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Positivity, accuracy, optimality and applications

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Abstract

This paper surveys some recent advances relating positivity, accuracy and optimal bases. In particular, high relative accuracy computations for some structured classes of matrices adequately parametrized are considered. Some applications of these classes of matrices are commented.

1 Introduction

In many mathematical models, positivity is one of the fundamental underlying hypotheses because the involved variables only have meaning when they are nonnegative. This also implies that nonnegative matrices also play a crucial role in many mathematical models dealing with problems of the real world or arising in other scientific or technical fields. Spectral properties of nonnegative matrices are also remarkable because, by the Perron–Frobenius theorem (cf. [12]), the nonnegativity is inherited by an eigenvalue of the matrix with maximum absolute value and by a corresponding eigenvector, properties that also play a key role in many mathematical models. For instance, in the Leontief input–output model, very important in Economy (cf. [12]). More recent applications of positivity are related with the advantages of positivity in

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the context of numerical computations in floating point arithmetic, as explained in Section 2. In this sense, we mention two main sources of such applications. On the one hand, applications related to the factorization of a nonnegative sparse matrix as a product of two nonnegative matrices, which are natural applications in the context of Big Data (cf. [114, 82]). On the other hand, applications related to the search of numerical methods adapted to the structure of the classes of matrices and leading to computations with high relative accuracy. This will be one of the topics surveyed in this paper. In spaces of nonnegative functions arise optimal bases under different viewpoints. This problem was first studied in the context of Total Positivity and later extended to a more general framework, and it is another topic considered in this paper.

The paper is organized as follows. Section 2 presents some basic concepts about the errors obtained when computing with floating point arithmetic. The classical error analysis involves concept such as growth factor and conditioning. This leads to present some optimal bases and to recall that, in some problems related with positivity, it is possible to find a parametrization of the data and an algorithm leading to small roundoff errors in spite of a bad conditioning with its initial parametrization. Section 3 is devoted to optimal bases for several properties in the context of Total Positivity and more general contexts. In particular, we point out some optimal properties of the Bernstein basis of the space of polynomials.

Up to now, methods with high relative accuracy for algebraic computations (such us the eigenvalues, singular values or inverses) independently of the conditioning have been found mainly for classes of matrices related with positivity and coming from one of the two following sources: generalizations of diagonally dominant matrices (considered in Section 4) and subclasses of totally positive matrices (considered in Section 5). In both cases, we comment the parameterizations of the matrices leading to the computations with high relative accuracy as well as some applications. These parameterizations lead to some matrix factorizations that are used by algorithms with high relative accuracy: rank revealing decompositions, obtained for classes of matrices generalizing diagonal dominance, and bidiagonal factorization, obtained for nonsingular totally positive matrices. In Section 5 we provide some details of the accurate computations and bidiagonal factorizations of matrices of matrices related to the following five subclasses of nonsingular totally positive matrices: Pascal matrices, rational Bernstein-Vandermonde matrices, Jacobi-Stirling matrices, Laguerre matrices and Bessel matrices.

2 Error analysis, optimal bases and high relative accuracy

When we apply an algorithm and perform the computations with floating point arithmetic, since we do not know the exact error performed with our computations, it is convenient to try to derive upper bounds of this error, usually known as *forward* error bounds. However, it is usually very difficult to get directly such bounds. An alternative approach that has been very successful in the field of Numerical Linear Algebra, and later in other fields, bounds the forward error through the backward error. If we consider that our computed solution is the exact solution of a perturbed problem, the *backward* error measures the distance between the perturbed problem and the initial problem. The backward error depends on the numerical method that we have used and it is well known that the growth factor of an algorithm is an indicator of its backward stability (cf. [68], [71]). Let us recall that the *growth factor* of a numerical algorithm is usually defined as the quotient between the maximal absolute value of all the elements that occur during the performance of the algorithm and the maximal absolute value of all the initial data. The optimal growth factor for a numerical method is 1. Hence optimal methods under this viewpoint are methods with growth factor 1 (cf. [106, 108, 27]).

The *conditioning* of the problem measures the effect of data perturbations on the solution of the problem. In general, when we have defined the corresponding forward error, backward error and the condition number for a given problem, one tries to prove the relation:

$$\text{forward error} \leq \text{condition number} \times \text{backward error},$$

which allows us to obtain a forward error bound through the backward error. Although the computed solution has a small backward error, it can be amplified by the condition number leading to a large forward error. So, in contrast to the backward error, which depends of the used method, the conditioning can become an intrinsic cause to get a large forward error bound. In conclusion, under this approach, in order to get a small forward error we need a small conditioning and using a method with small backward error.

In the problem of evaluating a real function of a finite dimensional vector space of functions, the conditioning depends on the basis that we use. The problem of finding bases with minimal condition number has been analyzed in the context of bases of nonnegative functions, which play an important role in many problems of approximation theory or computer aided geometric design (cf. [53]). The optimal bases in this context for the evaluation of a function are bases b of nonnegative functions that cannot be expressed, up to permutation and positive scaling, in the form vK, where v is another basis of nonnegative functions and K is a nonnegative matrix (see [25, 105, 106]). Examples of these optimal bases are the Bernstein basis [52] for the space of polynomials on a compact interval, the B-spline basis [101] or the bases of spaces of real multivariate functions given in [84]. Let us recall that the *Bernstein basis* $(b_0^n(t), \ldots, b_n^n(t))$ of the space P_n of polynomials of degree at most n on $[0,1]$ is given by

$$b_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad t \in [0,1],\ i = 0, 1, \ldots, n. \tag{1}$$

and it is the polynomial basis most used in computer aided geometric design. In the next section, we consider again this basis and other optimal properties that it satisfies. The conditioning for the representations associated to polynomial interpolation problems (depending on the nodes distribution and ordering) and to least squares approximations have been analyzed recently (see [16, 17, 18, 19]). Coming back to the problem of evaluating a function, in addition to bases with minimal condition number, backward stable methods are also required in order to obtain small forward errors. These backward stable methods and their corresponding error analysis can be seen in [85, 109, 110, 8, 9, 34, 36, 10, 38, 42].

