Rev. Real Academia de Ciencias. Zaragoza. **74:** 41–74, (2019).
ISSN: 0370-3207

Light-matter interaction in the nonperturbative regime: a lecture

David Zueco

http://complex.unizar.es/~zueco/
Instituto de Ciencia de Materiales de Aragón
and Departamento de Física de la Materia Condensada
CSIC-Universidad de Zaragoza, Pedro Cerbuna 12, 50009 Zaragoza, Spain
Fundación ARAID, Campus Río Ebro, 50018 Zaragoza, Spain

Abstract

Quantum electrodynamics (QED) is the understanding of how light and matter interact. In this lecture, I give an overview of the non-relativistic limit of the theory, which is the foundation of quantum optics. I introduce cavity and waveguide QED and explain its role in the construction of a quantum computer. Finally, I review the many-body physics emerging when light and matter interact beyond the perturbative regime. The lecture is focused on the theoretical tools typically used.

1 Introduction

Because of its extreme complexity, most physicists will be glad to see the end of QED.
P. A. M. Dirac, 1937.

Light and matter interact. Attracted by the Earth's magnetic field, charged particles pass through the atmosphere, ionising it and triggering the emission of visible light. This form the auroras and it is beautiful consequence of that interaction. Other examples are lasers, the scattering of an electron and positron in two muons, the functioning of a quantum computer or, in a healthy eye, the photodetection that allows to read this manuscript. These dissimilar phenomena are understood within the theory of quantum electrodynamics and is *perhaps the best fundamental physical theory that we have* [1].

[1]Here, I steal the words of Peskin and Schroeder in their popular book *An Introduction to Quantum Field Theory.*

In this manuscript, I will review the theory and its nonrelativistic limit, which is the basis of quantum optics. Then, I will discuss selected topics on quantum optics and the many-body phenomena occurring because of this interaction. I will spend some time to explain different theoretical tools used to do the calculations. To fit the lecture in a reasonable extension, I will focus on the strongly correlated phenomena occurring when the interaction enters into the nonperturbative regime.

1.1 The interaction (in a tiny nutshell)

Here, I will follow the excellent explanations of Nolting & Ramakanth [50, Sects. 2.2 and 2.3] and Snoke [74, Sect. 10.9] and the standard treatises of Peskin & Schroeder [54] and Tong [71].

In the modern viewpoint, *gauge invariance* is promoted to a general principle [28]. A gauge transformation is:

$$A'_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu \lambda(x)\ , \tag{1}$$

for the potential vector, $A_\mu = (\phi, \mathbf{A})$. Here, $\partial_\mu = (\frac{1}{c}\ \partial_t, \nabla)$ and $x_\mu = (ct, -\mathbf{x})$, e is the charge of an electron and c is the speed of light in vacuum. Besides, the wave function transforms as,

$$\psi'(x) = \psi(x) \mathrm{exp}[ie\lambda/\hbar c] \tag{2}$$

It turns out that the transformations (1) and (2) fix the interaction between light and matter. In particular, the *Dirac equation* for an electron, of mass m_e interacting with the electromagnetic field is

$$(i\hbar \not{D} - m_e c)\psi(x) = 0 \tag{3}$$

Here, we use the field theoretical notation $\not{D} \equiv \partial_\mu + ieA_\mu(x)$, which makes explicit the gauge invariance of the Dirac equation. Notice that $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ is a four-component spinor. The Dirac equation is really famous. It was invented to provide a relativistic version of the *Schrödinger equation*. It comes with the appearance of negative energy solutions which form the Dirac sea. In the non-relativistic limit, $v \ll c$ with v the the particle velocity, two components of the solutions for (3) are small enough so the the theory is effectively a two-component one. In this limit, the theory reduces to the *Pauli Hamiltonian*:

$$H = \frac{1}{2m_e}(\mathbf{p} + e\mathbf{A})^2 - e\phi + \frac{\mu_{\mathrm{B}}}{\hbar}\sigma \cdot \mathbf{B}\ . \tag{4}$$

Here, we have introduced the Bohr magneton $\mu_B = e\hbar/2m_e$, $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices [2] and $\mathbf{B} = \nabla \times \mathbf{A}$ is the magnetic field. It is difficult to exaggerate the importance of the Hamiltonian (4). It is the starting point of every quantum optics book. It is *the* light-matter Hamiltonian. Due to its relevance, it deserves to appreciate the way it was found. Notice that only gauge and Lorentz invariance (Dirac equation) and the nonrelativistic limit was used. Grounded in such a general laws, in its validity we trust.

1.2 Light Quantization

The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction.

Sidney Coleman.

Gauge invariance fixes the interaction form [cf. Eqs. (3) and (4)] and introduces a redundancy. The piece of reality that emerges from gauge invariance is that the phase space is enlarged, foliated by gauge orbits. Every point in the orbit must be reached by a gauge transform. Then, in our calculation, we pick one point from each orbit. Obviously, the output of the calculation must be independent of this choice. Picking a point in each orbit means to fix the gauge. Important examples are the Lorentz, Coulomb or dipole gauges. In particular, here we are interested in quantizing the EM field in (4). In this way we treat on equal footing both matter and light. In the quantization, it is convenient to choose the Coulomb gauge,

$$\nabla \cdot \mathbf{A} = 0\ .$$

[2] The Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In doing so, we have only the transversal component of the field.

The details of quantization would occupy a regular sized book, so I will just sketch the idea and I will argue how the result is reasonable. In other words, I am going to summarise the Cohen-Tannoudji, Dupont-Roc and Gryberg book [12] in a single paragraph. In these books the quantization is rigorously done by identifying the action in terms of the fields (at the classical level). Then, the canonical variables are found. Finally, they tackle the quantization program. Here, however, I will take a non-rigorous shortcut and quantize arguing why the electromagnetic field is a collection of harmonic oscillators (photons) and that $\mathbf{A}$ (and $\mathbf{B}$ and $\mathbf{E}$) are linear combinations of creation-anhilation operators. In the Coulomb gauge, the equation for $\mathbf{A}$ is the wave equation $\partial_\mu \partial^\mu \mathbf{A} = 0$, which is solved via separation of variables with the *ansatz*, $\mathbf{A} = \sum_l q_l(t) \mathbf{u_l}(\mathbf{r})$ with $\mathbf{u}_l(\mathbf{r})$ orthogonal functions (in free space they are Fourier series). Using the Maxwell equations, $\mathbf{E} = -\partial_t \mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$ we express the electromagnetic (EM) fields in terms of dimensionless time-functions q_k, $\dot{q}_k$. Introducing these expressions in the EM-energy, $U_{\mathrm{EM}} = \int dV \epsilon_0 \mathbf{E}^2(\mathbf{r})/2 + \mu_0 \mathbf{B}^2(\mathbf{r})/2$, we arrive to [64]

$$U_{\mathrm{EM}} = \hbar \sum_k \frac{1}{2} \dot{q}_k^2 + \frac{1}{2} \omega_k^2 q_k^2 \,, \tag{5}$$

where $\omega_k = c|\mathbf{k}_k|$ is the solution of the Helmhotz equation $-\nabla^2 \mathbf{u_k} = \mathbf{k}_k^2 \mathbf{u}_k$. Identifying $\dot{q}_k \sim p_k$ (p_k is the momentum) the above is nothing but the Hamiltonian of uncoupled harmonic oscillators. Introducing the creation operator $a_k^\dagger = \frac{1}{\sqrt{2\hbar\omega_k}}(\omega_k q_k + i p_k)$ and the Hermitean conjugate anhilation operators, a_k, we arrive to the quantum Hamiltonian for the EM:

$$H_{\mathrm{EM}} = \sum_l \omega_k a_k^\dagger a_k \,, \tag{6}$$

and the corresponding expressions for the fields **A**, **B** and **E**:

$$\mathbf{A} = \sum_l \sqrt{\frac{\hbar}{2\epsilon_0 V_k \omega_k}} \mathbf{u}_k \, (a_k + a_k^\dagger) \; , \tag{7}$$

$$\mathbf{B} = \sum_k \sqrt{\frac{\hbar}{2\epsilon_0 V_k \omega_k}} \nabla \times \mathbf{u}_k \, (a_k + a_k^\dagger) \; , \tag{8}$$

$$\mathbf{E} = \sum_k \sqrt{\frac{\hbar \omega_k}{2\epsilon_0 V_k \omega_k}} \mathbf{u}_k \, i(a_k - a_k^\dagger) \; . \tag{9}$$

A final word of caution. The quantum operators have been derived in the *Coulomb gauge*. Therefore, the quantum version of (4) with (7) must be named with the surname *in the Coulomb gauge*. Obviously, we can always change of gauge by the corresponding unitary transformation. We will come to this point later. For a general discussion see [12, Chap. IV].

1.3 What is Quantum Optics?

The quantum vacuum is not empty
Anonymous.

Quantum optics deal with the simplest objects: few level systems and photons. In particular, quantum optics *is* the entanglement of light and matter. The most basic phenomenon, the emission of light (with is certainly important for life and electrical companies) is a manifestation of entanglement and, in addition, an elementary example of particle creation from a field theoretical point of view. In simple words, light emission occurs when an atom changes from one state to another and, then, emits a photon. Let me describe the experimental fact and we will see how the light-matter entanglement builds up in the emission.

Consider a spinless neutral atom, e.g. strontium or helium. Therefore, the Zeeman coupling, the last term in (4), does not play any role. For further convenience, we write the rest of the Hamiltonian in the dipole gauge. This is done with the Power–Zienau–Woolley unitarty transformation yielding [12]

$$H^{(D)} = H_{\rm EM} + H_{\rm atom} + i\mathbf{d} \sum_l \sqrt{\frac{\hbar \omega_k}{2\epsilon_0}} u_k(\mathbf{r}_0) a_k^\dagger + \text{h.c.} \; . \tag{10}$$

Here $\mathbf{d}$ is the atom dipole. Besides, to make our life even simpler I consider that only two states of the atom are relevant in the dynamics. Thus, the atom in projected in a two level system (2LS) with energy difference Δ and a space spanned in the basis $\{|0\rangle, |1\rangle\}$ [3]. Typically, the atomic states have a well defined parity. Since the dipole operator $\mathbf{d}$ is an odd operator, its matrix elements are thus $\langle 0|\mathbf{d}|0\rangle = \langle 1|\mathbf{d}|1\rangle = 0$ and $\langle 0|\mathbf{d}|1\rangle = \langle 1|\mathbf{d}|0\rangle \equiv d$. So (10) can be rewritten (after the two level projection) as (from now on I will set $\hbar = 1$)

$$\mathcal{H}^{(D)} = \frac{\Delta}{2}\sigma^z + \sum_l \omega_k a_k^\dagger a_k + \sigma^x \sum_l c_k (a_k^\dagger + a_k) \; . \tag{11}$$

with $c_k = d\sqrt{\frac{\hbar\omega_k}{2\epsilon_0}} u_k(\mathbf{r}_0)$, cf. Eq. (10). This is an important and familiar model for the coupling of a two level system (or qubit) with the electromagnetic field. It is also well known beyond the quantum optics community. It models impurity models in condensed matter and it is the paradigmatic model in open systems [41, 72].

