability in $\mathbb{R}^n$ with compact support satisfies (2) with some constant $C_n > 2\sqrt{2}$ (which we can always assume), then any isotropic log-concave probability satisfies (2) with constant $5C_n$ and so any log-concave (non-necessarily isotropic) probability in $\mathbb{R}^n$ satisfies (2) with constant $5C_n$. Indeed, let μ be an isotropic log-concave probability, $d\mu = e^{-V(x)}dx$, with $V : \mathbb{R}^n \to (-\infty, \infty]$ convex and let f be any Lipschitz μ integrable function f. If we take K a convex body such that

- $\int_K e^{-V(x)}dx \geq \frac{1}{2}$
- $\int_K (f(x) - \mathbb{E}_\mu f(x))^2 d\mu(x) \geq \frac{1}{2} \int_{\mathbb{R}^n} (f(x) - \mathbb{E}_\mu f(x))^2 d\mu(x),$
- $(\mathbb{E}_{\mu_K} f - \mathbb{E}_\mu f)^2 \leq \mathbb{E}_\mu |\nabla f|^2$

denoting by μ_K the probability supported on K with density

$$d\mu_K(x) = \frac{e^{-V(x)}dx}{\int_K e^{-V(x)}dx},$$

and taking into account that for any log-concave probability ν one has that the operator norm of its covariance matrix verifies $\|\mathrm{Cov}_\nu\|_{\mathrm{op}} = \sup_{\theta \in S^{n-1}} \left(\mathbb{E}_\nu \langle x, \theta \rangle^2 - (\mathbb{E}_\nu \langle x, \theta \rangle)^2 \right)$, we obtain

$$\begin{aligned}
\mathrm{Var}_\mu f &\leq 2\sqrt{2}\mathrm{Var}_{\mu_K} f + 2\sqrt{2}(\mathbb{E}_{\mu_K} f - \mathbb{E}_\mu f)^2 \leq 2\sqrt{2}\mathrm{Var}_{\mu_K} f + 2\sqrt{2}\mathbb{E}_\mu |\nabla f|^2 \\
&\leq C_n \|\mathrm{Cov}_{\mu_K}\|_{\mathrm{op}} \mathbb{E}_{\mu_K} |\nabla f|^2 + 2\sqrt{2}\mathbb{E}_\mu |\nabla f|^2 \\
&= C_n \sup_{\theta \in S^{n-1}} \left(\mathbb{E}_{\mu_K} \langle x, \theta \rangle^2 - (\mathbb{E}_{\mu_K} \langle x, \theta \rangle)^2 \right) \mathbb{E}_{\mu_K} |\nabla f|^2 + 2\sqrt{2}\mathbb{E}_\mu |\nabla f|^2 \\
&\leq C_n \sup_{\theta \in S^{n-1}} \mathbb{E}_{\mu_K} \langle x, \theta \rangle^2 \mathbb{E}_{\mu_K} |\nabla f|^2 + 2\sqrt{2}\mathbb{E}_\mu |\nabla f|^2 \\
&\leq 4C_n \sup_{\theta \in S^{n-1}} \mathbb{E}_\mu \langle x, \theta \rangle^2 \mathbb{E}_\mu |\nabla f|^2 + 2\sqrt{2}\mathbb{E}_\mu |\nabla f|^2 \\
&= (4C_n + 2\sqrt{2})\mathbb{E}_\mu |\nabla f|^2 \leq 5C_n \mathbb{E}_\mu |\nabla f|^2.
\end{aligned}$$

Therefore, one can consider only compactly supported isotropic log-concave probabilities in $\mathbb{R}^n$ in order to prove (2).

By using a nice result by E. Milman, [EM], in order to prove (2) it is enough to give an upper bound of the variance of f by an absolute constant times $\|\nabla f\|_\infty^2$ for any Lipschitz μ integrable function. Besides, if f is a 1-Lipschitz μ-integrable function one has

$$\mathrm{Var}_\mu f \leq \mathbb{E}_\mu|f - f(0)|^2 \leq \mathbb{E}_\mu|x|^2 = n.$$

As a consequence one obtains that for every fixed $n \in \mathbb{N}$, the value of the constant such that (2) holds for every log-concave probability μ in $\mathbb{R}^n$ and Lipschitz μ-integrable function f is bounded by a constant C_n, depending on $\mathbb{N}$, Therefore, it is enough to prove Theorem 1.3 for every $n \in \mathbb{N}$ larger than some fixed n_0, since, changing the value of the constant C, one can immediately obtain the result for every dimension $n \in \mathbb{N}$.

In conclusion, one can consider only compactly supported log-concave isotropic probabilities in $\mathbb{R}^n$ for $n \geq n_0$ for some $n_0 \in \mathbb{N}$ in order to prove (2).

We will include some preliminary facts o results we are going to use.

Lemma 2.1. *Let μ be any probability on $\mathbb{R}^n$ and $z \in \mathbb{R}^n$, then*

$$\mathbb{E}_\mu\langle x - b_\mu, z\rangle^2 = \langle A_\mu z, z\rangle.$$

Proof. Simply expand both expressions. □

Proposition 2.2 (Reverse Hölder's inequality)**.** *There exists an absolute constant $C > 0$ such that for every log-concave probability μ on $\mathbb{R}^n$, any seminorm $g : \mathbb{R}^n \to \mathbb{R}$ and $1 \leq p \leq q$ we have*

$$\left(\mathbb{E}_\mu g^p\right)^{1/p} \leq \left(\mathbb{E}_\mu g^q\right)^{1/q} \leq C\frac{q}{p}\left(\mathbb{E}_\mu g^p\right)^{1/p}.$$

Proof. See [BGVV, Theorem 2.4.6.], □

The next result says that we only need to take into account Borel sets with probability 1/2.

Proposition 2.3. *Let μ be an isotropic log-concave probability on $\mathbb{R}^n$. Assume that there exist two positive numbers $\Theta, C > 0$ such that*

$$\mu(E^\Theta \setminus E) \geq C$$

for any Borel set $E \in \mathbb{R}^n$ *such that* $\mu(E) = \frac{1}{2}$*, where* E^Θ *is the* Θ*-dilation of* E*, i.e.* $E^\Theta = \{e + x \in \mathbb{R}^n : e \in E, |x| < \Theta\}$*. Then*

$$\mu^+(A) \geq \frac{C}{\Theta} min\{\mu(A), \mu(A^c)\} \qquad \text{for any Borel set } A \subset \mathbb{R}^n.$$

Proof. See [EM2]. □

In order to control the probability of dilations of Borel sets, the following concentration results for *more convex than Gaussian* probabilities can be applied

Proposition 2.4. *Let* ϕ *be a convex function* $\phi : \mathbb{R}^n \to \mathbb{R}$ *and let* $t > 0$*. Assume that*

$$d\mu(x) = e^{-\phi(x) - \frac{t}{2}|x|^2} dx,$$

is a centered probability on $\mathbb{R}^n$*. Then for every Borel set* $A \subset \mathbb{R}^n$ *such that*

$$\frac{1}{10} \leq \mu(A) \leq \frac{9}{10}$$

we have

$$\mu\left(A^{\frac{D}{\sqrt{t}}}\right) \geq \frac{95}{100},$$

where $D > 0$ *is a suitably chosen absolute constant independent of every other parameter and* $A^{D/\sqrt{t}}$ *is the* $D/\sqrt{t}$*-dilation of* A*.*

The proof of this fact follows from [BGVV, Theorem 14.6.6] (see also [AB1, Theorem 3.8]).