We now comment an alternative approach to assure small forward errors and that can be applied in fields where positivity plays a key role. In some problems it is possible to find a parametrization of the data and an algorithm leading to small forward error bounds in spite of a bad conditioning with its initial parametrization. The desired goal is to guarantee *high relative accuracy* (HRA). We say that we have performed an algorithm with HRA if the following formula holds: relative forward error $\leq Ku$, for some constant K, where u is the unit roundoff. It is not always possible to guarantee HRA for a given problem. An example of a simple problem for which an HRA algorithm cannot be found is provided by the sum of three real numbers $x+y+z$ (see [44]). For some structured classes of matrices, HRA algorithms can be found, as we shall recall in this paper. However, there are classes of structured matrices for which we again have that these algorithms cannot be found. For instance, accurate linear algebra for the problem of calculating determinants or minors is impossible on the class of Toeplitz matrices (see corollaries 3.43 and 3.45 of

[44]). Let us recall that a Toeplitz matrix B has the following simple structure:

$$B = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ a_{-1} & a_0 & \ddots & & a_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{-n+2} & & \ddots & \ddots & a_1 \\ a_{-n+1} & a_{-n+2} & \cdots & a_{-1} & a_0 \end{pmatrix}.$$

There exists a sufficient condition to assure the HRA of an algorithm that we now recall. Given an algorithm using only additions of numbers of the same sign, multiplications and divisions, and assuming that each initial real datum is known to HRA, then it is well known that the output of the algorithm can be computed to HRA (cf. [45, p. 52]). Moreover, in (well implemented) floating point arithmetic, HRA is also preserved even when we perform true subtractions when the operands are original (and so, exact) data (cf. p. 53 of [45]). So, the sufficient condition to assure the HRA of an algorithm is called "no inaccurate cancellation" (NIC) and it is satisfied if it only uses additions of numbers of the same sign, multiplications, divisions and subtractions (additions of numbers of different sign) of the initial data. In order to find algorithms with HRA for some classes of matrices, it is usually necessary to reparameterize the matrices belonging to these classes, since HRA will be satisfied independently of the conditioning of the matrices. Up to now , the main classes of matrices for which algorithms with HRA have been found are closely related with positivity and the corresponding algorithms are in fact NIC algorithms. We show many examples of these classes of matrices in sections 4 and 5.

3 Optimal bases

In the previous section, we have mentioned the bases of nonnegative functions that are optimal, with respect to the corresponding condition number, for the problem of evaluating a real function of a finite dimensional vector space of functions. Let us now recall this condition number. Given a basis $u = (u_0, \ldots, u_n)$ of a real vector space $\mathcal{U}$ of real functions defined on a subset S of $\mathbf{R}^m$ $(m \geq 1)$ and a function $f \in \mathcal{U}$, we can write $f(x) = \sum_{i=0}^n c_i u_i(x)$ for all $x \in S$, where $c_i \in \mathbf{R}$ for all $i = 0, \ldots, n$. The stability of the basis U with respect to the evaluation at a point is measured by the function $C_u : \mathcal{U} \times I \to \mathbf{R}_+$ given by

(2) $$C_u(f, x) := \sum_{i=0}^{n} |c_i u_i(x)|.$$

Given two bases of nonnegative functions u, v, let A the matrix of change of basis such that $v = uA$. The following result compares the conditioning of the bases by means of the nonnegativity of K. The result can be proved with the adaptation to the condition number (2) of the proof of Lemma 3.1 of [84], which proves a similar result for a relative condition number derived from the previous one.

Lemma 3.1 *Let $\mathcal{U}$ be a finite dimensional vector space of functions defined on a subset S of $\mathbf{R}^m$. Let u, v be two bases of nonnegative functions of $\mathcal{U}$. Then*

$$C_u(f, x) \le C_v(f, x), \quad \forall f \in \mathcal{U}, \quad \forall x \in S \tag{3}$$

if and only if the matrix A such that $v = uA$ is nonnegative.

In the case of the space P_n of polynomials of degree at most n on an interval $[0, 1]$, Theorem 3 of [52] leads to the optimality result of the Bernstein basis given by (1).

Theorem 3.2 *Let $b = (b_0^n, \dots, b_n^n)$ be the Bernstein basis. Then there does not exist (up to reordering and positive scaling) another basis $u = (u_0, \dots, u_n)$ of nonnegative functions in P_n such that $C_u(p, t) \le C_b(p, t)$ for all $t \in [0, 1]$ and $p \in P_n$.*

Similar results to the previous one are satisfied, for instance, by the B-spline basis [101] or by the bases of spaces of real multivariate functions given in [84] (see also [105]).