Consider now that, driven via an external field, the atom is excited from the ground state,

$$\begin{tikzcd}
\varrho_T(0) = \varrho \otimes \varrho_{\mathrm{EM}} \arrow[r] \arrow[d, dashed, "\mathrm{Tr}_{\mathrm{EM}}"] & \varrho_T(t) = U_T(t)\varrho_T(0)U_T^\dagger(t) \arrow[d, dashed, "\mathrm{Tr}_{\mathrm{EM}}"'] \\
\varrho \arrow[r] & \varrho(t) = \mathcal{E}\rho(0)
\end{tikzcd}$$

Figure 1: **Light-matter entanglement**. Given the state at $t = 0$ the system can be evolved with the total unitary operator $U_T = e^{-i\mathcal{H}^{(D)}t}$. If the observables act on the 2LS only trace over the EM-field is taken. Alternatively, we can take the trace at the beginning and find the non-unitary map $\mathcal{E}$.

$|0\rangle$, to the excited state $|1\rangle$ and that this driving is sufficiently fast to avoid correlations

[3] Through the rest of the manuscript I will restrict the discussion that the atoms can be model with two level systems (2LS). In the modern terminology they are named as qubits.

between the atom and field. Then, the density matrix at time t_0 can be written as $\varrho_{\rm T} = |1\rangle\langle 1| \otimes \varrho_{\rm EM}(t_0)$. $\varrho_{\rm EM}(t_0)$ is the equilibrium density matrix for the electromagnetic field at a given temperature T. I will consider the case $T = 0$ [4]. This initial density matrix may be evolved with the total Hamiltonian (11). Then, the time evolution of $P_e = \langle \sigma^+ \sigma^- \rangle$ can be computed. This way of computing is represented in the upper path in figure 1. Because P_e is a 2LS observable, an alternative is to find the map for the reduced density matrix $\varrho(t) = \mathcal{E}\varrho(0)$ in the space spanned by the two atomic relevant states $\{|0\rangle, |1\rangle\}$. This is the lower path indicated in the figure 1. During the time evolution some light-matter entanglement is built up, $\varrho(t)$ becomes a mixed state and the map cannot be unitary [5]. Let me sketch a simpler way of finding $\mathcal{E}$. I restrict myself to linear operators [24]. Adding an imaginary term in the energy, $P_e = e^{-i(\Delta - i\Gamma)t}$, is a phenomenological way of introducing dissipation. However, the corresponding equation $\dot\varrho = -i[\Delta/2\sigma^z, \varrho] - \{\Gamma/2\sigma^z, \varrho\} = -i[\Delta\sigma^+\sigma^-, \varrho] - \{\Gamma\sigma^+\sigma^-, \varrho\}$ does not conserves the trace, $\frac{d}{dt}{\rm Tr}(\varrho) = -\Gamma\varrho_{11}$. This is unacceptable. Then, we realize that, appart from dissipation, fluctuations should be introduced (fluctuation-dissipation theorem). These are jumps that transform $|1\rangle \to |0\rangle$ with some rate Γ'. They are conveniently written as the transformation for ϱ: $\Gamma'\sigma^-\varrho\sigma^+$. It turns out that imposing ${\rm Tr}(\dot\varrho) = 0$ yields $\Gamma' = \frac{1}{2}\Gamma$. Therefore, the differential form for the map $\mathcal{E}$ is

$$\dot\varrho = -i[\Delta\sigma^+\sigma^-, \varrho] + \frac{\Gamma}{2}\Big(\sigma^-\varrho\sigma^+ - \{\sigma^+\sigma^-, \varrho\}\Big) \tag{12}$$

This is a Lindblad-type master equation [6]. I argued that it provides a *bona fide* evolution, it was built ensuring all the density matrix properties. The only free parameter is the value for the spontaneus rate Γ. It is computed using the spin-boson model (11). Using

[4] Finite temperature can be done in a similar way.

[5] The exponential decay means that $\varrho_{11} = \exp(-\Gamma t)$ and $\varrho_{00} = 1 - \exp(-\Gamma t)$. Then, the purity ($\equiv {\rm Tr}(\varrho^2)$) equals to $1 + 2\exp(-2\Gamma t) - 2\exp(-\Gamma t) < 1$.

[6] The equation was found independently by Lindblad and Kossakowski, Gorini and Sudarshan in 1976. However, following the well know rule that regardless of what most physicists do there is always a Russian paper which "did it first". In 1969, Belavin, Zel'dovich, Perelomov and Popov found the Lindblad equation before Lindblad. Though unfair, I will use the standard notation and I will refer to (12) as the Lindblad equation.

perturbation theory the Fermi's golden rule is obtained:

$$\frac{\Gamma}{\Delta} = \frac{\Delta^2 d^2}{3\pi\epsilon_0 \hbar c^3} = \frac{4\alpha}{3}\frac{\Delta^2 r^2}{c^2} \, . \tag{13}$$

In the second equality I have introduced the fine-structure constant $\alpha = e^2/4\pi\epsilon_0\hbar c \cong 1/137$. The latter is a dimensionless combination of fundamental constants including the electron charge. It accounts for the coupling strength in electrodynamics. Besides, I have replaced the dipole $d = er$, with r being the inter-atomic distance. We can make $r \sim 10^{-10} m$ while the optical wavelength is of the order 10^7cm^{-1}, so $\Gamma/\Delta \sim 10^{-8}$. This is a rather small number saying that atoms and photons in free space are weakly coupled.

I have just sketched the most fundamental phenomenon in quantum optics. Let me enumerate another important examples that can be computed from (11) or more generally from (10) . In the interaction Hamiltonian the leading terms are $\sigma^- a_k + \text{h.c.}$, *i.e.* a one photon transition, thus 2LS are single photon emitters. Another, paradigmatic example is the amplification of the emission when a collection of atoms emit coherently. Consider N atoms place at the same point [7]. In this case, the coupling is through the total spin operator, $\sum \sigma^x = \sqrt{N} J_x$. Thus, in the second part of Eq. (12) $\sigma^\pm$ is replaced by $\sqrt{N} J^\pm$ and $\Gamma \to N\Gamma$, *i.e.* the emission is $N-$*enhanced*, this is nothing but *superradiance*. Finally, there is the Mollow triplet that occurs when an atom is driven with a sufficiently large intensity and the atom is dressed by the EM-field. For all of them, the ultimate responsibility is the light-matter entanglement.

2 Cavity and waveguide QED

In the early 1980's, reaching this situation, now called the strong coupling regime of cavity QED, became our Holy Grail.

Serge Haroche, 2012 (Nobel Prize lecture)

[7] This is a good approximation when the atom separations are smaller than the emitted light-wavelength

In free space, when a photon is emitted it does so in a irreversible way. Eq. (12) generates an irreversible time evolution [8]. Therefore it is not possible to generate interactions in a *coherent* way, *i.e.* exchanging light and matter in a periodic fashion. The solution to both problems is to trap photons in a cavity. Naively speaking, the photon would travel back and forward inside the cavity enlarging the effective atomic cross section. Besides, if the atom is excited and emits a photon, this cannot escape and eventually is re-absorbed obtaining the desired periodic (or coherent) coupling. Te setup of atoms interacting with cavity photons is known as cavity QED. The leaders of the first two groups that measured this coherent interaction,Haroche and Wineland, were awarded with the Nobel prize in 2012. In the following few lines I will explain the model for cavity QED and its main consequences.

A cavity is a box where the walls are made of mirrors. The finite volume of the cavity fixes the extension of the photon selecting its allowed wavelengths. You can think in a one cylin drical cavity with radial symmetry, see figure 2a). Then, the stationary waves have the frequencies $\omega_n = c\pi n/L$. The lowest frequency mode has $\omega_c = c\pi/L$ (it is named as the fundamental mode). It fits half of the wavelength, thus it is also known as the $\lambda/2$-mode. If an atom is placed inside the cavity the light-matter coupling is given by (11) where the EM-frequencies are restricted to ω_n. The difference between two consecutive normal modes is $\omega_{n+1} - \omega_n = \omega_0$. Since the light-matter coupling is sufficiently weak, the 2LS is mainly coupled to the mode which is closest to the atom level spacing Δ. Therefore, we can single out one mode from (11) *e.g.* ω_0, yielding the third Hamiltonian of this lecture, also quite important,

$$\mathcal{H}_{\rm qR} = \frac{\Delta}{2}\sigma^z + \omega_c a^\dagger a + g\sigma^x(a + a^\dagger) \tag{14}$$

This equation is known as the quantum Rabi model. Here, the key parameter is the single photon-atom coupling $g/(\omega_c + \Delta)$. It determines how fast light and matter exchange excitations. Its importance in the physics of cavity QED deserves to estimate its value. In the single-mode approximation, the electric field inside the cavity can be written as $\mathbf{E}(\mathbf{r}) = \mathcal{E}(\mathbf{r})(a^\dagger + a)$, see Eq. (9). Therefore, $g = \mathbf{d}\cdot\mathcal{E}(\mathbf{r})$. Besides, $\langle \mathbf{0}|\mathbf{E}(\mathbf{r})\cdot\mathbf{E}(\mathbf{r})|\mathbf{0}\rangle = |\mathcal{E}(\mathbf{r})|^2$,

[8] Recall that it was built by adding two terms, one of them is an anti-hermitean operator, see the discussion above Eq. (12).

and the vacuum EM energy is $\langle \mathbf{0}| \int dV \epsilon_0 \mathbf{E}^2(\mathbf{r})/2 + \mu_0 \mathbf{B}^2(\mathbf{r})/2 |\mathbf{0}\rangle = \frac{1}{2}\hbar\omega_c$. To give numbers, like in the previous section, I assume a cylindrical cavity with radius R. The volume of integration is $V = \pi R^2 \lambda/2$, with λ the wavelength of the fundamental mode. Putting all together an upper bound for the coupling is found,

$$\frac{g}{\omega_c} \leq \frac{r}{R}\frac{2\alpha}{\pi} \; . \tag{15}$$

As in the spontaneous emission calculation (13), I took the dipole: $d = er$. Again, the interaction strength is bounded by the fine structure and the ratio between two magnitudes, the interatomic one r and the radial dimension of the cavity R. This bound is reasonable. The photon occupies the whole cavity so the interaction must depends on this ratio between lengths. Up to 2004, experiments confirmed that the coupling was rather small 2c). In this case, the Hamiltonian can be approximated by the so called Jaynes-Cummings model, obtained in perturbation theory up to terms $(g/\omega_c)^2$,

$$\mathcal{H}_{\rm qR} \cong \mathcal{H}_{\rm JC} = \frac{\Delta}{2}\sigma^z + \omega_c a^\dagger a + g(\sigma^+ a + \sigma^- a^\dagger) \; . \tag{16}$$