Next we are going to describe some results on Itô processes we are going to use. (see for instance, [O], [Kle]).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and $(\mathcal{F}_t)_{t \in [0,T]}$ a filtration in Ω, i.e., a family of sub-σ-algebras on Ω such that $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2} \subseteq \mathcal{F}$, whenever $0 \leq t_1 \leq t_2 \leq T$.

A one-dimensional Itô process $(X(t))_{t \in [0,T]}$ on Ω is a real stochastic process having the form

$$X(t) = X(0) + \int_0^t U(s) ds + \int_0^t V(s) dW(s), \qquad 0 \leq t \leq T,$$

where $X(0)$ is $\mathcal{F}_0$-measurable and the processes $U(t)$ and $V(t)$ are $\mathcal{F}_t$-adapted and such that $\mathbb{E}_\mathbb{P} \int_0^T |U(t)| dt < \infty$, $\mathbb{E}_\mathbb{P} \int_0^T V^2(t) dt < \infty$, and $(W(t))_{t \geq 0}$ is a Wiener process (or Brownian motion). It is said that the process $(X(t))_{t\in[0,T]}$ has the stochastic differential on $[0,T]$

$$dX(t) = U(t)dt + V(t)dW(t), \qquad 0 \le t \le T.$$

The process $(U(t))_{t\in[0,T]}$ is called the drift and $(V(t))_{t\in[0,T]}$ is called the diffusion of $(X(t))_{t\in[0,T]}$. Note that the processes $(U(t))_{t\in[0,T]}$ and $(V(t))_{t\in[0,T]}$ may (and often do) depend on $(X(t))_{t\in[0,T]}$ or the Wiener process $(W(t))_{t\ge 0}$ as well.

In the case that the processes $(U(t))_{t\in[0,T]}$ is $\mathbb{R}^n$-valued, $(V(t))_{t\in[0,T]}$ is an $(n\times n)$ matrix and $(W(t))_{t\ge 0}$ is an n-dimensional Wiener process, we say that X is an n-dimensional Itô process.

Let $(X_1(t))_{t\in[0,T]}, (X_2(t))_{t\in[0,T]}$ be two 1-dimensional Itô processes. The quadratic covariation of $[X_1, X_2]_t$ is defined by

$$[X_1, X_2]_t = \lim_{\|P\|\to 0} \sum_{k=0}^{N} \big(X_1(\tau_{k+1}) - X_1(\tau_k)\big)\big(X_2(\tau_{k+1}) - X_2(\tau_k)\big)$$

where $P = \{0 = \tau_0 \le \tau_1 \le \cdots \le \tau_N \le T\}$ is a stochastic partition of the non-negative real numbers, $\|P\| = \max(\tau_n - \tau_{n-1})$ is called the mesh of P and the limit is defined using convergence in probability. If $X_2 = X_1$ we will denote $[X_1]_t := [X_1, X_1]_t$ for every $t \in [0,T]$.

In the case where $dX_i(t) = U_i(t)dt + \langle V_i(t), dW(t)\rangle$, for $i = 1,2$, where $(U_i(t))_{t\in[0,T]}$ and $(V_i(t))_{t\in[0,T]}$ $(i = 1,2)$, are n-dimensional adapted stochastic processes and $(W(t))_{t\ge 0}$ is an n-dimensional Wiener process $[X_1, X_2]_t$ is also an Itô process without diffusion and

$$d[X_1, X_2]_t = \langle V_1(t), V_2(t)\rangle dt. \tag{7}$$

Proposition 2.5 (Itô's formula). *Let $(X(t))_{t\in[0,T]}$ be an n-dimensional Itô process given by $dX(t) = U(t)dt + V(t)dW(t)$, where $U(t) \in \mathbb{R}^n$, $V(t)$ is an $n\times n$ matrix and $W(t)$ is a n-dimensional Wiener process. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a function with $g \in \mathcal{C}^{2)}(\mathbb{R}^n)$. Then the stochastic process $(Y(t))_{t\in[0,T]}$ given by $Y(t) = g(X(t))$ verifies*

$$\begin{aligned} dY(t) = dg(X_1(t),\ldots,X_n(t)) &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} g(X_1(t),\ldots,X_n(t))dX_i(t) \\ &+ \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} g(X_1(t),\ldots,X_n(t))d[X_i, X_j]_t. \end{aligned}$$

Proposition 2.6 (Reflection principle)**.** *Given a Wiener process* $(W(t))_{t\geq 0}$*,* $\gamma > 0$*, and* $T \geq 0$*, then we have that*

$$\mathbb{P}\left(\max_{s\in[0,T]} W(s) \geq \gamma\right) = 2\mathbb{P}\big(W(T) \geq \gamma\big).$$

It is important to remark that any Itô process without drift is also a martingale. Conversely, we have

Proposition 2.7 (Dambis, Dubins-Schwarz)**.** *Every continuous local martingale,* $(M(t))_{t\geq 0}$ *can be obtained as a Brownian motion* $(\bar{W}_{[M]_t})_{t\geq 0}$*, i.e.*

$$M(t) - M(0) = \bar{W}_{[M]_t} \qquad \forall\, t \geq 0.$$

3 Eldan's localization scheme

In the sequel we are going to prove Theorem 1.3. As mentioned in Section2 we may assume that μ is compactly supported.

In his work [E1], R. Eldan introduced the stochastic localization scheme through the following system of stochastic differential equations. Given an isotropic log-concave distribution $d\mu(x) = f(x)dx$ in $\mathbb{R}^n$, consider

$$\begin{aligned} dc_t &= C_t b_t dt + C_t^{1/2} dW_t, \qquad && c_0 = 0 \\ dB_t &= C_t dt, && B_0 = 0 \end{aligned}$$

where $C_t \in \mathbb{R}^n \times \mathbb{R}^n$ is a symmetrical positive definite matrix to be precised later, W_t a n-dimensional Wiener process and b_t is the barycenter of the density $f_t(x)$ given by

$$f_t(x) = \frac{e^{\langle c_t, x\rangle - \frac{1}{2}\langle B_t x, x\rangle} f(x)}{\int_{\mathbb{R}^n} e^{\langle c_t, x\rangle - \frac{1}{2}\langle B_t x, x\rangle} f(x)dx}, \qquad b_t = \int_{\mathbb{R}^n} x f_t(x)dx.$$

Eldan's choice for C_t is the matrix A_t^{-1}, the inverse of the covariance matrix of the probability $d\mu_t = f_t(x)dx$, $A_t = \mathbb{E}_{\mu_t}(x - b_t) \otimes (x - b_t)$, while Lee & Vempala use $C_t = I_n$ (the identity $(n \times n)$ matrix).