Now we shall focus on optimal bases for shape preservation in Computer Aided Geometric Design (CAGD). We shall see that they are closely related with the optimal bases for the evaluation commented previously. We start with the definition of collocation matrix of a system of functions, which will be used later to introduce special bases. Given a system of functions $U = (u_0, \dots, u_n)$ defined on $I \subseteq \mathbf{R}$, the *collocation matrix* of U at $t_0 < \cdots < t_m$ in I is given by

$$M \begin{pmatrix} u_0, \dots, u_n \\ t_0, \dots, t_m \end{pmatrix} := (u_j(t_i))_{i=0,\dots,m; j=0,\dots,n}. \tag{4}$$

We now recall some concepts of CAGD. Given a sequence of functions $(u_0, \dots, u_n)$ on $I = [a, b]$, and a sequence of points $(C_0, \dots, C_n)$ in $\mathbf{R}^k$, we may define a parametric curve

$$\gamma(t) = \sum_{i=0}^{n} C_i u_i(t), \quad t \in [a, b].$$

The points C_i, $i = 0, \dots, n$, are called *control points*. The *control polygon* of the curve γ is the polygonal arc with vertices $C_0, \dots, C_n$. In CAGD, it is usually required that the functions u_i, $i = 0, \ldots, n$, are nonnegative and $\sum_{i=0}^{n} u_i(t) = 1$ for all $t \in I$ (i.e., the system $u = (u_0, \ldots, u_n)$ is *normalized*, or equivalently, the functions form a partition of unity). A normalized system of nonnegative functions is called a *blending* system. An important property for curve design is the *convex hull property*: for any control polygon, the curve lies always in the convex hull of the control polygon. The convex hull property holds if and only if the system of functions is blending. These geometric properties correspond to some properties of the collocation matrices of the system of functions. Observe that u is blending if and only if all its collocation matrices are stochastic (that is, nonnegative and such that the sum of the entries of each row is one).

More shape preserving properties of the systems of functions require in turn additional properties to their corresponding collocation matrices. In interactive design we also desire that the shape of a parametrically defined polynomial curve mimics the shape of its control polygon in order to predict or manipulate the shape of the curve by suitably choosing or changing the control polygon. This leads to the concept of totally positive systems and totally positive matrices, due to the variation diminishing property of these matrices. A matrix is *totally positive* (TP) if all its minors are nonnegative and a system of functions is totally positive (TP) if all its collocation matrices (4) are totally positive. If all minors of a matrix are positive, then the matrix is called *strictly totally positive* (STP) matrix. TP and STP matrices have been also called in the literature as totally nonnegative and totally positive, respectively. If a system u is normalized totally positive (NTP) then the curve γ inherits many shape properties of the control polygon.Let us mention that, in addition to the space of polynomials on a compact internal and polynomial spline spaces, many other spaces containing algebraic, trigonometric and hyperbolic polynomials also posses NTP bases (cf. [90, 86, 20, 87, 88, 21, 22]).

Now we consider the problem of comparing two NTP systems of the same space. Given two NTP bases $(p_0, \ldots, p_n)$ and $(b_0, \ldots, b_n)$ of a space of functions $\mathcal{U}$, let K be the nonsingular matrix given by

$$(p_0, \ldots, p_n) = (b_0, \ldots, b_n)K.$$

Since both bases are normalized we conclude that each row of K has sum 1. If we now assume that the matrix K of change of basis is TP, then it is an stochastic nonsingular TP matrix The following properties of the control polygons $B_0 \cdots B_n$ (with respect to $(b_0, \ldots, b_n)$) and $P_0 \cdots P_n$ (with respect to $(p_0, \ldots, p_n)$) can be obtained (see [65] and [67]):

1. If $P_0 \cdots P_n$ is convex, then so are $B_0 \ldots B_n$ and the curve γ, and $B_0 \cdots B_n$ lies between $P_0 \cdots P_n$ and γ.
2. Length $\gamma \leq$ length $B_0 \cdots B_n \leq$ length $P_0 \cdots P_n$.
3. If $P_0 \cdots P_n$ turns through an angle $< \pi$, then $I(\gamma) \leq I(B_0 \cdots B_n) \leq I(P_0 \cdots P_n)$, where $I(\beta)$ denotes the number of inflexions of a curve β.
4. $\theta(\gamma) \leq \theta(B_0 \cdots B_n) \leq \theta(P_0 \cdots P_n)$, where $\theta(\beta)$ denotes the angular variation of a curve β.

Therefore, the curve γ imitates better the form of the control polygon $B_0 \ldots B_n$ than the form of the control polygon $P_0 \ldots P_n$. This motivates that an NTP basis with optimal shape preserving properties satisfies the following definition.

Definition 3.3 *Let $(u_0, \ldots, u_n)$ be a TP basis of a space $\mathcal{U}$. Then $(u_0, \ldots, u_n)$ is a B-basis if for any other TP basis $(v_0, \ldots, v_n)$ of $\mathcal{U}$ the matrix K of change of basis*

$$(v_0, \ldots, v_n) = (u_0, \ldots, u_n)K$$

is TP.

By the previous reasoning, a normalized B-basis has *optimal shape preserving properties*. A space with an NTP basis has a unique normalized B-basis (see [24]). Example of normalized B-bases are the Bernstein basis (1) and the B-spline basis (see [23, 24]).