In $\mathcal{H}_{\rm JC}$ there are not the counterrotating terms $\sigma^+ a^\dagger + \sigma^- a$, both of them create (annihilate) one photon and one atom excitation at the same time and they are irrelevant when $g/\omega_c \ll 1$. $\mathcal{H}_{\rm JC}$ is simpler to solve than $\mathcal{H}_{\rm qR}$ because $[\mathcal{H}_{\rm JC}, N] = 0$, with $N = a^\dagger a + \sigma^+\sigma^-$ being the number of excitations. The diagonalization is done in subspaces with fixed N. They are two dimensional: $\{|n0\rangle, |n-1, 1\rangle\}$, with $a^\dagger a|n\rangle = n|n\rangle$. Recall that $|0\rangle$ and $|1\rangle$ are the fundamental and first excited state of the 2LS respectively. Besides, the ground state is the trivial vacuum $|GS\rangle = |0; 0\rangle$. In particular, at resonance $\Delta = \omega_c$, the eigenvectors, conveniently labelled by the number of excitations are

$$|\psi_{n,\pm}\rangle = \frac{1}{\sqrt{2}}\Big(|n1;\rangle \pm |n-1, 0\rangle\Big) \; , \tag{17}$$

with energies

$$E_{n,\pm} = n\omega_c \pm \sqrt{n}g \; . \tag{18}$$

Notice that these states are light-matter entangled states. In the literature they are called *polaritons*. Knowing the spectrum, the dynamics can be easily obtained. Consider as initial condition that the atom is excited, $|\psi(0)\rangle = |0;1\rangle$ then (this state overlaps equally with $|\psi_{1,\pm}\rangle$)

$$P_e = \mathrm{Tr}_c(|1\rangle\langle 1| \, |\psi(t)\rangle\langle\psi(t)|) = \cos(2gt) \; . \tag{19}$$

These are the quantum Rabi oscillations. The frequency of the oscillations is given by $2g$.

Why was it so difficult to observe these oscillations? (the first experimental observation was achieved in 1992). The reason is that the cavity is not perfect and it has some leakage. In addition, the two level system can decay in other channels apart from the cavity photons. Both the leakage and the non-radiative channels can be modelled as a coupling to a continuum set of modes, see Appendix A and Eq. (12). The full dynamics, including these dissipative channels, is governed by a master equation of the form

$$\dot{\varrho} = -i[\mathcal{H}_{\mathrm{JC}}, \varrho] - \kappa(a\varrho a^\dagger - \tfrac{1}{2}\{a^\dagger a, \varrho\}) - \gamma(\sigma^- \varrho \sigma^- - \tfrac{1}{2}\{\sigma^+\sigma^-, \varrho\}) \; , \tag{20}$$

with κ and γ are decay rates. Solving the master equation, the Rabi oscillations decay on time,

$$P_e = e^{-(\kappa+\gamma)t} \cos\Big(2\sqrt{g^2 - (\gamma-\kappa)^2/4}t\Big) \; . \tag{21}$$

Therefore, for resolving the Rabi oscillations g must be greater the losses κ and γ. This is the *strong* coupling regime. Reaching this limit opens the possibility of doing quantum operations at the single photon limit and winning a Nobel prize.

In 2004, in Yale, instead atoms and cavities they used superconducting circuits, see figure 2d) [76] . The cavity used was a superconducting coplanar waveguide (CPW) and the atom was a charge qubit[9]. Circuits mimicking cavity QED are named circuit QED. The coupling measured was $g/\omega_c \sim 10^{-3}$, See 2c). The reason of such a sizeable coupling is that the CPW was esentially one-dimensional (reducing the cavity volume) and that the artificial atom was huge, few microns-size. Thus, lengths in (15) approach each other.

In 2010, in the Walther Meissner Institut, a new milestone occur. A coupling $g/\omega_c \cong 0.1$ was reached by increasing the cross talk between the superconducting circuit (in this case it

[9]Superconductors are used to minimise loses and to use Josephson junctions for having nonlinearities [15]. All of these circuits operate in the microwave regime.

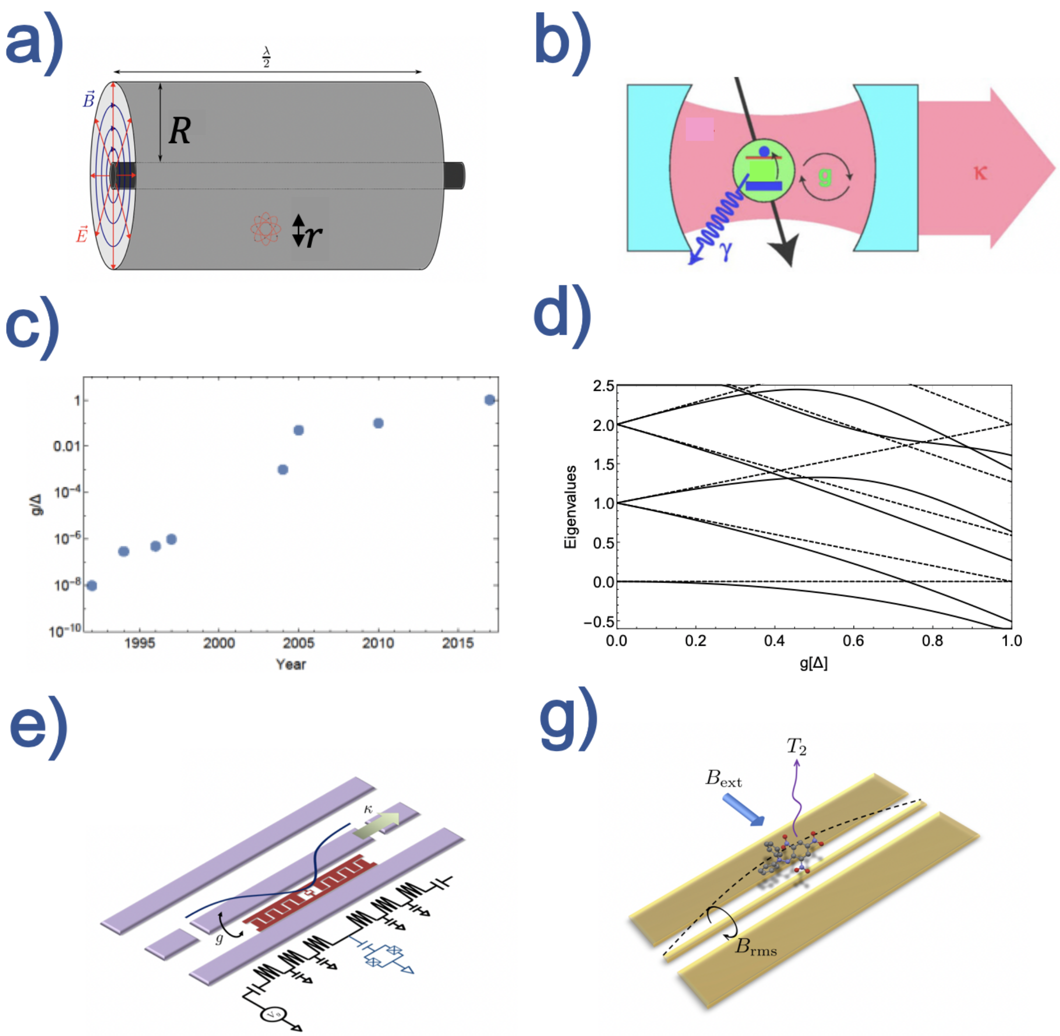

Figure 2: **Cavity QED** a) Cylindrical cavity. b) A generic cavity QED system with the main rates indicated. c) The time evolution for the maximum coupling achieved in the lab between a two level system and a single mode cavity (data taken from [21]. d) Spectrum for the quantum Rabi model (14) [solid lines] and the JC model (16) [shaded lines]. c) A circuit QED sketch (the qubit, in red, is not in scale). g) a magnetic molecule coupled to a superconducting cavity.

was of a flux-type) and the microwave cavity [49]. Apart from this number, the interesting feature is that the experimental results can not be understood within the JC-model $\mathcal{H}_{\rm JC}$ but require the full quantum Rabi model. When this occurs, *i.e.* that the full model is needed, the light-matter in said to be in the ultrastrong coupling regime (USC). In a similar experiment, in Delft, the USC was also reached using an LC-resonator [22] [10].

Model (14) is known to be integrable since a few years ago [7]. However its solution is not practical, but one can always diagonalize it in a computer. The eigenvalues for the quantum Rabi, compared to the ones of the JC, are shown in figure 2d). Several features need to be discussed. At very large coupling, the eigenstates of $H_{\rm qR}$ are two-fold degenerate. In the limit of large coupling, the full model can be approximated by $\mathcal{H}_{\rm qR} \cong \omega_c a^\dagger a + g\sigma^z(a^\dagger + a)$, *i.e.* a displaced harmonic oscillator. This displacement can be positive or negative depend on the eigenvalues of σ^z. The energy do not depend on the sign of the displacement, explaining the degeneracy. Besides, the ground state is not longer the trivial but $|gs\rangle = \sum_2 nc_n |2n, 0\rangle + c'_n |2n-1, 1\rangle$. Notice that this is a sum over states with an even number of excitations. The reason is that the model is parity-conserving, $[\mathcal{H}_{\rm qR}, P] = 0$, with $P = \sigma^z e^{i\pi a^\dagger a}$. In a more pedestrian way, this symmetry can be appreciated by noticing that the interaction term in (14) creates (destroys) excitations by pairs. Finally, a striking feature in the USC is the subtlety with the gauge principle [14, 16]. In short, the gauge principle must be satisfied (of course). However a problem arises when using the dipole and the Coulomb gauges within the two level approximation as they yield non equivalent Hamiltonians. The solution to the puzzle, is to show that the two level approximation is accurate in the dipole gauge but not in the Coulomb one. Therefore, the two level projection should be done in the latter gauge and transformed to the Coulomb one. The transformation must be projected in the two level subspace. Finally, the correct gauge-invariant Hamiltonian in the Coulomb gauge was found [16]. Due to its importance, let me write the correct quantum Rabi model in the Coulomb gauge

$$\mathcal{H}^{(\rm C)}_{\rm qR} = \omega_c a^\dagger a + \frac{\Delta}{2}\left(\sigma^z \cos\left[\frac{2g}{\Delta}(a + a^\dagger)\right] + \sigma^y \sin\left[\frac{2g}{\Delta}(a^\dagger + a)\right]\right) \tag{22}$$

Expanding the cosine and sine in powers of g/Δ we recover (14), thus both Coulomb and dipole gauges coincide and the issues disappear. Therefore, the USC also serves for testing the gauge invariance in cavity QED.