Our idea now is to use the same approach to get both results. We will follow Lee &

Vempala's choice. So the system of stochastic differential equations is

$$dc_t = b_t dt + dW_t, \qquad c_0 = 0 \tag{8}$$

where W_t a n-dimensional Wiener process and b_t is the barycenter of the density $f_t(x)$ given by

$$f_t(x) = \frac{e^{\langle c_t, x\rangle - \frac{t}{2}|x|^2} f(x)}{\int_{\mathbb{R}^n} e^{\langle c_t, x\rangle - \frac{t}{2}|x|^2} f(x) dx} \qquad b_t = \int_{\mathbb{R}^n} x f_t(x) dx. \tag{9}$$

The probability measure with density $f_t(x)$ will be denote by μ_t.

Lemma 3.1 (Existence and uniqueness)**.** *Assume $f(x)$ is a compactly supported function on $\mathbb{R}^n$, then the stochastic system of differential equations (8) has a unique solution for all $t > 0$.*

Proof. See, for instance [O]. □

Proposition 3.2. *Given the system of stochastic differential equations* (8)*, the density $f_t(x)$ defined by* (9) *is log-concave as a function of x and for every $x \in \mathbb{R}^n$ it is an Itô process verifying*

$$df_t(x) = f_t(x)\langle x - b_t, dW_t\rangle.$$

Proof. We denote

$$Y_t = \langle c_t, x\rangle - \frac{t}{2}|x|^2, \qquad Z_t = e^{Y_t}$$

for fixed $x \in \mathbb{R}^n$. Then Y_t is an Itô process

$$dY_t = \langle dc_t, x\rangle - \frac{1}{2}|x|^2 dt = (\langle b_t, x\rangle - \frac{1}{2}|x|^2)dt + \langle x, dW_t\rangle$$

and $d[Y]_t = |x|^2\, dt$. So, by Itô's formula,

$$dZ_t = e^{Y_t}(dY_t + \frac{1}{2}d[Y]_t) = Z_t \left\langle x, b_t dt + dW_t\right\rangle.$$

If $V_t = \int_{\mathbb{R}^n} Z_t(x) f(x) dx$, the function

$$f_t(x) = \frac{Z_t}{V_t}\, f(x),$$

as a function of x is a log-concave density in $\mathbb{R}^n$. Applying Itô's formula for fixed $x \in \mathbb{R}^n$ we have

$$\begin{aligned} df_t(x) &= f(x)\left(\frac{dZ_t}{V_t} - \frac{Z_t}{V_t^2}\,dV_t + \frac{1}{2}\left(-2\frac{1}{V_t^2}[dZ_t, dV_t] + \frac{2Z_t}{V_t^3}d[V]_t\right)\right) \\ &= f_t(x)\left(\frac{dZ_t}{Z_t} - \frac{dV_t}{V_t} - \frac{[dZ_t, dV_t]}{Z_t V_t} + \frac{d[V]_t}{V_t^2}\right). \end{aligned}$$

We compute dV_t using Itô's formula:

$$\begin{aligned} dV_t &= d\left(\int_{\mathbb{R}^n} Z_t(x)f(x)dx\right) = \int_{\mathbb{R}^n} f(x)dZ_t(x)dx \\ &= \int_{\mathbb{R}^n} f(x)Z_t\left\langle x, b_t dt + dW_t\right\rangle dx = \left\langle \int_{\mathbb{R}^n} xf(x)Z_t dx, b_t dt + dW_t \right\rangle \\ &= V_t\left\langle b_t, b_t dt + dW_t\right\rangle \end{aligned}$$

Then

$$\frac{[dZ_t, dV_t]}{Z_t V_t} = \langle x, b_t dt + dW_t\rangle\langle b_t, b_t dt + dW_t\rangle = \langle x, dW_t\rangle\langle b_t, dW_t\rangle = \langle x, b_t\rangle dt$$

and

$$\frac{d[V]_t}{V_t^2} = |b_t|^2\,dt.$$

Hence

$$df_t(x) = f_t(x)\left(\langle x - b_t, b_t dt + dW_t\rangle - \langle x, b_t\rangle dt + \langle b_t, b_t\rangle\,dt\right) = f_t(x)\langle x - b_t, dW_t\rangle.$$

□

In the following proposition we analyze how the covariance matrix evolves.

Proposition 3.3. *Given the system of stochastic differential equations* (8)*, let b_t be the barycenter and A_t the covariance matrix of the measure μ_t defined by* (9)*. Then*

$$dA_t = \left\langle \mathbb{E}_{\mu_t}(x - b_t) \otimes (x - b_t)(x - b_t), dW_t \right\rangle - A_t^2\,dt.$$

Proof. First we compute the variation of the barycenter

$$db_t = \int_{\mathbb{R}^n} x df_t(x)dx = \int_{\mathbb{R}^n} x f_t(x)\langle x - b_t, dW_t\rangle dx = \mathbb{E}_{\mu_t}\langle x - b_t, dW_t\rangle x \tag{10}$$

$$= \mathbb{E}_{\mu_t}(x - b_t) \otimes x dW_t = \left(\mathbb{E}_{\mu_t}(x - b_t) \otimes (x - b_t)\right) dW_t = A_t dW_t, \tag{11}$$

since $(\mathbb{E}_{\mu_t}(x - b_t) \otimes b_t) = 0$ and A_t is the covariance matrix of μ_t, i.e., $A_t = \mathbb{E}_{\mu_t}(x - b_t) \otimes (x - b_t)$.

It is clear that

$$dA_t = \int_{\mathbb{R}^n} d\left((x - b_t) \otimes (x - b_t) f_t(x)\right) dx.$$

In order to use Itô's formula in matrix calculus we introduce, for every $x \in \mathbb{R}^n$, the matrix $g(u, v)$, defined by $g(u, v) = v(x - u) \otimes (x - u)$ where $u \in \mathbb{R}^n$ and $v \in \mathbb{R}$. One can check that

$$\begin{aligned}
\frac{\partial g}{\partial u_k} &= -v \sum_{i,j=1}^{n} (\delta_{ik}(x_j - u_j) + \delta_{jk}(x_i - u_i)) e_i \otimes e_j \\
&= -v \sum_{j=1}^{n} (x_j - u_j) e_k \otimes e_j - v \sum_{i=1}^{n} (x_i - u_i) e_i \otimes e_k, \\
\frac{\partial g}{\partial v} &= (x - u) \otimes (x - u), \\
\frac{\partial^2 g}{\partial v^2} &= 0, \\
\frac{\partial^2 g}{\partial v \partial u_k} &= -\sum_{j=1}^{n} (x_j - u_j) e_k \otimes e_j - \sum_{i=1}^{n} (x_i - u_i) e_i \otimes e_k, \\
\frac{\partial^2 g}{\partial u_k \partial u_l} &= v \sum_{i,j=1}^{n} \delta_{ik}\delta_{jl} e_{ij} = v(e_k \otimes e_l + e_l \otimes e_k).
\end{aligned}$$

where $(e_i)_{i=1}^n$ is the canonical basis and δ_{ik} is the Kronecker's delta. Then, using Itô's formula,

$$\begin{aligned} dA_t = & -\mathbb{E}_{\mu_t} db_t \otimes (x-b_t) - \mathbb{E}_{\mu_t}(x-b_t) \otimes db_t + \int_{\mathbb{R}^n} (x-b_t) \otimes (x-b_t) df_t(x)dx \\ & - \int_{\mathbb{R}^n} (db_t \otimes (x-b_t) + (x-b_t) \otimes db_t)\, df_t(x)dx \\ & + \int_{\mathbb{R}^n} f_t(x) db_t \otimes db_t dx. \end{aligned}$$