Another optimal property of normalized B-basis is related to the *progressive iteration approximation* property (see [35]), which is satisfied by NTP bases. Given a sequence of points $(P_i)_{i=0}^n$ such that the ith point is assigned to a parameter value t_i for $i = 0, 1, \ldots, n$ and a basis $(u_0, \ldots, u_n)$, we construct a starting curve $\gamma^0(t) = \sum_{i=0}^n P_i^0 u_i(t)$ with $P_i^0 = P_i$ for all $i \in \{0, 1, \ldots, n\}$. Then, computing the adjusting vector $\Delta_i^0 = P_i - \gamma^0(t_i)$ we can take $P_i^1 = P_i^0 + \Delta_i^0$, for $i = 0, 1, \ldots, n$, and construct a new curve as $\gamma^1(t) = \sum_{i=0}^n P_i^1 u_i(t)$. Iterating this process we can get a sequence of curves $\{\gamma^k\}_{k=0}^\infty$. The progressive iteration approximation property holds when this curve sequence converges to the polynomial curve interpolating the given initial sequence of points. This property holds for NTP bases and we proved in Theorem 4 of [35] the optimal convergence speed of the normalized B-basis.

Theorem 3.4 *The normalized B-basis of a space $\mathcal{U}$ with an NTP basis provides a progressive iterative approximation with the fastest convergence rate among all NTP bases of $\mathcal{U}$.*

Other optimal properties of normalized B-bases can be found in [25, 102].

Now, we finish this section presenting the optimal conditioning of the collocation matrices of the Bernstein basis. Given a nonsingular matrix $A = (a_{ij})_{1\le i,j\le n}$, let us consider the classical condition number

$$\kappa_\infty(A) := \|A\|_\infty \, \|A^{-1}\|_\infty.$$

Denoting by $|A|$ the matrix whose (i,j)-entry is $|a_{ij}|$, the Skeel condition number of a nonsingular matrix A is defined as

$$\mathrm{Cond}(A) := \| \, |A^{-1}| \, |A| \, \|_\infty.$$

The following result corresponds to Theorem 2.1 of [37]. It shows that the collocation matrices of the Bernstein basis are the best conditioned among all the corresponding collocation matrices of NTP bases of the space P_n of polynomials of degree at most n on $[0,1]$, and a similar result using the Skeel condition number of the transposes of the collocation matrices.

Theorem 3.5 *Let $(b_0^n, \dots, b_n^n)$ be the Bernstein basis, let $(v_0, \dots, v_n)$ be another NTP basis of P_n on $[0,1]$, let $0 \le t_0 < t_1 < \cdots < t_n \le 1$ and $V := M\binom{v_0,\dots,v_n}{t_0,\dots,t_n}$ and $B := M\binom{b_0^n,\dots,b_n^n}{t_0,\dots,t_n}$. Then:*

$$\kappa_\infty(B) \le \kappa_\infty(V), \quad \mathrm{Cond}(B^T) \le \mathrm{Cond}(V^T).$$

The previous result deals with the conditioning of certain totally positive matrices, In Section 5 we shall show some subclasses of totally positive matrices for which many algebraic computations can be performed with high relative accuracy (HRA).

4 Generalizing diagonal dominance: computations with HRA and applications

For some classes of matrices closely related with diagonal dominance, some algebraic calculations can be performed with HRA: singular values, inverses, the solution of some linear system and, in some cases, even the eigenvalues. An adequate parametrization of the matrices has been needed. Let us first recall some related classes of matrices. A square real matrix is called a *P-matrix* if all its principal minors are positive (the principal minors use the same rows and columns). Examples of subclasses of P-matrices with many applications are the nonsingular TP matrices (considered in the next section) and the nonsingular M-matrices. A real matrix with nonpositive off-diagonal entries is called a *Z-matrix.* A matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ is *(row) diagonally dominant* (resp., *strictly (row) diagonally dominant*) if, for each $i = 1, \dots, n$, $|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$ (reps., $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$). If A^T is row diagonally dominant, then we say that A is column diagonally dominant. Given a matrix $M = (m_{ij})_{1 \leq i,j \leq n}$, its *comparison matrix* $\tilde{M} = (\tilde{m}_{ij})_{1 \leq i,j \leq n}$ is the Z-matrix defined by $\tilde{m}_{ii} := |m_{ii}|$ and $\tilde{m}_{ij} := -|m_{ij}|$ if $i \neq j$, $1 \leq i,j \leq n$. Let us recall (cf. [12]) that if a Z-matrix A can be expressed as $A = sI - B$, with $B \geq 0$ and $s \geq \rho(B)$ (where $\rho(B)$ is the spectral radius of B), then it is called an *M-matrix.* A Z-matrix A is a nonsingular M-matrix if and only if A^{-1} is nonnegative (cf. [12]). Nonsingular M-matrices have important applications, for instance, in iterative methods in numerical analysis, in the analysis of dynamical systems, in economics and in mathematical programming (see [12]). Finally, we say that a matrix is an *H-matrix* if its comparison matrix is a nonsingular M-matrix. A is a nonsingular H-matrix if and only if there exists a diagonal matrix D such that AD is strictly diagonally dominant, and so these matrices are also called generalized diagonally dominant matrices.

An important concept related with the construction of algorithms with HRA for the computation of the singular values of a matrix is the rank revealing decomposition. A *rank revealing decomposition* of a matrix A is defined in [45] as a decomposition $A = XDY^T$, where X, Y are well conditioned and D is a diagonal matrix. In [45] it was proved that the singular value decomposition can be computed with HRA and efficiently for matrices possessing rank revealing decompositions with HRA. We have mentioned previously the need to reparametrize matrices in order to obtain accurate computations. In the class of diagonally dominant M-matrices, the natural parameters that permit obtaining efficient algorithms with HRA are the off-diagonal entries and the row sums (or the column sums): see [2] and [3]. These parameters can even have a meaningful interpretation when such matrices arise in in the field of digital electrical circuits: the column sums are given by the quotient between the conductance and capacitance of each node (see [2]). For $n \times n$ diagonally dominant M-matrices, an algorithm of [3] computes to HRA the LDU factorization if the off-diagonal entries and the row sums are given. It modifies Gaussian elimination to compute the off-diagonal entries and the row sums of each Schur complement without performing subtractions.