[10]In this history, I have only discussed circuits because they are exact realisations of the qR model (a two level system coupled to a single mode cavity). Other light-matter systems as exciton polaritons or intersuband polaritons have also reached couplings of 0.1 ω_c and beyond. A full history of the USC regime can be read in two recent reviews [21, 34].

We can still insist in our desire to connect distant atoms through the EM field. Several people do. Propagating photons are the ideal carriers of information. To enhance the atom cross section, the electromagnetic field is confined in one dimensional waveguides [58]. The waveguide QED Hamiltonian in the dipole gauge is given by (11). It turns out that the the spin boson is classified with the spectral density,

$$J(\omega) = 2\pi \sum_k c_k^2 \delta(\omega - \omega_k) \; . \tag{23}$$

In particular, the spontaneous emission (13) in the line is $\Gamma_{\text{line}} = J(\Delta)$. The factor $\beta = \frac{\Gamma_{\text{line}}}{\Gamma_{\text{line}}+\gamma}$ measures the atom-waveguide coupling compared to another channels of atom dissipation with rate γ. Experiments in waveguide QED are done with superconducting circuits [4, 73, 43, 26], optical waveguides [19] among others [44, 11] with β-factors approaching one. Thus, waveguide QED has an clear application for emitting single photons. Another advantage of being one dimensional is that a single atom can act as a perfect mirror [4] for single photons. This paves the way to control the transport of photons or create atomic cavities. Besides, it is also quantifiable the induced interaction between distant atoms mediated the propagating photons in the waveguide [18, 81]. After tracing out the waveguide modes the the master equation [See App. A] is obtained,

$$\dot{\varrho} = -i[\Delta(\sigma_1^z + \sigma_2^z + J_{12}\sigma_i^+\sigma_j^-, \varrho] - \sum_{i,j=1,2} \gamma_{ij}(\sigma_i \varrho \sigma_j^\dagger - \tfrac{1}{2}\{\sigma_i^+\sigma_j^-, \varrho\}) \; . \tag{24}$$

Two spin-spin interactions are generated. A coherent tight-binding interaction between atoms with strength $J_{ij} = J(\Delta)\cos(\Delta/vd_{12})$ and a cross-dissipation rate given by $\gamma_{12} = J(\Delta)\sin(\Delta/vd_{12})$. Here, v is the light propagation velocity in the waveguide and $\gamma_{11} = \gamma_{22} = J(\Delta)$. Both interactions, the coherent and the dissipative can be qualitatively understood. The former, occurs for $\lambda/4$ distances. The first atom emits a photon. The wavepacket is maximum in the second atom. The latter, however, happens when the atoms are separated $\lambda/2$. In this case, both points are equivalent (except a phase) and the atoms emit collectively [cf. the discussion on enhanced transmission in Section 1.3]. This interaction could be used to generate entanglement or gates. However they are not optimal, since there is always a dissipative term (the last term in (24). In fact, one can guess that only 50% of efficiency can be reached at maximum. When one atom emits a photon half of it goes in the *bad* direction (the one which is opposite to the other atom). This can be fixed using mirrors [61] or chiral waveguides [45]. Another alternative is by looking for waveguides supporting bound states [69, 9, 63]. These are dressed light-matter states localized (non propagating) arround the qubit. A necessary condition for their existence is the photonic band to be finite. The bound state energies must lie outside the photonic band. See figure 3b). With the help of bound states several spin-like models can be obtained where all the interactions are dissipationless making them an attractive approach for, *e.g.* quantum simulations.

2.3 The quantum technology

Quantum information science has put quantum optics into the spotlight of modern physics

Ulf Leonhardt in *Essential Quantum Optics* [42].

While writing this manuscript, newspapers arround the world announced that the *quantum supremacy* has been reached. That's it, the demonstration that a computational task that can be done in a quantum computer cannot be made in a classical computer without spending thousands of years. John Martinis and his group in the Google lab were able to generate a random quantum state of 53-qubits and measuring (calculate) the bit distribution [3]. This is extremely hard for a classical computer. This problem may not seem the most interesting one but building a quantum computer of 53 qubits is a remarkable technological milestone. Importantly enough for this manuscript, the functioning of the computer is based on the light-matter-like interaction.

A quantum computer is a set of two level systems that can be coupled and decoupled performing logical gates. Besides, qubit preparation and readout is also needed. In the superconducting prototypes, the qubits are artificial atoms that may interact via cavity photons or directly because of the cross talk between them [5, 31]. The coupling-decoupling is done by tuning on/off resonance the qubits. The readout is done via their coupling to a cavity. In the case of the Google prototype, each qubit is coupled to a cavity mode, see [5]. If the two level system and the cavity mode are conveniently detuned they do not share real but virtual excitations. In this regime, called dispersive, the Hamiltonian (14) is

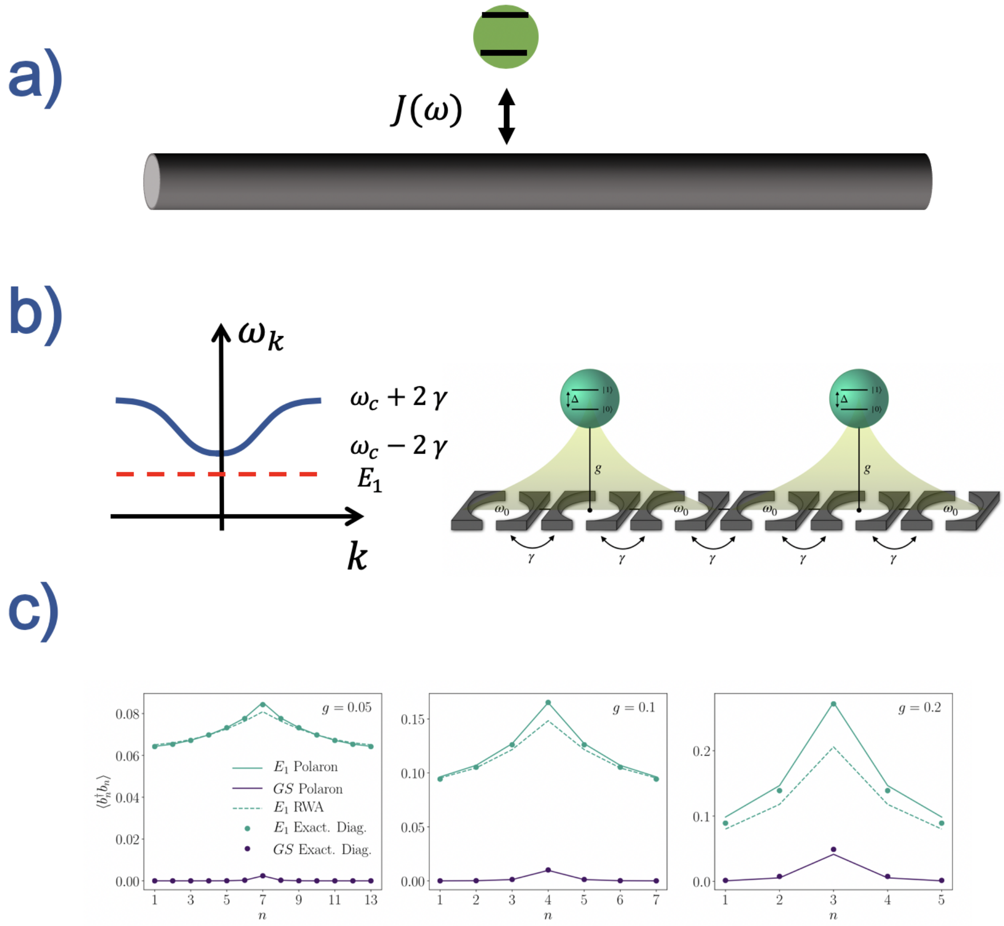

Figure 3: **Waveguide QED** a) Sketech of waveguide QED: an atom (here a two level system) coupled to a one dimensional waveguide. b) Qubits coupled to a cavity array. Each one is coupled to one cavity of the array with strenght g [cf. (14)]. The cavity array has the Hamiltonian $H = \omega_0 \sum_n^N a_n^\dagger a_n - \gamma \sum_n^N \left(a_n^\dagger a_{n-1} + \text{H.c.}\right)$ that gives the dispersion relation $\omega_k = \omega_c - 2\gamma\cos(k)$. Each two level system coupled to the cavity array contribute to one bound state E_1. If the qubits are closer enough E_1 lifts its degeneracy and the bound states are coupled. c) Comparison of spatial boson distributions from the PT, the RWA and exact diagonalisation (small chains, here are of 12 sites, can be diagonaized within a classical computer). They correspond to $\Delta = 0.3$, and $g = 0.05$, $g = 0.1$ and $g = 0.2$ respectively from left to right. Solid lines are used to indicate polaron results, dashed lines for RWA results and dots for exact diagonalisation results.

equivalent to [82]

$$\mathcal{H}_{\text{qR}} \cong \frac{\Delta}{2}\sigma^z + \omega_c a^\dagger a + g^2 \left(\frac{1}{\Delta - \omega_c} + \frac{1}{\Delta + \omega_c}\right)\sigma^z a^\dagger a \tag{25}$$

I emphasize that this Hamiltonian is valid provided $g \ll |\Delta - \omega_c|$ Notice that the frequency of the cavity is shifted by $\pm g^2\left(\frac{1}{\Delta-\omega_c}+\frac{1}{\Delta+\omega_c}\right)$ depending on the state of the qubit. This can be used to do the readout. At the same time, if a second qubit is coupled to the cavity mode and both are in the dispersive regime they couple through a term $\sim J_{ij}\sigma_i^\dagger\sigma_j + \text{h.c.}$ [Cf. Eq. (24)] [82]

$$J_{ij} = g_i g_j \left(\frac{1}{\Delta_i - \omega_c} + \frac{1}{\Delta_j - \omega_c} - \frac{1}{\Delta_i + \omega_c} - \frac{1}{\Delta_j + \omega_c}\right) \tag{26}$$

This has been used to generate interactions between the qubits via the cavity wich plays the role of a quantum bus [47]. Therefore, both the interaction and the readout, ingredients of a quantum computer are based on the light-matter Hamiltonian (14). In this manuscript, I have focused on the superconducting circuit architecture. Other quantum computers are based on the ion-trap technology. In that case, the functioning is based on the coupling between ions and light [78].