As above, $\mathbb{E}_{\mu_t} db_t \otimes (x-b_t) = db_t \otimes \mathbb{E}_{\mu_t}(x-b_t) = 0$. Also, by (7) and (10), the entry (i,j) of the matrix $db_t \otimes db_t$ is $\sum_{k,l=1}^{n} (A_t)_{ik}(A_t)_{jl} dt$, so, $db_t \otimes db_t = A_t A_t dt$. Eventually

$$\int_{\mathbb{R}^n} db_t \otimes (x-b_t) df_t(x)dx = db_t \otimes \int_{\mathbb{R}^n} (x-b_t) df_t(x)dx = db_t \otimes db_t.$$

Gluing all this computations we obtain the result.

□

4 Strategy

Let μ be an isotropic log-concave probability. Our goal is to find two values $\Theta, C > 0$ such that for any Borel set $E \subseteq \mathbb{R}^n$ with $\mu(E) = 1/2$.

$$\mu(E^\Theta \setminus E) \geq C$$

in order to apply Proposition 2.3 ([EM2]).

In the sequel $d\mu(x) = f(x)dx$ is an isotropic log-concave probability on $\mathbb{R}^n$ and E is a fixed Borel set in $\mathbb{R}^n$ such that $\mu(E) = 1/2$. We introduce the stochastic process

$$g_E(t) = g(t) = \mu_t(E) = \int_E f_t(x)dx, \qquad t \geq 0,$$

where μ_t and $f_t(x)$ are defined by the system of stochastic differential equations (8) and by (9). It is obvious that $g(0) = 1/2$, $\forall \omega \in \Omega$, since $f_0(x) = f(x)$ for every $\omega \in \Omega$ and also that g is an Itô process with

$$dg(t) = \int_E df_t(x)dx = \left\langle \int_E f_t(x)(x - b_t)dx, dW_t \right\rangle.$$

In particular the process $(g(t))_{t\geq 0}$ is a martingale and for every $t \geq 0$ the expected value of $g(t)$ is $\mathbb{E}_{\mathbb{P}} g(t) = 1/2$.

Let $T > 0$ be a time to be precised later and notice that for any $\Theta > 0$, since also $(g_{E^\Theta \setminus E}(t))_{t\geq 0}$ is a martingale,

$$\begin{aligned}\mu(E^\Theta \setminus E) &= \int_{E^\Theta \setminus E} f(x)dx = \int_{E^\Theta \setminus E} \mathbb{E}_{\mathbb{P}} f_T(x)dx \\ &= \mathbb{E}_{\mathbb{P}} \int_{E^\Theta \setminus E} f_T(x)dx = \mathbb{E}_{\mathbb{P}} \mu_T(E^\Theta \setminus E).\end{aligned}$$

In order to apply the preceding propositions we will consider the event $\mathcal{G} = \{\omega \in \Omega; |g(T) - 1/2| \leq 1/4\}$. By Proposition 2.4 and the way that the densities f_t are defined, we will have that there exists some absolute constant $D > 0$ such that for $\omega \in \mathcal{G}$ we will have $\mu_T(E^{D/\sqrt{T}}) \geq 0.95$ and therefore, by Markov's inequality,

$$\mu(E^{D/\sqrt{T}} \setminus E) = \mathbb{E}_{\mathbb{P}} \mu_T(E^{D/\sqrt{T}} \setminus E) \geq (0.95 - 0.5)\mathbb{P}(\mathcal{G}) = \frac{9}{20}\mathbb{P}(\mathcal{G}).$$

Hence, if we find $T, C_1 > 0$ independent of E such that $\mathbb{P}(\mathcal{G}) > C_1$ then we will get that

$$\mu^+(A) \geq \frac{C_1 D}{\sqrt{T}} \min\{\mu(A), \mu(A^c)\} \qquad \forall A \text{ Borel set } \subset \mathbb{R}^n.$$

5 Computing the trace

We will bound $\|A_t\|_{\text{op}}$ by computing the trace of A_t raised to some power. The main result is the following

Proposition 5.1. *Given the system of stochastic differential equations* (8)*, let A_t be the covariance matrix of the measure μ_t defined by* (9)*. Let $p \geq 2$ be an integer. Then*

$$d(\mathit{Tr}(A_t^p)) = \delta_t dt + \langle v_t, dW_t \rangle$$

where δ_t is an adapted, with bounded variation process, such that

$$\delta_t \leq \begin{cases} C\, p^2\sigma_n^2 \log n\, Tr(A_t^p)^{1+\frac{1}{p}}, & \text{if } p \geq 3 \\ C\, Tr(A_t^2)^{3/2}, & \text{if } p = 2 \end{cases}$$

and

$$|v_t| \leq Cp\, Tr(A_t^p)^{1+\frac{1}{2p}} \qquad \forall p \geq 2,$$

where $C > 0$ is an absolute constant and $\sigma_n^2 = \sup \mathbb{E}\big||X| - \sqrt{n}\big|^2$ and the sup *runs over all isotropic log-concave random vectors in $\mathbb{R}^n$.*

Proof. We follow Eldan's method to compute $d(\mathrm{Tr}(A_t^p))$. However, for $p = 2$ we will use the idea given by Lee-Vempala. In order to do that we will express A_t in terms of a special orthonormal basis.