In order to obtain, rank reveling decompositions, pivoting strategies were later used. For a diagonally dominant M-matrix A, a symmetric pivoting leading to an LDU-decomposition of A is equivalent to the following factorization of A: $PAP^T = LDU$, where P is the permutation matrix associated to the pivoting strategy. Symmetric complete pivoting was used in [46] to compute well conditioned L and U factors because U is row diagonally dominant and the off-diagonal entries of L have absolute value less than 1. This factorization is a special case of a rank revealing decomposition. To implement symmetric complete pivoting, the algorithm in [46] calculates all the diagonal entries and all Schur complements and this increases the cost in $\mathcal{O}(n^3)$ flops with respect to standard Gaussian elimination. In [107] another symmetric pivoting strategy (called diagonally dominant pivoting) was used, also with a subtraction-free implementation and a similar computational cost, but improving the conditioning of L because it leads to both triangular matrices L and U column and row diagonally dominant, respectively. In [6], an accurate algorithm for the same LDU-decomposition of [107], but requiring $\mathcal{O}(n^2)$ elementary operations beyond the cost of Gaussian elimination (instead of $\mathcal{O}(n^3)$), was presented. This method can also be applied for diagonally dominant matrices satisfying certain sign patterns: with off-diagonal entries of the same sign or satisfying a chessboard pattern. The problem of computing an accurate LDU decomposition of diagonally dominant matrices from adequate parameters has been solved by Ye in [117], although in this case it is not used a subtraction-free algorithm. Diagonally dominant matrices with arbitrary sign patterns were also considered in [48]. For a class of $n \times n$ nonsingular almost row diagonally dominant Z-matrices, and given adequate parameters, an efficient method to compute its LDU decomposition with HRA is provided in [7]. It adds an additional cost of $\mathcal{O}(n^2)$ elementary operations over the computational cost of Gaussian elimination. In all these cases, we can later apply the method of [45] to calculate all the singular values with HRA.

Let us now recall another class of matrices generalizing diagonally dominant. Let us start by defining the concept of a Nekrasov matrix (see [115]). We can define recursively for a complex matrix $A = (a_{ij})_{1\leq i,j\leq n}$ with $a_{ii} \neq 0$, for all $i = 1, \ldots, n$,

$$(5) \qquad h_1(A) := \sum_{j\neq 1} |a_{1j}|, \;\; h_i(A) := \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=i+1}^{n} |a_{ij}|, \quad i = 2, \ldots, n.$$

We say that A is a *Nekrasov matrix* if $|a_{ii}| > h_i(A)$ for all $i = 1, \ldots, n$. A Nekrasov matrix is a nonsingular H-matrix [115]. Therefore, a Nekrasov Z-matrix with positive diagonal entries is a nonsingular M-matrix. In [98], computations with HRA for the class of Nekrasov Z-matrices were studied. The proposed n^2 parameters used for an $n \times n$ Nekrasov Z-matrix $A = (a_{ij})_{1 \le i,j \le n}$ with positive diagonal are:

$$\begin{cases} a_{ij}, & i \neq j \\ \Delta_j(A) := a_{jj} - h_j(A), & j \in N \end{cases} \tag{6}$$

A is a Nekrasov Z-matrix with positive diagonal if and only if the first $n^2 - n$ parameters are nonpositive and the last n parameters $\Delta_j(A)$ $(j = 1, \ldots, n)$ are positive. In [98], this parametrization was used to compute the inverse of a Nekrasov Z-matrix with positive diagonal with HRA.

Let us now focus on some applications of the tools and classes of matrices considered in this section. For instance, the graph Laplacian matrices (see [95]) are positive semidefinite symmetric diagonally dominant M-matrices with zero row sums and zero column sums. Let us recall that these matrices and their spectral properties have important applications to chemistry, mathematical biology, information theory, quantum graphs or pattern recognition problems.

Diagonal dominance is closely related with the obtention of results for localizing the eigenvalues of a matrix. It is well known that the nonsingularity of a strictly diagonally dominant matrix is equivalent to the first part of the Gerschgorin circles Theorem for the localization of the eigenvalues of a matrix. More general nonsingularity conditions than diagonal dominance lead to sharper localization regions of the eigenvalues. On the other hand, it is also well known that a strictly diagonally dominant matrix with positive diagonal entries has positive determinant. In [15], it was proved that a matrix with positive row sums and all its off-diagonal elements bounded above by their corresponding row means has also positive determinant. This condition was used in [103] for the localization of the real eigenvalues of real matrices, which complement the information provided by the Gerschgorin circles. Sharper conditions were obtained in [104, 28, 29].