3 Many-body quantum optics in the non perturbative regime

I:*Mom, how do I know that the water is boiling?*
Mom: *You will.*

My mom, giving the best explanation of a phase transition the day I introduced myself in the fine art of cooking an egg.

There are many works doing quantum many-body physics with light-matter systems [27, 2, 40]. In this section, I focus on the strongly correlated phenomena occurring because the coupling between light and matter enters into the nonperturbative or USC regime.

3.1 Many body cavity QED

A two level system coupled to a single mode cavity is not a many-body system. A quantum many-body system can be build by coupling N 2LS to a single mode cavity. The Dicke model is the generalisation of (14) to N-qubits,

$$\mathcal{H}_{\text{Dicke}} = \frac{\Delta}{2}\sum_j^N \sigma_j^z + \omega_C a^\dagger a + g\sum_j^N \sigma_j^x (a + a^\dagger)\,. \tag{27}$$

For simplicity, let me consider equal atoms that are equally coupled to the cavity. It turns out that this model has a phase transition, called superradiant. Understanding this transition is easy. In the mean field approximation (27) $a + a^\dagger \to \alpha$, with α a real number. The ground state energy is given by $E = \omega\alpha^2 - \frac{N}{2}\sqrt{\Delta^2 + 4g^2\alpha^2}$. Energy minimisation yields that if $g < g_c = \sqrt{\omega_c \Delta}/2\sqrt{N}$ then $\alpha = 0$ while if $g > g_c$, $\alpha \neq 0$. α is the order parameter and this phase transition belongs to the mean-field Ising universality class [33]. This transition has not been observed yet. The reason is that the realization of (27) is not trivial. Model (27) does not account for N independent atoms in single mode cavity. In the dipole gauge it reads however,

$$\mathcal{H}^{(D)} = \mathcal{H}_{\text{Dicke}} + \frac{g^2}{\Delta} \sum \sigma_i^x \sigma_j^x \; . \tag{28}$$

This term ruins the phase transition. Notice, that pretty much like in the gauge issues discussed in section 2.1, this term is important whenever $g\sqrt{N}$ is large enough. This is the regime where the transition is expected to occur. Currently, there is an enormous interest for searching systems to observe this phase transition [13].

Many-body spin-spin interactions is mediated by the cavity field (remind the previous section) [1, 75, 51]. In the detuned cavity-atoms case the interaction is of the tipe $\sigma_i^x \sigma_j^x$ with the coupling constants given in (26). Recently, we have found a novel way of obtaining a hybrid spin-spin-boson model [46]. The starting point is again the Dicke model with a parity breaking (second term below) for the spins [Cf. Eq. (14)]

$$\mathcal{H} = \frac{\Delta}{2} \sum \sigma_j^z + \frac{\epsilon}{2} \sum \sigma_j^z + \omega_c \, a^\dagger a + g(\hat{a} + \hat{a}^\dagger) \sum \sigma_j^x \; . \tag{29}$$

When $\epsilon \neq 0$, $\mathcal{H}$ can couple states differing by an odd number of excitations. For example, an avoided level crossing, originating from the coupling of the states $\hat{a}^\dagger |0, j, -j\rangle \leftrightarrow \hat{J}_+^2 |0, j, -j\rangle$, is expected when the resonance frequency of the cavity $\omega_c \simeq 2\omega_q = 2\sqrt{\Delta^2 + \epsilon^2}$ [23]. We label the states as $|n, j, m\rangle$, where the quantum number n describes the Fock states of the cavity, and $j = N/2$ is the total angular momentum and $m = -j + N_{\text{exc}}$ is the $\sum \sigma_j^z$ eigenstate, where N_{exc} describes the number of excited atoms. Notice that this transition is allowed only if the counter-rotating terms are included. Using perturbation theory [66] the following effective interaction Hamiltonian can be obtained:

$$\hat{H}_{\rm eff} = g_{\rm eff} \left(\hat{a} \hat{J}_+^2 + \hat{a}^\dagger \hat{J}_-^2 \right) \,, \tag{30}$$

where

$$g_{\rm eff} = -\frac{4 g^3 \cos^2\theta \sin\theta}{3\omega_q^2} \,, \tag{31}$$

with $\sin\theta = \epsilon/\sqrt{\Delta^2+\epsilon^2}$. This procedure also gives rise to a renormalization of the atomic frequencies, which can be reabsorbed into ω_q. This yields a two-axis twisting like-interaction among the spins wich can be used to generate macroscopic spin entanglement [46].

3.2 Many body with spin-boson type models

In the previous section I was considering a single mode cavity. Now, let me discuss the coupling with a collection of modes. This is a more general case. It covers many situations described with the Hamiltonian (11), by choosing different functional forms for $J(\omega)$, Eq. (23). Three examples are drawn in figure 4. The first is a one dimensional waveguide, figure 3a). There, $J(\omega) \sim \omega$ [53]. A second example is the cavity QED considering the leakage of photons, figure 2. In this case, $J(\omega)$ is a Lorentzian-type function peaked in the cavity frequency ω_r. Finally, a cavity array is considered, figure 3b) where the density of states diverge at the band edges [62, 59].

In all its generality, the spin-boson Hamiltonian (11) does not have a known solution. See [72, Chap. 18.1] and [41] . However, if the coupling constant is small enough, the rotating wave approximation (RWA) can be used, by which the interaction term becomes [Cf. Eq. (16)]

$$\sum_k c_k \left(\sigma^- a_k^\dagger + \sigma^+ a_k \right) . \tag{32}$$

It is clear now that ground state (GS) is $|GS\rangle = |0; \mathbf{0}\rangle$. In addition, the Hamiltonian preserves the number of excitations N, $[\mathcal{H}, N] = 0$ with $N = \sum_k a_k^\dagger a_k + \sigma^+\sigma^-$. Owing to these properties, within the RWA, the g.s physics and the single excitation dynamics is trivial. Notice that this is the generalization of the JC-model (16) to the multimode case. In this lecture, however I am interested in the regime where the RWA fails. Different

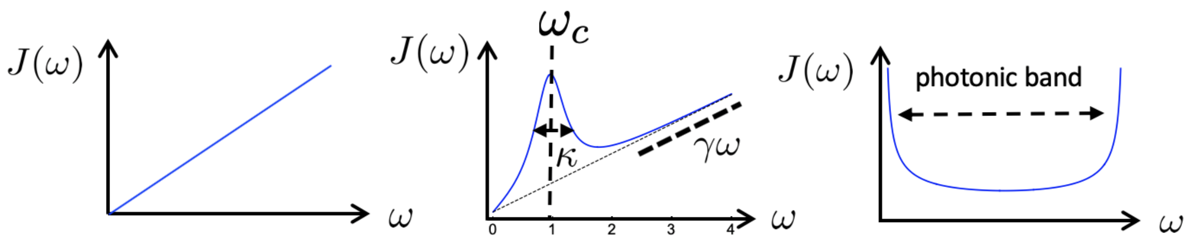

Figure 4: **Spectral density** $J(\omega)$ for different situations. They correspond to coupling to a one dimensional transmission line, Fig. 3a), to a dissipative cavity, Fig 2 and to a cavity array Fig. 3b) respectively from left to right.

numerical techniques have been used to solve (11) beyond the RWA. These are matrix-product state (MPS) [53, 62, 59] , density matrix renormalization group (DMRG) [56] or path integral approaches [25, 39]. Analytical treatments are also used. They are based on different varational anstazs: polaron-like [70, 6, 17, 67, 80, 60] or Gaussian ones [68]. In this overview I will sketch the polaron-like approach that allows to find, in an analytical way, the low energy equilibrium and non-equilibrium dynamics.

In order to understand the motivation behind the polaron *ansatz*, it is convenient to analyse the asymptotic limits where the Hamiltonian is exactly solvable. In the case where $c_k = 0$, the coupling vanishes and the problem splits into a 2LS and a bosonic bath which can both be solved independently. The GS is therefore $|GS\rangle = |0; \mathbf{0}\rangle$, which is clearly localised, in the sense that the spin ket is an eigenstate of σ^z. In the opposite case, $\Delta = 0$, the Hamiltonian becomes that of a set of displaced oscillators. One can choose a state of the form

$$|\psi\rangle = |\pm\rangle \otimes |osc.\rangle \,, \tag{33}$$

where $|\pm\rangle$ can be either eigenstate of σ^x and $|osc.\rangle$ is an unknown state for the oscillators.

Now it can be seen that

$$H|\psi\rangle = |\pm\rangle \otimes \left(\sum_k \omega_k a_k^\dagger a_k \pm \sum_k c_k \left(a_k^\dagger + a_k\right)\right)|osc.\rangle\,, \tag{34}$$

so the direction of the displacement is determined by the qubit state. By defining new bosonic operators: $A_k = A_k \pm c_k/\omega_k$ and $A_k^\dagger = A_k^\dagger \pm c_k/\omega_k$ with $[A_k, A_k^\dagger] = 1$, one arrives at

$$H|\psi\rangle = |\pm\rangle \otimes \left(\sum_k \omega_k A_k^\dagger A_k - \sum_k \frac{c_k^2}{\omega_k}\right)|osc.\rangle\,. \tag{35}$$

It is clear now that the GS for this Hamiltonian is $|\pm\rangle\,, \mathbf{0}$ and the energy is independent on the spin state and equal to $-\sum_k c_k^2/\omega_k$. In this instance the GS is delocalised, as the qubit state can be in either the symmetric or antysimmetric superpositions of the eigenstates of σ^z [11]. Here I have introduced the redefined bosonic operators rather *ad hoc*, but they can also be reached by means of the displacement operator, which is presented below in its most general form,

$$D(\alpha) = \exp\left[\alpha a^\dagger - \alpha^* a\right]. \tag{36}$$

The displacement operator is a unitary transformation, since $D(\alpha)^\dagger \equiv D(-\alpha)$, which justifies that the energy deduced from the transformed Hamiltonian is the same as for the original one. If the operator is tweaked to produce displacements corresponding to the redefined bosonic operators one has

$$D\left(\pm\frac{c_k}{\omega_k}\right) = \exp\left[\sigma^x \sum_k \frac{c_k}{\omega_k}\left(a_k^\dagger - a_k\right)\right]. \tag{37}$$

The presence of σ^x in the exponent serves to generate the plus or minus sign that determines the direction of displacement, upon acting on the corresponding eigenstate.