Let $t_0 \geq 0$ be a fixed time. Let $(v_i)_{i=1}^n$ be an orthonormal basis composed by the eigenvectors of A_{t_0} and $(\alpha_{ii}(t_0))_{i=1}^n$ the corresponding eigenvalues. Assume that the orthonormal basis $(v_i)_{i=1}^n$ is ordered in such way that $\alpha_{11}(t_0) \geq \alpha_{22}(t_0) \geq \cdots \geq \alpha_{nn}(t_0)$. Let also, for any $t \geq 0$, $\alpha_{i,j} = \alpha_{i,j}(t) := \langle A_t v_i, v_j\rangle$. We can express, for any $t \geq 0$,

$$A_t = \sum_{i,j=1}^n \alpha_{ij} v_i \otimes v_j.$$

It is not difficult to see that for any natural number $p \geq 2$ and for any $t \geq 0$,

$$\mathrm{Tr}(A_t^p) = \sum \alpha_{i_1 i_2}\alpha_{i_2 i_3}\dots\alpha_{i_p i_1},$$

where the sum runs over all indices $i_1, \dots i_p \in \{1, \dots, n\}$. Notice that if $t = t_0$ then $\alpha_{ij}(t_0) = \langle A_{t_0} v_i, v_j\rangle = \delta ij$, the Kronecker delta. Therefore, differentiating at $t = t_0$,

$$\begin{aligned} d(\mathrm{Tr}(A_t^p))|_{t=t_0} &= \sum d(\alpha_{i_1 i_2}\alpha_{i_2 i_3}\dots\alpha_{i_p i_1})\big|_{t=t_0} \\ &= \sum_{i=1}^n d(\alpha_{ii}^p)|_{t=t_0} + \sum_{\substack{i\neq j \\ k_1+k_2+k_3=p-2}} d(\alpha_{ii}^{k_1}\alpha_{ij}\alpha_{jj}^{k_2}\alpha_{ji}\alpha_{ii}^{k_3})\big|_{t=t_0} \\ &= \sum_{i=1}^n d(\alpha_{ii}^p)|_{t=t_0} + \sum_{\substack{i\neq j \\ 0\leq k\leq p-2}} d(\alpha_{ii}^{k}\alpha_{jj}^{p-k-2}\alpha_{ij}^2)\Big|_{t=t_0} \end{aligned}$$

(the rest of the terms are 0 by Itô's formula). According to the expression of $d(A_t)$ we have that

$$
\begin{aligned}
d(\alpha_{ij})|_{t=t_0} &= \langle\, d(A_t)|_{t=t_0}\, v_i, v_j\rangle \\
&= \langle \mathbb{E}_{\mu_{t_0}}\langle x-b_{t_0}, v_i\rangle\langle x-b_{t_0}, v_j\rangle(x-b_{t_0}), dW_t\rangle - \langle A_{t_0}^2 v_i, v_j\rangle dt \\
&= \langle \xi_{ij}, dW_t\rangle - \langle A_{t_0}v_i, A_{t_0}v_j\rangle dt = \langle \xi_{ij}, dW_t\rangle - \alpha_{ii}\alpha_{jj}\delta_{ij}dt,
\end{aligned}
$$

where ξ_{ij} are the vectors $\xi_{ij} = \xi_{i,j}(t_0) = \mathbb{E}_{\mu_{t_0}}\langle x-b_{t_0}, v_i\rangle\langle x-b_{t_0}, v_j\rangle(x-b_{t_0}) \in \mathbb{R}^n$

By Itô's formula we obtain the following estimates

$$
\begin{aligned}
d(\alpha_{ii}^p)|_{t=t_0} &= p\alpha_{ii}^{p-1}\, d\alpha_{ii}|_{t=t_0} + \frac{1}{2}p(p-1)\alpha_{ii}^{p-2}d[\alpha_{ii}]_{t_0} \\
&= \left(\frac{1}{2}p(p-1)\alpha_{ii}^p\frac{|\xi_{ii}|^2}{\alpha_{ii}^2} - p\alpha_{ii}^{p+1}\right)dt + p\alpha_{ii}^p\left\langle \frac{\xi_{ii}}{\alpha_{ii}}, dW_t\right\rangle
\end{aligned}
$$

and for $i<j$ and $0\le k\le p-2$, since $\alpha_{ij}=0$ and $\alpha_{ii}\ge\alpha_{jj}$,

$$
\begin{aligned}
d((\alpha_{ii})^k(\alpha_{jj})^{p-k-2}(\alpha_{ij})^2) &= (\alpha_{ii})^k(\alpha_{jj})^{p-k-2}d[\alpha_{ij}]_t \\
&= (\alpha_{ii})^{k+1}(\alpha_{jj})^{p-k-1}\frac{|\xi_{ij}|^2}{\alpha_{ii}\alpha_{jj}}dt \le (\alpha_{ii})^p\frac{|\xi_{ij}|^2}{\alpha_{ii}\alpha_{jj}}dt.
\end{aligned}
$$

Therefore, $\mathrm{Tr}(A_t^p)$ is an Itô process with

$$
d(\mathrm{Tr}(A_t^p)) = \delta_t dt + \langle v_t, dW_t\rangle,
$$

where for any $t=t_0$

$$
\begin{aligned}
\delta_{t_0} = &\frac{1}{2}p(p-1)\sum_{i=1}^n(\alpha_{ii})^p\frac{|\xi_{ii}|^2}{(\alpha_{ii})^2} - p\sum_{i=1}^n(\alpha_{ii})^{p+1} \\
&+ \sum_{\substack{i\neq j\\ 0\le k\le p-2}}(\alpha_{ii})^{k+1}(\alpha_{jj})^{p-k-1}\frac{|\xi_{ij}|^2}{\alpha_{ii}\alpha_{jj}}
\end{aligned}
$$

and

$$
v_{t_0} = p\sum_{i=1}^n\alpha_{ii}^p\frac{\xi_{ii}}{\alpha_{ii}}.
$$

It is now enough to bound form above δ_t and $|v_t|$ at each particular $t=t_0$. First of all we estimate $|v_t|$.

By using Cauchy-Schwartz and Borell's reverse Hölder inequalities (Proposition 2.2), there exists an absolute constant $C > 0$ such that for every $1 \leq i \leq n$

$$\begin{aligned} |\xi_{ii}| &= \left\langle \xi_{ii}, \frac{\xi_{ii}}{|\xi_{ii}|} \right\rangle = \mathbb{E}_{\mu_t} \langle x - b_t, v_i \rangle^2 \left\langle x - b_t, \frac{\xi_{ii}}{|\xi_{ii}|} \right\rangle \\ &\leq \sqrt{\mathbb{E}_{\mu_t} \langle x - b_t, v_i \rangle^4} \sqrt{\mathbb{E}_{\mu_t} \left\langle x - b_t, \frac{\xi_{ii}}{|\xi_{ii}|} \right\rangle^2} \\ &\leq C \, \mathbb{E}_{\mu_t} \langle x - b_t, v_i \rangle^2 \sqrt{\mathbb{E}_{\mu_t} \left\langle x - b_t, \frac{\xi_{ii}}{|\xi_{ii}|} \right\rangle^2} \end{aligned}$$

Taking into account that, by Lemma 2.1,

$$\mathbb{E}_{\mu_t} \langle x - b_t, z \rangle^2 = \langle A_t z, z \rangle \qquad \forall \, z \in \mathbb{R}^n$$

we obtain that for every $1 \leq i \leq n$

$$\begin{aligned} |\xi_{ii}| &\leq C \langle A_t v_i, v_i \rangle \left\langle A_t \frac{\xi_{ii}}{|\xi_{ii}|}, \frac{\xi_{ii}}{|\xi_{ii}|} \right\rangle^{1/2} \\ &\leq C \alpha_{ii} \|A_t\|_{\text{op}}^{1/2}. \end{aligned}$$

Hence

$$|v_t| \leq Cp \|A_t\|_{\text{op}}^{1/2} \text{Tr}(A_t^p) \leq Cp (\text{Tr}(A_t^p))^{1+1/(2p)},$$

for some absolute constant $C > 0$.