The next application corresponds to the field of optimization. Let us recall the linear complementarity (LC) problem. The LC problem consists of finding vectors $x \in \mathbf{R}^n$ satisfying

$$Mx + q \ge 0, \quad x \ge 0, \quad x^T(Mx + q) = 0, \tag{7}$$

where M is an $n \times n$ real matrix and $q \in \mathbf{R}^n$. We denote this problem by LCP(M, q) and its solutions by x^*. A linear complementarity problem has always a unique solution if and only if the associate matrix M is a P-matrix. Many problems can be posed in the form (7): problems in linear and quadratic programming, the problem of finding a Nash equilibrium point of a bimatrix game or some free boundary problems of fluid mechanics (see Chapter 10 of [12], [94] and [30], and references therein). It is well-known that an H-matrix with positive diagonals is a P-matrix (see, for instance, Theorem 2.3 of Chapter 6 of [12]) and that a strictly diagonally dominant matrix is an H-matrix. In [94], error bounds for $\|x - x^*\|$ were derived when M in (1.1) is a P-matrix. When M in (7) is an H-matrix with positive diagonals, sharper error bounds were obtained in [26]. Sharper bounds can be obtained for particular subclasses of H-matrices or P-matrices: see, for instance, [56, 59], [57, 99] for Nekrasov matrices, or [58] for B_π^R-matrices. The class of B_π^R-matrices (introduced in [97]) has also been extended in [100] to define a new class of tensors. Tensors (also called hypermatrices) provide, joint with the Kronecker product of matrices, a very useful tool for the treatment of Big Data (see also [81, 111]).

5 Totally positive matrices: computations with HRA and applications

Let us recall that totally positive (TP) matrices are matrices whose minors are all nonnegative (see Section 3, where STP matrices are also defined). These matrices present important applications (see [72], [55], [5], [60], [51], [112]) in many fields such as Approximation Theory, Biology, Economics, Combinatorics, Statistics, Differential Equations, Mechanics or Computer Aided Geometric Design (CAGD). TP and STP matrices satisfy some remarkable properties, such as the variation diminishing property (see Section 5 of [5]), which is fundamental in their applications. They also satisfy nice spectral properties (see Section 6 of [5]), for instance the nonnegativity of the eigenvalues of TP matrices or the positivity of the eigenvalues of STP matrices

The parametrization of TP matrices leading to HRA algorithms is provided by their bidiagonal factorizations, which are in turn closely related to an elimination procedure known as Neville elimination. In some papers by M. Gasca and G. Mühlbach ([61], for example) on the relationship between interpolation formulas and elimination techniques, it became clear that what they called *Neville elimination* had special interest for TP matrices. It is a procedure to make zeros in a column of a matrix by adding to each row an appropriate multiple of the previous one and had been already used in some of the first papers on TP matrices. However, in later papers such as [62] and [63], a better knowledge of the properties of Neville elimination was developed and permitted to improve many previous results on those matrices. Given a nonsingular matrix $A = (a_{ij})_{1\le i,j\le n}$, the *Neville elimination* (NE) procedure consists of $n-1$ steps and leads to the following sequence of matrices:

$$A =: A^{(1)} \to \widetilde{A}^{(1)} \to A^{(2)} \to \widetilde{A}^{(2)} \to \cdots \to A^{(n)} = \widetilde{A}^{(n)} = U, \tag{8}$$

where U is an upper triangular matrix. The matrix $\widetilde{A}^{(k)} = (\widetilde{a}_{ij}^{(k)})_{1\le i,j\le n}$ is obtained from the matrix $A^{(k)} = (a_{ij}^{(k)})_{1\le i,j\le n}$ by a row permutation that moves to the bottom the rows with a zero entry in column k below the main diagonal. For nonsingular TP matrices, it is always possible to perform NE without row exchanges (see [62]). If a row permutation is not necessary at the k-th step, we have that $\widetilde{A}^{(k)} = A^{(k)}$. The entries of $A^{(k+1)} = (a_{ij}^{(k+1)})_{1\le i,j\le n}$ can be obtained from $\widetilde{A}^{(k)} = (\widetilde{a}_{ij}^{(k)})_{1\le i,j\le n}$ using the formula:

$$a_{ij}^{(k+1)} = \begin{cases} \widetilde{a}_{ij}^{(k)} - \frac{\widetilde{a}_{ik}^{(k)}}{\widetilde{a}_{i-1,k}^{(k)}}\widetilde{a}_{i-1,j}^{(k)}, & \text{if } k \le j < i \le n \text{ and } \widetilde{a}_{i-1,k}^{(k)} \neq 0, \\ \widetilde{a}_{ij}^{(k)}, & \text{otherwise,} \end{cases} \tag{9}$$

for $k = 1, \ldots, n-1$. The (i,j) *pivot* of the NE of A is given by

$$p_{ij} = \widetilde{a}_{ij}^{(j)}, \quad 1 \le j \le i \le n.$$

If $i = j$ we say that p_{ii} is a *diagonal pivot*. The (i,j) *multiplier* of the NE of A, with $1 \le j \le i \le n$, is defined as

$$m_{ij} = \begin{cases} \frac{\widetilde{a}_{ij}^{(j)}}{\widetilde{a}_{i-1,j}^{(j)}} = \frac{p_{ij}}{p_{i-1,j}}, & \text{if } \widetilde{a}_{i-1,j}^{(j)} \neq 0, \\ 0, & \text{if } \widetilde{a}_{i-1,j}^{(j)} = 0. \end{cases}$$

The multipliers satisfy that

$$m_{ij} = 0 \Rightarrow m_{hj} = 0 \quad \forall h > i.$$

Pivots and multipliers of the NE of A and A^T characterize nonsingular TP and STP matrices, as the following result shows. It follows from theorems 4.1 and 4.2 of [63] and p. 116 of [63].