---

[11] For those who have a condensed-matter background, the spin-boson is paradigmatical in impurity models. In those formulations that naturally lead to a double-well interpretation of the 2LS, the roles of σ^x and σ^z are switched in the Hamiltonian. In that case, $c_k = 0$ is viewed as the delocalised regime whereas $\Delta = 0$ is viewed as the localised regime.

Therefore, one seeks a unitary transformation that renders the Hamiltonian diagonalisable, with a non-trivial ground state that merges the two asymptotic solutions. Unfortunately, the former condition is not completely satisfied by the polaron transform, but it does produce a quasi-solvable Hamiltonian whose GS can be calculated with the use of a simple *ansatz*. The transformation is

$$U_P = \exp\left[-\sigma^x \sum_k f_k a_k^\dagger - f_k^* a_k\right], \tag{38}$$

and the variational *ansatz* is

$$|GS\rangle = U_P \,|S; \mathbf{0}\rangle\,. \tag{39}$$

Where $|S\rangle$ is an unknown spin state and f_k are free parameters to be determined by the application of the variational method. Minimization yields the self-consistent relation

$$f_k = \frac{-c_k/2}{\Delta_r + \omega_k} \;\text{ with }\; \Delta_r = \Delta\, e^{-2\sum_k f_k^2}\,. \tag{40}$$

We have seen that in the polaron picture the GS is trivial, then it is expected that the low energy dynamics consists on single particle excitations over this GS. Exactly, the unitary transformed $H_p = U_p^\dagger H U_p$ reads

$$\begin{aligned} H_p = {} & \Delta_r \sigma^+\sigma^- + \sum_{k=1}^N \omega_k a_k^\dagger a_k - 2\Delta_r \left(\sigma^+ \sum_{k=1}^N f_k a_k + \text{H.c.}\right) \\ & - 2\Delta_r \sigma_z \sum_{k,p=1}^N f_k^* f_p a_k^\dagger a_p \\ & + \frac{\Delta}{2} + \sum_{k=1}^N (\omega_k |f_k|^2 - g_k^* f_k - f_k^* g_k) + \text{h.o.t.} \end{aligned} \tag{41}$$

Here, h.o.t. stands for higher-order terms of order $\mathcal{O}(f^3)$ with two and more excitations. Note how the transformed Hamiltonian conserves the number of excitations and can be treated analytically like in the RWA approximation. With this tools at hand, we can overview the main features for the low energy physics of the spin boson model, rather independently of $J(\omega)$, both at equilibrium and non-equilibrium.

Ground state properties.- Notice that due to the coupling to the photons the qubit frequency is renormalized to Δ_r, Cf. (41) and [41]. Besides, from (40), it renormalizes to zero. Rewriting Δ_r in terms of $J(\omega)$

$$\Delta_r = \Delta \mathrm{e}^{-1/2 \int_0^{\omega_c} J(\omega)/(\omega+\Delta_r)^2} \tag{42}$$

is obtained. Notice that Δ_r means localised (delocalised) 2LS (see footnote 11). Whether this occurs through a phase transition or not depends only on $J(\omega)$. The most studied cases are $J(\omega) \sim \omega^s$. Depending on the value of s, there exist three distinct cases known as sub-Ohmic ($s < 1$), Ohmic ($s = 1$), and super-Ohmic ($s > 1$) regimes. It has been shown that the transition is of second order in the sub-Ohmic regime and Kosterlitz-Thouless type in the Ohmic regime. In the super-Ohmic regime there is no phase transition [79]. Other $J(\omega)$ are currently been investigated together with their critical properties.

Vacuum emission.- Apart from this phase transition, another interesting phenomena occuring in the spin boson model is its vacuum emission as we recently calculated [60]. The idea is that the ground state (39) depends on f_k. The latter depend on the light-matter coupling strenght. Physically, the ground state photon occupation is different from zero around the qubit, see fig (3)c). Then, by modifying the light-matter interaction non adiabatically the ground state emits light, pretty much like in the Casimir effect [10, 38, 48, 37, 77].

Spontaneous emission.- Within the polaron picture H_p, the model resembles the weak coupling-RWA one. In this case, it is expected that standard perturbative techniques hold here too. In fact, in [80] we have shown that the spontaneous emission is given by

$$\Gamma_{\rm line} = J(\Delta_r) \tag{43}$$

recovering Fermi's golden rule at the weak-coupling limit and a quenching of the emission at large couplings, which recalls the effective decoupling in the USC regime [21, 34].

Spin-spin interactions in waveguide QED.- Generalising the polaron transformation to several qubits coupled to a one dimensional field, a direct spin-spin interaction of the form, $J\sigma_i^x\sigma_j^x$ emerges. Our calculations with the cavity array, figure 3b) confirm that J decays exponentially with the distance. Therefore, we expect the occurrence of transitions belonging to the Ising class type (recall our discussion on the superradiance transition) [35]. Finally, in the polaron picture (41), bound states can be also computed. It turns out that the existence of bound states is guaranteed at any coupling strength and that can be computed, see figure 3c). Therefore, non dissipative spin-spin interactions build up also in the USC regime of waveguide QED.

Therefore, the polaron transformation is a convenient tool for discussing the physics of waveguide QED in the USC. This regime is still in its infancy, here I just listed the first calculations but further and more exotic phenomena are currently been investigated [55, 20].

4 Concluding remarks

All right.
Ignatius Farray.

A lot of stuff was not explained here, maybe too much. For example, I did not say anything about magnetic molecules coupled to the cavity field through the last term in (4) [29], see figure 2f). Here, some issues on the gauge principle and/or the superradiance transition may be of relevance. Besides, they are also proposed as prototypes of a quantum computer [30]. I omitted most of the phenomena of quantum optics as the generation of squeezed light, measurement or timely topics as topology and photonics. I have written a naive explanation of a quantum computer. Famous reports on the topic and its relation with the light-matter coupling are Refs. [52, 36]. In addition, I have done an extremely partisan review of the many body physics done with light. As said, I focus on how many body effects emerge due to the entrance in the USC. However, light-matter systems are being used to generate strongly correlated models even without the USC as explained in Refs. [40, 11].

With respect to the topics covered, some of them are still incomplete and need further investigation. Some of them are the occurrence of a localized-delocalized phase transition for $J(\omega)$ besides $J(\omega) \sim \omega^s$, the extension of the polaron technique for finite temperature systems and the study of multimode cavities or going beyond the two level paradigm, for *e.g.* to build quantum simulators for higher spin systems are some examples. Another topic of current interest is to tune the physical and chemical properties of quantum materials inside quantum cavities [65, 32].

Let me say goodbye with a reflection. Quantum mechanics is reaching the level of a technological solution. Apart from the understanding of the fundamental interactions young physicists should acknowledge the big heroes of physics for the existence of job offers where knowing quantum physics is mandatory.

Acknowledgments. I am indebted with so many people that this section would be too large. So, in the spirit of this overview *in a nutshell*, I will do my best to shorten the list. It is an honour to acknowledge the members of the *Real Academia de Ciencias de Zaragoza* for this prize. Thanks to my students. Specially to the QED crew: Fernando Quijandría, Eduardo Sánchez-Burillo, Virginia Ciriano, Juan Román-Roche and Sebas Roca. I learnt a lot from you. To my colleagues-friends. Specially to José Luis García-Palacios, Fernando Luis, Juanjo García-Palacios, Luis Martín-Moreno, Charles Downing, Salvatore Savasta, Franco Nori and Peter Hänggi. It is a pleasure to do physics with you. Finally, I also acknowledge the funding from the Spanish Goverment, the EU comission and the Aragón regional funds.

A The master equation

Even if one can argue that the Universe follows a Schrödinger-type equation in some limit, it is difficult to defend that practical calculations must include the Hamiltonian for the whole Universe. In fact, most of the physicists are interested in a corner (typically small) of the Universe. This corner interacts with the rest. Denoting it as *the system of interest* and the rest is denoted *the environment* the *total* Hamiltonian is splitted:

$$H_{\rm T} = H_{\rm S} + H_{\rm E} + H_{\rm I} \tag{44}$$

And the relevant object is the reduced density matrix $\varrho \equiv Tr_{\rm E}(\varrho_T)$. I remind you that, for any observable acting on the system ($O_{\rm S} = O_{\rm S} \otimes \mathbb{I}_E$), its average is given by $\langle O_{\rm S} \rangle = {\rm Tr}(O_{\rm S} \varrho)$. Writen like this, $H_{\rm S}$ describes an open system and it would be nice to have a dynamical equation for ϱ. The latter is a master equation. Below, within the style of this lecture, I will sketch the derivation of a generic master equation with emphasis in its conceptual roots rather than in the algebra.

Notice the *exact* time evolution for the *reduced density matrix* given the $H_{\rm T}$ in (44) [Cf.

figure 1],

$$\varrho(t) = \mathrm{Tr}_{\mathrm{E}}\Big(U_{\mathrm{T}}(t,t_0)\varrho_{\mathrm{T}}(t_0)U^{\dagger}_{\mathrm{T}}(t,t_0)\Big)\ . \tag{45}$$

Consider first the case where both system and environment are in a product state at t_0 (they are not correlated; neither classically nor quantum). Then, $\varrho_{\mathrm{T}}(t_0) = \varrho_{\mathrm{S}}(t_0)\otimes\varrho_{\mathrm{B}}(t_0)$. In this case,

$$\varrho(t) = \sum_{\alpha,\beta} K^{\dagger}_{\alpha,\beta}(t,t_0)\varrho_{\mathrm{S}}(t_0)K_{\alpha,\beta}(t,t_0) \equiv \mathcal{E}(t,t_0)\varrho(t_0) \tag{46}$$

with $K_{\alpha,\beta}(t,t_0) = \sqrt{\lambda_\alpha}\langle\lambda_\alpha|U(t,t_0)|\lambda_\beta\rangle$. This last formula is rather direct from (45) using the diagonalization for the bath initial density matrix $\varrho_{\mathrm{B}}(t_0) = \sum_\alpha \lambda_\alpha|\lambda_\alpha\rangle\langle\lambda_\alpha|$. The last equivalence is a convenient notation that highlights the fact that this is nothing but a map that transforms a density matrix in another density matrix. We emphasize that this result is exact and that the operators K (so the map $\mathcal{E}$) are independent of the state. However, it lies in the assumption that at t_0 the density matrix was a product state. This, marks t_0 as an special time. In general, however the state is

$$\varrho_{\mathrm{T}}(t_0) = \rho_{\mathrm{S}}(t_0)\otimes\rho_{\mathrm{E}}(t_0) + \delta_\varrho \tag{47}$$