Next we will estimate δ_t

i) Case $p = 2$

Note that this necessarily implies $k = 0$, and so we have a simpler expression for δ_t,

$$\delta_t = \sum_{i,j=1}^n |\xi_{ij}|^2 - 2 \sum_{i=1}^n (\alpha_{ii})^3.$$

Thus, using again Borell's reverse Hölder's inequality twice

$$\begin{aligned}
\delta_t &\leq \sum_{i,j=1}^{n} |\xi_{ij}|^2 = \sum_{i,j=1}^{n} \left|\mathbb{E}_{\mu_t} \langle x - b_t, v_i\rangle\langle x - b_t, v_j\rangle(x - b_t)\right|^2 \\
&= \sum_{i,j,k=1}^{n} |\mathbb{E}_{\mu_t} \langle x - b_t, v_i\rangle\langle x - b_t, v_j\rangle\langle x - b_t, v_k\rangle|^2 \\
&= \mathbb{E}_{\mu_{t,x}\otimes\mu_{t,y}} \langle x - b_t, y - b_t\rangle^3 \leq C\mathbb{E}_{\mu_{t,x}} \left(\mathbb{E}_{\mu_{y,t}} \langle x - b_t, y - b_t\rangle^2\right)^{3/2} \\
&= C\mathbb{E}_{\mu_{t,x}} \langle A_t(x - b_t), x - b_t\rangle^{3/2} = C\mathbb{E}_{\mu_{t,x}} |A_t^{1/2}(x - b_t)|^3 \\
&\leq C^2 \left(\mathbb{E}_{\mu_{t,x}} |A_t^{1/2}(x - b_t)|^2\right)^{3/2} \leq C^2 \left(\mathbb{E}_{\mu_{t,x}} \langle A_t(x - b_t), x - b_t\rangle\right)^{3/2} \\
&= C^2(\mathrm{Tr}(A_t^2))^{3/2}.
\end{aligned}$$

ii) Case $p \geq 3$. Now

$$\begin{aligned}
\delta_t &\leq \frac{1}{2} p(p-1) \sum_{i=1}^{n} (\alpha_{ii})^p \frac{|\xi_{ij}|^2}{(\alpha_{ii})^2} + p(p-1) \sum_{1\leq i<j\leq n} (\alpha_{ii})^p \frac{|\xi_{ij}|^2}{\alpha_{ii}\alpha_{jj}} \\
&\leq p(p-1) \sum_{i=1}^{n} (\alpha_{ii})^p \sum_{j=1}^{n} \frac{|\xi_{ij}|^2}{\alpha_{ii}\alpha_{jj}}.
\end{aligned}$$

Let us fix $1 \leq i \leq n$. Then

$$\sum_{j=1}^{n} \frac{|\xi_{ij}|^2}{\alpha_{ii}\alpha_{jj}} = \sum_{j=1}^{n} \left|\mathbb{E}_{\mu_t} \left\langle x - b_t, \frac{v_i}{\sqrt{\alpha_{ii}}}\right\rangle \left\langle x - b_t, \frac{v_j}{\sqrt{\alpha_{jj}}}\right\rangle (x - b_t)\right|^2 = (\star)$$

We perform in the integral defining the expectation the change of variables $x - b_t = A_t^{1/2} y$. The integral with respect to the new variable y can be regarded as an expectation with respect to a probability ν_t, which is isotropic and, since the vectors $(\eta_i)_{i=1}^n$ with $\eta_i =$

$A_t^{1/2}v_i/\sqrt{\alpha_{ii}}$ form an orthonormal basis, we have

$$
\begin{aligned}
(\star) &= \sum_{j=1}^{n} \left| \mathbb{E}_{\nu_t} \left\langle A_t^{1/2} y, \frac{v_i}{\sqrt{\alpha_{ii}}} \right\rangle \left\langle A_t^{1/2} y, \frac{v_j}{\sqrt{\alpha_{jj}}} \right\rangle A_t^{1/2} y \right|^2 \\
&= \sum_{j=1}^{n} \left| A_t^{1/2} \left(\mathbb{E}_{\nu_t} \langle y, \eta_i \rangle \langle y, \eta_j \rangle y \right) \right|^2 \\
&\leq \sum_{j=1}^{n} \|A_t^{1/2}\|_{\text{op}}^2 \, |\mathbb{E}_{\nu_t} \langle y, \eta_i \rangle \langle y, \eta_j \rangle y|^2 \\
&\leq \|A_t\|_{\text{op}} \sup_{\theta \in S^{n-1}} \sum_{j=1}^{n} |\mathbb{E}_{\nu_t} \langle y, \theta \rangle \langle y, \eta_j \rangle y|^2 \\
&= \|A_t\|_{\text{op}} \sup_{\theta \in S^{n-1}} \|\mathbb{E}_{\nu_t} y \otimes y \langle y, \theta \rangle\|_{HS}^2
\end{aligned}
$$

Thus

$$
\delta_t \leq C\, p^2 \|A_t\|_{\text{op}} \mathrm{Tr}(A_t^p) \sup_{\theta \in S^{n-1}} \|\mathbb{E}_{\nu_t} y \otimes y \langle y, \theta \rangle\|_{HS}^2
$$

Eldan proved in [E1, Lemma 1.6] that the expression before is bounded from above by

$$
\sup_{\theta \in S^{n-1}} \|\mathbb{E}_{\nu_t} y \otimes y \langle y, \theta \rangle\|_{HS}^2 \leq C \sigma_n^2 \log n
$$

which gives us the corresponding estimate.

□

Proposition 5.2. *There exist $n_0 \in \mathbb{N}$ and $C > 0$ such that for any $n \geq n_0$ we have*

$$
\mathbb{P}\{\|A_t\|_{op} \leq 4, \forall t \in [0,T]\} > 0.9, \qquad \textit{for} \qquad T = \frac{1}{4C(\log n)^2 \sigma_n^2}
$$

and

$$
\mathbb{P}\left\{ \|A_t\|_{op} \leq \frac{\sqrt{51}}{7}\sqrt{n}, \forall t \in [0,T] \right\} \geq 0.9, \qquad \textit{for} \qquad T = \frac{1}{256 C \sqrt{n}}
$$

Proof. Let $p \geq 2$, fixed. Consider the function $\Phi(t) = -(n + \mathrm{Tr}(A_t^p))^{-1/p}$. Then $\Phi(t)$ is an

Itô Process and

$$\begin{aligned} d(\Phi(t)) &= \frac{1}{p}\frac{d(\mathrm{Tr}(A_t^p))}{(n+\mathrm{Tr}(A_t^p))^{1+1/p}} - \frac{1}{2p}\left(1+\frac{1}{p}\right)\frac{d[\mathrm{Tr}(A_t^p)]_t}{(n+\mathrm{Tr}(A_t^p))^{2+1/p}} \\ &= \left(\frac{1}{p}\frac{\delta_t}{(n+\mathrm{Tr}(A_t^p))^{1+1/p}} - \frac{1+p}{2p^2}\frac{|v_t|^2}{(n+\mathrm{Tr}(A_t^p))^{2+1/p}}\right)dt \\ &+ \left\langle \frac{v_t}{p(n+\mathrm{Tr}(A_t^p))^{1+1/p}}, dW_t \right\rangle = \alpha_t dt + dZ_t, \end{aligned}$$

where α_t is an adapted process of bounded variation and Z_t a martingale term with $Z_0 = 0$.