Theorem 5.1 *A matrix A is nonsingular TP (STP, respectively) if and only if the NE of A and A^T can be performed without row exchanges, all the mutipliers of the NE of A and A^T are nonnegative (positive, respectively) and all the diagonal pivots of the NE of A are positive.*

A direct consequence of the well known Cauchy-Binet identity for minors of the product of matrices (see formula (1.23) of [5]) is that the product of TP matrices is again a TP matrix. Therefore, one of the topics in the literature of TP matrices has been their decomposition as products of simpler TP matrices. In particular, in view of applications, the most interesting factorization seems to be in terms of bidiagonal nonnegative matrices which, obviously, are always TP matrices. In addition, this factorization provides the mentioned parametrization for the HRA algorithms with TP matrices. In fact, nonsingular TP matrices can be expressed as a product of nonnegative bidiagonal matrices using again the pivots and multipliers of the NE of A and A^T. The following theorem (see Theorem 4.2 and p. 120 of [63]) presents this factorization, which is called the *bidiagonal decomposition*.

Theorem 5.2 *(cf. Theorem 4.2 of [63]) Let* $A = (a_{ij})_{1\le i,j\le n}$ *be a nonsingular TP matrix. Then* A *admits the following representation:*

$$A = F_{n-1}F_{n-2}\cdots F_1 D G_1 \cdots G_{n-2}G_{n-1}, \tag{10}$$

where D *is the diagonal matrix* $\mathrm{diag}(p_{11},\ldots,p_{nn})$ *with positive diagonal entries and* F_i, G_i *are the nonnegative bidiagonal matrices given by*

$$F_i = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & m_{i+1,1} & 1 & & \\ & & & & \ddots & \ddots & \\ & & & & & m_{n,n-i} & 1 \end{pmatrix}, \tag{11}$$

$$G_i = \begin{pmatrix} 1 & 0 & & & & & \\ & 1 & \ddots & & & & \\ & & \ddots & 0 & & & \\ & & & 1 & \widetilde{m}_{i+1,1} & & \\ & & & & 1 & \ddots & \\ & & & & & \ddots & \widetilde{m}_{n,n-i} \\ & & & & & & 1 \end{pmatrix}, \tag{12}$$

for all $i \in \{1, \ldots, n-1\}$. If, in addition, the entries m_{ij} and $\widetilde{m}_{ij}$ satisfy

$$\begin{aligned} m_{ij} = 0 \Rightarrow m_{hj} = 0 \quad \forall h > i, \\ \widetilde{m}_{ij} = 0 \Rightarrow \widetilde{m}_{hj} = 0 \quad \forall h > i, \end{aligned} \tag{13}$$

then the decomposition is unique.

In the bidiagonal decomposition given by (10), (11) and (12), the entries m_{ij} and p_{ii} are the multipliers and diagonal pivots, respectively, corresponding to the NE of A (see Theorem 4.2 of [63] and the comment below it) and the entries $\widetilde{m}_{ij}$ are the multipliers of the NE of A^T (see p. 116 of [63]). In [76] the following matrix notation $\mathcal{BD}(A)$ was introduced to represent the bidiagonal decomposition of a nonsingular TP matrix

$$(\mathcal{BD}(A))_{ij} = \begin{cases} m_{ij}, & \text{if} i > j, \\ \widetilde{m}_{ji}, & \text{if} i < j, \\ p_{ii}, & \text{if} i = j. \end{cases} \tag{14}$$

In the particular case that the nonsingular $n \times n$ TP matrix A is also stochastic (the entries of every row sum up to 1), the bidiagonal factorization of A can be transformed into a bidiagonal factorization of $n-1$ lower triangular bidiagonal stochastic matrices and $n-1$ upper triangular bidiagonal stochastic matrices. This idea had been used in important applications of several fields. For instance, through this factorization Frydman and Singer ([54], Theorem 1) showed that the class of transition matrices for the finite state time-inhomogeneous birth and death processes coincides with the class of nonsingular stochastic TP matrices. The fact that those transition matrices for birth and death processes are all stochastic TP had already been pointed out inby Karlin and Mc Gregor (see [73] and [74]) with probabilistic arguments. All these results have been surveyed in 1986 by G. Goodman [64], who extended them to compound matrices, that is, matrices whose elements are the values of the minors of a certain order m of a given matrix A. On the other hand, factorizations of stochastic TP matrices as product of bidiagonal stochastic TP matrices are also important in the field of Computer Aided Geometric Design (CAGD). In fact, the main family of algorithms used in this field, called corner cutting algorithms, can be represented in this way. In [66], Goodman and Micchelli showed, again through the mentioned factorization of stochastic TP matrices, that the existence of a corner cutting algorithm transforming a control polygon of a curve into another one with the same number of vertices was equivalent to the fact that both polygons were related by a nonsingular stochastic TP matrix.

More recently, the bidiagonal factorization has been used to compute accurately with these matrices (see [75, 76]). In fact, if we have the $\mathcal{BD}(A)$ of a nonsingular TP matrix with HRA, then we can perform many computations of A with HRA, such as computing its inverse or computing its eigenvalues or its singular values (cf. [76, 77]). There are some subclasses of nonsingular TP matrices for which this factorization can be obtained to HRA, and so, the computations mentioned previously can be also performed with HRA. For instance, for Vandermonde positive matrices [47], for Cauchy-Vandermonde positive matrices [93], arising in rational interpolation and that include the famous example of ill-conditioned matrices given by Hilbert matrices, or for Schoenmakers-Coffey matrices [33], arising in Finance. From now on, we shall illustrate other subclasses of TP matrices for which the $\mathcal{BD}(A)$ can be computed with HRA or for which we can perform computations with HRA.