Here, $\varrho_{\mathrm{S}} = \mathrm{Tr}_{\mathrm{E}}(\varrho_{\mathrm{T}})$, $\varrho_{\mathrm{E}} = \mathrm{Tr}_{\mathrm{S}}(\varrho_{\mathrm{T}})$ and $\delta_\varrho = \rho - \rho_{\mathrm{S}}\otimes\rho_{\mathrm{E}}$ wich encapsulates the system-bath correlations. Introducing the latter in (45) the expression for $\varrho(t)$ adds the extra term $\mathrm{Tr}_{\mathrm{E}}\Big(U_{\mathrm{T}}(t,t_0)\delta_\varrho U^{\dagger}_{\mathrm{T}}(t,t_0)\Big)$. This term depends on the initial condition δ_ϱ. Therefore, is not always possible to find a Universal dynamical model (this is the standard name) independent on the state of the system density matrix at time t_0. This is annoying. In general, it is not possible to find a differential operator such that, $\dot\varrho = \mathcal{L}[\varrho]$. In other words, we cannot always find a local equation for the evolution of an open system . This is not surprising, the same occurs for classical systems. Thus, an approximation is necessary. I am going to assume that the map (46) is Markovian, wich means that satisfy the composition $(\forall t_2, t_0, t_1)$,

$$\mathcal{E}(t_2,t_0) = \mathcal{E}(t_2,t_1)\ \mathcal{E}(t_1,t_0) \tag{48}$$

In practice, this means that (46) can be used for any t and t_0. Obviously this is not true, however let me continue and justify the Markovian approximation at the end. The differential version for (46) can be found noticing that

$$\frac{d\varrho}{dt} = \lim_{\epsilon \to 0} \frac{\mathcal{E}(t+\epsilon, t) - \epsilon}{\epsilon} \varrho(t) \equiv \mathcal{L}[\varrho] \ . \tag{49}$$

Doing two pages of calculations (not shown here) it is found that

$$\frac{d\varrho}{dt} = -i[H_{\rm S}, \varrho] + \sum_{i,j}^{N^2-1} \gamma_{ij} \Big(L_i \varrho L_j^\dagger - \frac{1}{2} \{ L_i^\dagger L_j, \varrho \} \Big) \tag{50}$$

This is the Lindblad equation discussed in the main text. Here the operators L form an orthogonal basis in the space N^2 and the coefficients are related to the coefficients for the expansion. See [8, Chapter 3] and [57, Chapter 4] for a more detailed discussion.

My final comment on the main approximation used: Eq. (50) has been found under the Markovian condition. This approximation, in practice, neglects the correlated part δ_ϱ in (47). This seems to contradict my discussion on the importance of entanglement done in Sect. 1.3. However, it does not. We are *not* neglecting the entanglement, which would mean that $\varrho_{\rm S}$ is a pure state all the time. We are neglecting δ_ϱ, a correction to the density matrix that is, at least, of the order of the system-bath coupling. Therefore, Eq. (50) is expected to hold whenever the system and bath are weakly coupled.

References

[1] G. Agarwal, R. Puri, and R. Singh, *Atomic Schrödinger cat states*, Phys. Rev. A **56** (1997), no. 3, 2249–2254.

[2] Dimitris G Angelakis, Marcelo Franca Santos, and Sougato Bose, *Photon-blockade-induced mott transitions and x y spin models in coupled cavity arrays*, Physical Review A **76** (2007), no. 3, 031805.

[3] Frank Arute, Kunal Arya, Ryan Babbush, Dave Bacon, Joseph C Bardin, Rami Barends, Rupak Biswas, Sergio Boixo, Fernando GSL Brandao, David A Buell, et al., *Quantum supremacy using a programmable superconducting processor*, Nature **574** (2019), no. 7779, 505–510.

[4] O Astafiev, Alexandre M Zagoskin, AA Abdumalikov, Yu A Pashkin, T Yamamoto, K Inomata, Y Nakamura, and JS Tsai, *Resonance fluorescence of a single artificial atom*, Science **327** (2010), no. 5967, 840–843.

[5] Rami Barends, Julian Kelly, Anthony Megrant, Andrzej Veitia, Daniel Sank, Evan Jeffrey, Ted C White, Josh Mutus, Austin G Fowler, Brooks Campbell, et al., *Superconducting quantum circuits at the surface code threshold for fault tolerance*, Nature **508** (2014), no. 7497, 500.

[6] Soumya Bera, Ahsan Nazir, Alex W. Chin, Harold U. Baranger, and Serge Florens, *Generalized multipolaron expansion for the spin-boson model: Environmental entanglement and the biased two-state system*, Physical Review B **90** (2014), no. 7.

[7] Daniel Braak, *Integrability of the rabi model*, Physical Review Letters **107** (2011), no. 10, 100401.

[8] H.P. Breuer, P.I.H.P. Breuer, F. Petruccione, and S.P.A.P.F. Petruccione, *The theory of open quantum systems*, Oxford University Press, 2002.

[9] Giuseppe Calajó, Francesco Ciccarello, Darrick Chang, and Peter Rabl, *Atom-field dressed states in slow-light waveguide qed*, Physical Review A **93** (2016), no. 3, 033833.

[10] H. B. G. Casimir, *On the attraction between two perfectly conducting plates*, Proc. K. Ned. Akad. Wet **51** (1948), 739.

[11] DE Chang, JS Douglas, Alejandro González-Tudela, C-L Hung, and HJ Kimble, *Colloquium: Quantum matter built from nanoscopic lattices of atoms and photons*, Reviews of Modern Physics **90** (2018), no. 3, 031002.

[12] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Photons and atoms: Introduction to quantum electrodynamics*, Wiley, 1989.

[13] Daniele De Bernardis, Tuomas Jaako, and Peter Rabl, *Cavity quantum electrodynamics in the nonperturbative regime*, Physical Review A **97** (2018), no. 4, 043820.

[14] Daniele De Bernardis, Philipp Pilar, Tuomas Jaako, Simone De Liberato, and Peter Rabl, *Breakdown of gauge invariance in ultrastrong-coupling cavity qed*, Physical Review A **98** (2018), no. 5, 053819.

[15] Michel H Devoret, Andreas Wallraff, and John M Martinis, *Superconducting qubits: A short review*, arXiv preprint cond-mat/0411174 (2004).

[16] Omar Di Stefano, Alessio Settineri, Vincenzo Macrì, Luigi Garziano, Roberto Stassi, Salvatore Savasta, and Franco Nori, *Resolution of gauge ambiguities in ultrastrong-coupling cavity quantum electrodynamics*, Nature Physics (2019), 1.

[17] Guillermo Díaz-Camacho, Alejandro Bermudez, and Juan José García-Ripoll, *Dynamical polaron ansatz: A theoretical tool for the ultrastrong-coupling regime of circuit qed*, Phys. Rev. A **93** (2016), 043843.

[18] David Dzsotjan, Jürgen Kästel, and Michael Fleischhauer, *Dipole-dipole shift of quantum emitters coupled to surface plasmons of a nanowire*, Physical Review B **84** (2011), no. 7, 075419.

[19] Sanli Faez, Pierre Türschmann, Harald R Haakh, Stephan Götzinger, and Vahid Sandoghdar, *Coherent interaction of light and single molecules in a dielectric nanoguide*, Physical review letters **113** (2014), no. 21, 213601.

[20] Adrian Feiguin, Juan Jose Garcia-Ripoll, and Alejandro Gonzalez-Tudela, *Qubit-photon corner states in all dimensions*, arXiv preprint arXiv:1910.00824 (2019).

[21] P Forn-Díaz, L Lamata, E Rico, J Kono, and E Solano, *Ultrastrong coupling regimes of light-matter interaction*, Reviews of Modern Physics **91** (2019), no. 2, 025005.

[22] Pol Forn-Díaz, Jürgen Lisenfeld, David Marcos, Juan José Garcia-Ripoll, Enrique Solano, CJPM Harmans, and JE Mooij, *Observation of the bloch-siegert shift in a qubit-oscillator system in the ultrastrong coupling regime*, Physical review letters **105** (2010), no. 23, 237001.

[23] L. Garziano, V. Macrì, R. Stassi, O. Di Stefano, F. Nori, and S. Savasta, *One Photon Can Simultaneously Excite Two or More Atoms*, Phys. Rev. Lett. **117** (2016), 043601.

[24] N. Gisin, *Weinberg's non-linear quantum mechanics and supraluminal communications*, Physics Letters A **143** (1990), no. 1-2, 1–2.

[25] Milena Grifoni and Peter Hänggi, *Driven quantum tunneling*, Physics Reports **304** (1998), no. 5-6, 229–354.

[26] Xiu Gu, Anton Frisk Kockum, Adam Miranowicz, Yu-xi Liu, and Franco Nori, *Microwave photonics with superconducting quantum circuits*, Physics Reports **718** (2017), 1–102.

[27] Michael J Hartmann, Fernando GSL Brandao, and Martin B Plenio, *Strongly interacting polaritons in coupled arrays of cavities*, Nature Physics **2** (2006), no. 12, 849.

[28] J. D. Jackson and L. B. Okun, *Historical roots of gauge invariance*, Reviews of Modern Physics **73** (2001), no. 3, 663–680.

[29] Mark Jenkins, Thomas Hümmer, María José Martínez-Pérez, Juanjo García-Ripoll, David Zueco, and Fernando Luis, *Coupling single-molecule magnets to quantum circuits*, New journal of physics **15** (2013), no. 9, 095007.

[30] MD Jenkins, David Zueco, Olivier Roubeau, Guillem Aromí, J Majer, and Fernando Luis, *A scalable architecture for quantum computation with molecular nanomagnets*, Dalton Transactions **45** (2016), no. 42, 16682–16693.

[31] Abhinav Kandala, Antonio Mezzacapo, Kristan Temme, Maika Takita, Markus Brink, Jerry M Chow, and Jay M Gambetta, *Hardware-efficient variational quantum eigensolver for small molecules and quantum magnets*, Nature **549** (2017), no. 7671, 242.

[32] Martin Kiffner, Jonathan R. Coulthard, Frank Schlawin, Arzhang Ardavan, and Dieter Jaksch, *Manipulating quantum materials with quantum light*, Physical Review B **99** (2019), no. 8.

[33] Peter Kirton, Mor M Roses, Jonathan Keeling, and Emanuele G Dalla Torre, *Introduction to the dicke model: from equilibrium to nonequilibrium, and vice versa*, Advanced Quantum Technologies **2** (2019), no. 1-2, 1800043.