By the preceding Proposition $\delta_t \leq L_p(\mathrm{Tr}(A_t^p))^{1+1/p}$ where L_p is a different expression depending on whether $p = 2$ or $p \geq 3$. Therefore,

$$\alpha_t \leq \frac{L_p(\mathrm{Tr}(A_t^p))^{1+1/p}}{p(n+\mathrm{Tr}(A_t^p))^{1+1/p}} \leq \frac{L_p}{p} = \begin{cases} C\,p\,\sigma_n^2 \log n, & \text{if } p \geq 3 \\ C, & \text{if } p = 2. \end{cases}.$$

The quadratic variation of Z_t is

$$d[Z]_t = \frac{|v_t|^2}{p^2(n+\mathrm{Tr}(A_t^p))^{2+2/p}}dt \leq C\frac{(\mathrm{Tr}(A_t^p))^{2+1/p}}{(n+\mathrm{Tr}(A_t^p))^{2+2/p}}dt \leq \frac{C}{n^{1/p}}dt.$$

Then

$$\Phi(t) - \Phi(0) = \int_0^t \alpha_s ds + Z_t \qquad \forall t \geq 0.$$

We fix $T > 0$, then

$$\max_{0\leq t\leq T} \Phi(t) + (2n)^{-1/p} \leq \frac{L_p}{p}T + \max_{0\leq t\leq T} Z_t.$$

By the Dambis and Dubins-Schwarz theorem (see Proposition 2.7) we know that Z_t is equal in law to a Brownian motion $\tilde{W}_{[Z]_t}$, so for any $\gamma > 0$ we have, by the reflection principle (see Proposition 2.6), that

$$\begin{aligned} &\mathbb{P}\left\{\max_{0\leq t\leq T} \Phi(t) + (2n)^{-1/p} - \frac{L_p}{p}T > \gamma\right\} \leq \mathbb{P}\left\{\max_{0\leq t\leq T} Z_t > \gamma\right\} \\ &= \mathbb{P}\left\{\max_{0\leq t\leq T} \tilde{W}_{[Z]_t} > \gamma\right\} \leq \mathbb{P}\left\{\max_{0\leq s\leq CTn^{-1/p}} \tilde{W}_s > \gamma\right\} \\ &= 2\mathbb{P}\left\{\tilde{W}_{CTn^{-1/p}} > \gamma\right\} \leq 2\exp\left(-\frac{\gamma^2}{2CTn^{-1/p}}\right) \end{aligned}$$

We take $\gamma = \dfrac{1}{4n^{1/p}}$ and $T = \dfrac{p}{256L_p n^{1/p}}$ and we achieve

$$\mathbb{P}\left\{\max_{0\leq t\leq T}\Phi(t) > n^{-1/p}\left(\frac{1}{4}+\frac{1}{256}-2^{-1/p}\right)\right\} \leq 2\exp\left(-\frac{8L_p}{pC}\right)$$

Since $p \geq 2$, we have $\dfrac{1}{4} < 2^{-1/p} - \dfrac{1}{4} - \dfrac{1}{256} < \dfrac{7}{10}$ and then

$$\mathbb{P}\left\{\max_{0\leq t\leq T}\Phi(t) > -\frac{7}{10}n^{-1/p}\right\} \leq 2\exp\left(-\frac{8L_p}{pC}\right).$$

We remark that

$$\max_{0\leq t\leq T}\Phi(t) > -\frac{7}{10}n^{-1/p} \iff \max_{[0,T]}\mathrm{Tr}(A_t^p) \geq \left(\left(\frac{10}{7}\right)^p - 1\right)n$$

Hence we obtain that

$$\mathbb{P}\left\{\max_{[0,T]}\mathrm{Tr}(A_t^p) \geq \left(\left(\frac{10}{7}\right)^p - 1\right)n\right\} \leq 2\exp\left(-\frac{8L_p}{pC}\right)$$

Eventually we will consider two values of p in order to get our result. On the one hand, if we choose $p = 2$ we have $T = \dfrac{1}{256C\sqrt{n}}$ and

$$\mathbb{P}\left\{\max_{0\leq t\leq T}\mathrm{Tr}(A_t^2) > \frac{51}{49}n\right\} \leq 2\exp\left(-8\right)$$

and

$$\mathbb{P}\left\{\max_{0\leq t\leq T}\|A_t\|_{op} > \frac{\sqrt{51}}{7}\sqrt{n}\right\} \leq 2\exp\left(-8\right)$$

On the other hand, if we choose $p = \log n$. Then $T = \dfrac{1}{256C\sigma_n^2(\log n)^2}$ and

$$\mathbb{P}\left\{\max_{0\leq t\leq T}\mathrm{Tr}(A_t^{\log n}) > \left(\left(\frac{10}{7}\right)^{\log n} - 1\right)n\right\} \leq 2\exp\left(-8\sigma_n^2(\log n)^2\right)$$

Note that, considering the Gaussian distribution, we obtain $\sigma_n^2 \geq 1/2$ and then

$$\mathbb{P}\left\{\max_{0\leq t\leq T} \mathrm{Tr}(A_t^{\log n}) > \left(\frac{10}{7}\right)^{\log n} n\right\} \leq 2\exp\left(-2(\log n)^2\right)$$

and

$$\mathbb{P}\left\{\max_{0\leq t\leq T} \|A_t\|_{\mathrm{op}} > \frac{10e}{7}\right\} \leq 2\exp\left(-2(\log n)^2\right)$$

□

6 Gluing the estimates

Proposition 6.1. *There exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, μ is an isotropic log-concave probability measure on $\mathbb{R}^n$ and E is a Borel set $\mu(E) = \frac{1}{2}$, given the system of stochastic differential equations* (8)*, μ_t be the measure defined by* (9)*, $g(t) = \mu_t(E)$, and*

$$T = \frac{1}{4C\,\sigma_n^2(\log n)^2} \qquad \textit{or} \qquad T = \frac{1}{256C\sqrt{n}},$$

then we have that

$$\mathbb{P}\left\{\left|g(T) - \frac{1}{2}\right| > \frac{1}{4}\right\} \leq 0.2.$$

Proof. We know that

$$g(T) - \frac{1}{2} = g(T) - g(0) = \int_0^T dg(t) = \int_0^T \langle \eta_t, dW_t\rangle$$

where $\eta_t = \int_E f_t(x)(x - b_t)dx$, being f_t the density of the probability measure μ_t.

The function $g(t)$ is a martingale and so, by Dambis, Dubins-Schwarz theorem, Proposition 2.7, we have that in distribution

$$g(T) - g(0) = \bar{W}_{[g]_T}, \qquad t \geq 0$$

where $\bar{W}_s$ is a Wiener process and $[g]_T$ is the quadratic variation of g, which is,

$$[g]_T = \int_0^T |\eta_t|^2 dt.$$

Hence, for any $M > 0$,

$$\mathbb{P}\{|g(T) - 1/2| > 1/4\} = \mathbb{P}\{|\bar{W}_{[g]_T}| > 1/4\} \le \mathbb{P}\{[g]_T > M\} + \mathbb{P}\left\{\max_{0\le t\le M} |\bar{W}_t| > \frac{1}{4}\right\}.$$

We will bound both summands from above. Taking into account that for every $t \ge 0$

$$\begin{aligned} |\eta_t| = \left\langle \eta_t, \frac{\eta_t}{|\eta_t|} \right\rangle &= \int_E f_t(x) \left\langle (x - b_t), \frac{\eta_t}{|\eta_t|} \right\rangle dx \\ &\le \sqrt{\mathbb{E}_{\mu_t} \left\langle (x - b_t), \frac{\eta_t}{|\eta_t|} \right\rangle^2} = \sqrt{\left\langle A_t \frac{\eta_t}{|\eta_t|}, \frac{\eta_t}{|\eta_t|} \right\rangle} \le \sqrt{\|A_t\|_{\mathrm{op}}} \end{aligned}$$

we have that

$$[g]_T \le \int_0^T \|A_t\|_{\mathrm{op}}\, dt \le T \max_{0\le t\le T} \|A_t\|_{\mathrm{op}}.$$

and then

$$\mathbb{P}\{[g]_T > M\} \le \mathbb{P}\left\{\max_{0\le t\le T} \|A_t\|_{\mathrm{op}} > \frac{M}{T}\right\} \tag{12}$$

On the other hand, $\left(-\bar{W}_t\right)_{t\ge 0}$ is also a Brownian motion and then we have

$$\begin{aligned} \mathbb{P}\left\{\max_{0\le t\le M} |\bar{W}_t| > \frac{1}{4}\right\} &\le \mathbb{P}\left\{\max_{0\le t\le M} \bar{W}_t > \frac{1}{4}\right\} + \mathbb{P}\left\{\max_{0\le t\le M} -\bar{W}_t > \frac{1}{4}\right\} \\ &= 4\mathbb{P}\left\{\bar{W}_M > \frac{1}{4}\right\} \le 4\exp\left(-\frac{1}{32M}\right). \end{aligned} \tag{13}$$

We consider now two cases:

In the case $T = \dfrac{1}{4C\sigma_n^2(\log n)^2}$ we choose $M = 4T$ and then, by (12) and Proposition 5.2

$$\mathbb{P}\{[g]_T > M\} \le \mathbb{P}\left\{\max_{0\le t\le T} \|A_t\|_{\mathrm{op}} > \frac{1}{4}\right\} \le 0.1.$$

and by (13)

$$\mathbb{P}\left\{\max_{0\le t\le M} |\bar{W}_t| > \frac{1}{4}\right\} \le 4\exp\left(-\frac{C^2\sigma_n^2(\log n)^2}{32}\right) \le 0.1$$

for n large enough.

In the case $T = \dfrac{1}{256C\sqrt{n}}$ we choose $M = \dfrac{1}{128C}$. Then by (12) and Proposition 5.2

$$\mathbb{P}\{[g]_T > M\} \le \mathbb{P}\left\{\max_{0\le t\le T} \|A_t\|_{\mathrm{op}} > 2\sqrt{n}\right\} \le 0.1$$

and by (13)

$$\mathbb{P}\left\{\max_{0\le t\le M} |\bar{W}_t| \le 4\right\} \le \exp\left(-4C\right) \le 0.1,$$

assuming that $C > 2$, which we can assume without loss of generality.

□

The latter result, together with the discussion in Section 4, give the proof of Theorem 1.3.

Acknowledgments. This work is partially supported by MINECO/MICINN projects MTM2016-77710-P, PID2019-105979GB-I00 and DGA project E48_20R.

References

[AB1] D. Alonso-Gutiérrez, J. Bastero *Approaching the Kannan-Lovász-Simonovits and variance conjectures.* Lecture Notes in Mathematics, Springer. Monograph. **2131**, (2015).

[AB2] D. Alonso-Gutiérrez, J. Bastero *Sobre la conjetura de salto espectral de Kannan, Lovász y Simonovits.* Gaceta de la Real Sociedad Matemática Española **23** no. 1 (2020), 49–64.

[BK] S. Bobkov, A. Koldobsky, *On the central limit property of convex bodies*, Geometric aspects of functional analysis. Lecture Notes in Math. **1807**, 44–52. (Springer, Berlin, 2003)

[B] C. Borell, *Convex measures on locally convex spaces*, Ark. Math. **12** (1974), 239–252.

[BGVV] S. Brazitikos, A. Giannopoulos, P. Valettas, B. H. Vritsiou, *Geometry of Isotropic Convex Bodies*, Mathematical Surveys and Monographs **196** (American Mathematical Society, Providence, RI., 2014)

[E1] R. Eldan, *Thin shell implies spectral gap up to polylog via a stochastic localization scheme.* Geom. Funct. Anal. (GAFA) **23** (2013), 532–569

[E2] R. Eldan, *lecture Notes - from stochastic calculus to geometric inequalities, http://www.wisdom.weizmann.ac.il/ ronene/GFANotes.pdf.*

[GM] O. Guedon, E. Milman, *Interpolating thin-shell and sharp large-deviation estimates for isotropic log-concave measures*, Geom. Funct, Anal. **21**, (2011), 1043–1068

[K] B. Klartag, *Power-law estimates for the central limit theorem for convex sets.* J. Funct. Anal., Vol.**245**, (2007), 284–310.

[KLS] R. Kannan, L. Lovász, M. Simonovits *Isoperimetric problems for con- vex bodies and a localization lemma.* Discrete Comput. Geom. 13, no. 3-4, (1995), 541–559.

[Kle] Fima C. Klebaner, *Introduction to Stochastic calculus with Applications. Second Edition.* Imperial College Press, 2005

[LV1] Y. Lee, S. Vempala, Eldan's Stochastic Localization and the KLS hyperplane conjecture: an improved lower bound for expansion, 58^{th} Annual IEEE Symposim on Foundations of Computer Science-FOCS 2017. 998–1007, IEEE Computer Soc., 2017.

[LV2] Y. Lee, S. Vempala, The Kannan-Lovász-Simonovits Conjecture, Survey for Current Developments in Mathematics (CDM 2017)

[EM] E. Milman, *On the role of convexity in isoperimetry, spectral gap and concentration.* Invent. Math. 177 (2009), no. 1, 1–43.

[EM2] E. Milman, *Isoperimetric bounds on convex manifolds*, Proceedings of the Workshop on Concentration, Functional Inequalities and Isoperimetry, Contemporary Math. **545**, 195–208, (2011)

[O] B. Oksendal. *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media, 2013.