[34] Anton Frisk Kockum, Adam Miranowicz, Simone De Liberato, Salvatore Savasta, and Franco Nori, *Ultrastrong coupling between light and matter*, Nature Reviews Physics **1** (2019), no. 1, 19.

[35] Andreas Kurcz, Alejandro Bermudez, and Juan José García-Ripoll, *Hybrid quantum magnetism in circuit qed: from spin-photon waves to many-body spectroscopy*, Physical review letters **112** (2014), no. 18, 180405.

[36] Thaddeus D Ladd, Fedor Jelezko, Raymond Laflamme, Yasunobu Nakamura, Christopher Monroe, and Jeremy Lloyd O'Brien, *Quantum computers*, nature **464** (2010), no. 7285, 45.

[37] Pasi Lähteenmäki, G. S. Paraoanu, Juha Hassel, and Pertti J. Hakonen, *Dynamical casimir effect in a josephson metamaterial*, Proceedings of the National Academy of Sciences **110** (2013), no. 11, 4234–4238.

[38] Steve K. Lamoreaux, *Casimir forces: Still surprising after 60 years*, Physics Today **60** (2007), no. 2, 40–45.

[39] Karyn Le Hur, *Quantum phase transitions in spin-boson systems: Dissipa-tion and light phenomena*, Understanding Quantum Phase Transitions, CRC Press, 2010, pp. 245–268.

[40] Karyn Le Hur, Loïc Henriet, Alexandru Petrescu, Kirill Plekhanov, Guillaume Roux, and Marco Schiró, *Many-body quantum electrodynamics networks: Non-equilibrium condensed matter physics with light*, Comptes Rendus Physique **17** (2016), no. 8, 808–835.

[41] A. J. Leggett, S. Chakravarty, A. T. Dorsey, Matthew P. A. Fisher, Anupam Garg, and W. Zwerger, *Dynamics of the dissipative two-state system*, Reviews of Modern Physics **59** (1987), no. 1, 1–85.

[42] Ulf Leonhardt, *Essential quantum optics: from quantum measurements to black holes*, Cambridge University Press, 2010.

[43] Yanbing Liu and Andrew A Houck, *Quantum electrodynamics near a photonic bandgap*, Nature Physics **13** (2017), no. 1, 48.

[44] Peter Lodahl, Sahand Mahmoodian, and Søren Stobbe, *Interfacing single photons and single quantum dots with photonic nanostructures*, Reviews of Modern Physics **87** (2015), no. 2, 347.

[45] Peter Lodahl, Sahand Mahmoodian, Søren Stobbe, Arno Rauschenbeutel, Philipp Schneeweiss, Jürgen Volz, Hannes Pichler, and Peter Zoller, *Chiral quantum optics*, Nature **541** (2017), no. 7638, 473.

[46] Vincenzo Macrí, Franco Nori, Salvatore Savasta, and David Zueco, *Optimal spin squeezing in cavity qed based systems*, arXiv preprint arXiv:1902.10377 (2019).

[47] J Majer, JM Chow, JM Gambetta, Jens Koch, BR Johnson, JA Schreier, L Frunzio, DI Schuster, Andrew Addison Houck, Andreas Wallraff, et al., *Coupling superconducting qubits via a cavity bus*, Nature **449** (2007), no. 7161, 443.

[48] Gerald T. Moore, *Quantum theory of the electromagnetic field in a variable-length one-dimensional cavity*, Journal of Mathematical Physics **11** (1970), no. 9, 2679–2691.

[49] T. Niemczyk, F. Deppe, H. Huebl, E. P. Menzel, F. Hocke, M. J. Schwarz, J. J. Garcia-Ripoll, D. Zueco, T. Hümmer, E. Solano, A. Marx, and R. Gross, *Circuit quantum electrodynamics in the ultrastrong-coupling regime*, Nature Physics **6** (2010), 772–776.

[50] W. Nolting and A. Ramakanth, *Quantum theory of magnetism*, Springer Berlin Heidelberg, 2009.

[51] M.A. Norcia, R.J. Lewis-Swan, J.R.K. Cline, B. Zhu, A.M. Rey, and J.K. Thompson, *Cavity-mediated collective spin-exchange interactions in a strontium superradiant laser*, Science **361** (2018), no. 6399, 259–262.

[52] Jeremy L O'brien, Akira Furusawa, and Jelena Vučković, *Photonic quantum technologies*, Nature Photonics **3** (2009), no. 12, 687.

[53] Borja Peropadre, David Zueco, Diego Porras, and Juan José García-Ripoll, *Nonequilibrium and nonperturbative dynamics of ultrastrong coupling in open lines*, Physical review letters **111** (2013), no. 24, 243602.

[54] M.E. Peskin and D.V. Schroeder, *An introduction to quantum field theory*, Frontiers in Physics, Avalon Publishing, 1995.

[55] M Pino and Juan José García-Ripoll, *Mediator assisted cooling in quantum annealing*, arXiv preprint arXiv:1910.13459 (2019).

[56] Javier Prior, Alex W Chin, Susana F Huelga, and Martin B Plenio, *Efficient simulation of strong system-environment interactions*, Physical review letters **105** (2010), no. 5, 050404.

[57] Á. Rivas and S.F. Huelga, *Open quantum systems: An introduction*, SpringerBriefs in Physics, Springer Berlin Heidelberg, 2011.

[58] Dibyendu Roy, Christopher M Wilson, and Ofer Firstenberg, *Colloquium: Strongly interacting photons in one-dimensional continuum*, Reviews of Modern Physics **89** (2017), no. 2, 021001.

[59] Eduardo Sánchez-Burillo, Juanjo García-Ripoll, Luis Martín-Moreno, and David Zueco, *Nonlinear quantum optics in the (ultra) strong light–matter coupling*, Faraday discussions **178** (2015), 335–356.

[60] Eduardo Sánchez-Burillo, L Martín-Moreno, JJ García-Ripoll, and D Zueco, *Single photons by quenching the vacuum*, Physical review letters **123** (2019), no. 1, 013601.

[61] Eduardo Sánchez-Burillo, Luis Martín-Moreno, Juan José García-Ripoll, and David Zueco, *Full two-photon down-conversion of a single photon*, Physical Review A **94** (2016), no. 5, 053814.

[62] Eduardo Sánchez-Burillo, David Zueco, JJ Garcia-Ripoll, and Luis Martin-Moreno, *Scattering in the ultrastrong regime: nonlinear optics with one photon*, Physical review letters **113** (2014), no. 26, 263604.

[63] Eduardo Sánchez-Burillo, David Zueco, Luis Martín-Moreno, and Juan José García-Ripoll, *Dynamical signatures of bound states in waveguide qed*, Physical Review A **96** (2017), no. 2, 023831.

[64] W.P. Schleich, *Quantum optics in phase space*, Wiley, 2011.

[65] M. A. Sentef, M. Ruggenthaler, and A. Rubio, *Cavity quantum-electrodynamical polaritonically enhanced electron-phonon coupling and its influence on superconductivity*, Science Advances **4** (2018), no. 11, eaau6969.

[66] W. Shao, C. Wu, and X.L. Feng, *Generalized James' effective Hamiltonian method*, Phys. Rev. A **95** (2017), 032124.

[67] Tao Shi, Yue Chang, and Juan José García-Ripoll, *Ultrastrong coupling few-photon scattering theory*, Physical Review Letters **120** (2018), no. 15.

[68] Tao Shi, Eugene Demler, and J Ignacio Cirac, *Variational study of fermionic and bosonic systems with non-gaussian states: Theory and applications*, Annals of Physics **390** (2018), 245–302.

[69] Tao Shi, Ying-Hai Wu, Alejandro González-Tudela, and J Ignacio Cirac, *Bound states in boson impurity models*, Physical Review X **6** (2016), no. 2, 021027.

[70] Robert Silbey and Robert A Harris, *Variational calculation of the dynamics of a two level system interacting with a bath*, The Journal of chemical physics **80** (1984), no. 6, 2615–2617.

[71] David Tong, *Lecture on quantum field theory.*

[72] W. Ulrich, *Quantum dissipative systems (second edition)*, Series In Modern Condensed Matter Physics, World Scientific Publishing Company, 1999.

[73] Arjan F Van Loo, Arkady Fedorov, Kevin Lalumière, Barry C Sanders, Alexandre Blais, and Andreas Wallraff, *Photon-mediated interactions between distant artificial atoms*, Science **342** (2013), no. 6165, 1494–1496.

[74] S.D. W, *Solid state physics: Essential concepts*, Pearson Education, 2009.

[75] Michael L Wall, Arghavan Safavi-Naini, and Ana Maria Rey, *Boson-mediated quantum spin simulators in transverse fields: X y model and spin-boson entanglement*, Physical Review A **95** (2017), no. 1, 013602.

[76] A. Wallraff, D. I. Schuster, A. Blais, L. Frunzio, R.-S. Huang, J. Majer, S. Kumar, S. M. Girvin, and R. J. Schoelkopf, *Strong coupling of a single photon to a superconducting qubit using circuit quantum electrodynamics*, Nature **431** (2004), no. 7005, 162–167.

[77] C. M. Wilson, G. Johansson, A. Pourkabirian, M. Simoen, J. R. Johansson, T. Duty, F. Nori, and P. Delsing, *Observation of the dynamical casimir effect in a superconducting circuit*, Nature **479** (2011), no. 7373, 376–379.

[78] Jiehang Zhang, Guido Pagano, Paul W Hess, Antonis Kyprianidis, Patrick Becker, Harvey Kaplan, Alexey V Gorshkov, Z-X Gong, and Christopher Monroe, *Observation of a many-body dynamical phase transition with a 53-qubit quantum simulator*, Nature **551** (2017), no. 7682, 601.

[79] Nengji Zhou, Lipeng Chen, Dazhi Xu, Vladimir Chernyak, and Yang Zhao, *Symmetry and the critical phase of the two-bath spin-boson model: Ground-state properties*, Physical Review B **91** (2015), no. 19, 195129.

[80] David Zueco and Juanjo García-Ripoll, *Ultrastrongly dissipative quantum rabi model*, Physical Review A **99** (2019), no. 1, 013807.

[81] David Zueco, Juan J Mazo, Enrique Solano, and Juan José García-Ripoll, *Microwave photonics with josephson junction arrays: negative refraction index and entanglement through disorder*, Physical Review B **86** (2012), no. 2, 024503.

[82] David Zueco, Georg M Reuther, Sigmund Kohler, and Peter Hänggi, *Qubit-oscillator dynamics in the dispersive regime: Analytical theory beyond the rotating-wave approximation*, Physical Review A **80** (2009), no. 3, 033846.