5.1 Bidiagonal decomposition and HRA with Pascal matrices

A *Pascal matrix* of order n is the symmetric matrix

$$P = (p_{ij})_{1\leq i,j\leq n}; \quad p_{ij} := \binom{i+j-2}{j-1}. \tag{15}$$

Pascal matrices have a long history (cf. [1, 49, 83]) and arise in important applications in filter design and image and signal processing (cf. [49]), as well as in probability, combinatorics, numerical analysis and electrical engineering (cf. [13]), among other fields.

Conditioning and the bidiagonal factorization of Pascal matrices was analyzed in [4] (see also [76]). The $\mathcal{BD}(P)$ can be obviously computed with HRA because it is extremely simple:

$$\mathcal{BD}(P) = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}.$$

5.2 Bidiagonal decomposition and HRA with rational Bernstein-Vandermonde and related matrices

A collocation matrix of the Bernstein basis (1) is called a *Bernstein-Vandermonde matrix*. The bidiagonal factorization and computations with HRA of Bernstein-

Vandermonde matrices were obtained in [91]. The Said-Ball basis is another interesting basis used in CAGD and their collocation matrices are called Said-Ball-Vandermonde matrices. Their bidiagonal factorization and computations with HRA were obtained in [92]. Now we shall consider the corresponding rational bases and matrices, whose bidiagonal factorizations and computations with HRA were considered in [39].

Given a basis $u = (u_0^n, \dots, u_n^n)$ of nonnegative functions on $[a, b]$ and a sequence of strictly positive weights $(w_i)_{i=0}^n$, we can construct a rational basis $r = (r_0^n, \dots, r_n^n)$ defined by

$$r_i^n(t) = \frac{w_i\, u_i^n(t)}{W(t)}, \quad t \in [a, b], \quad i \in \{0, 1, \dots, n\}, \tag{16}$$

where $W(t) = \sum_{j=0}^n w_j\, u_j^n(t)$. If the initial basis u is the Bernstein basis, then the corresponding rational basis r is called the *rational Bernstein basis*. In CAGD, the usual representation of a polynomial curve is the so called Bernstein-Bézier form, that is, these curves are expressed in terms of the Bernstein basis (1). The Bernstein basis is a rational Bernstein basis with all weights equal to 1: $w_i = 1$ for all $i = 0, \dots, n$. Bidiagonal decompositions and computations with HRA have been also obtained for other bases used in CAGD and closely related with the Bernstein basis. For instance, for the q-Bernstein basis [41], for the Lupaş basis [43] or the Bernstein-like bases (which are normalized B-bases, see Section 3) of the spaces mixing algebraic, trigonometric and hyperbolic polynomials [89].

The corresponding square collocation matrices of the rational Bernstein basis at a sequence of parameters $0 < t_0 < t_1 < \ldots < t_n < 1$, given by $(r_j^n(t_i))_{0 \le i,j \le n}$, where functions r_i^n are given by (16) with $u_i^n = b_i^n$ for $i = 0, 1, \dots, n$, will be called *rational Bernstein-Vandermonde* (RBV) matrices.

In [39], the HRA calculations with RBV matrices through their bidiagonal decompositions and those of their inverses in terms of the diagonal pivots and multipliers of their Neville elimination and the multipliers of the Neville elimination of their transposes were obtained. We now recall the bidiagonal decomposition of a RBV matrix.

Theorem 5.3 *Let $A = (w_j b_j^n(t_i)/W(t_i))_{0 \le i,j \le n}$ be a RBV matrix whose nodes satisfy $0 < t_0 < t_1 < \cdots < t_n < 1$. Then A admits a factorization of the form*

$$A = \overline{F}_n \overline{F}_{n-1} \cdots \overline{F}_1 D \overline{G}_1 \cdots \overline{G}_n, \tag{17}$$

where $\overline{F}_i$ and $\overline{G}_i$, $i \in \{1,\ldots,n\}$, are the lower and upper triangular bidiagonal matrices given by

$$\overline{F}_i = \begin{pmatrix} 1 & & & & & & & & \\ 0 & 1 & & & & & & & \\ & \ddots & \ddots & & & & & & \\ & & 0 & 1 & & & & & \\ & & & m_{i0} & 1 & & & & \\ & & & & m_{i+1,1} & 1 & & & \\ & & & & & \ddots & \ddots & & \\ & & & & & & m_{n,n-i} & 1 \end{pmatrix},$$

$$\overline{G}_i^T = \begin{pmatrix} 1 & & & & & & & & \\ 0 & 1 & & & & & & & \\ & \ddots & \ddots & & & & & & \\ & & 0 & 1 & & & & & \\ & & & \widetilde{m}_{i0} & 1 & & & & \\ & & & & \widetilde{m}_{i+1,1} & 1 & & & \\ & & & & & \ddots & \ddots & & \\ & & & & & & \widetilde{m}_{n,n-i} & 1 \end{pmatrix},$$

and D the diagonal matrix $diag(p_{00}, p_{11} \ldots, p_{nn})$. The entries m_{ij}, $\widetilde{m}_{ij}$ and p_{ii} are given by

$$m_{ij} = \tfrac{W(t_{i-1})}{W(t_i)} \tfrac{(1-t_i)^{n-j}(1-t_{i-j-1})}{(1-t_{i-1})^{n-j+1}} \tfrac{\prod_{k=i-j}^{i-1}(t_i-t_k)}{\prod_{k=i-j-1}^{i-2}(t_{i-1}-t_k)}, \quad \text{for } 0 \le j < i \le n, \tag{18}$$

$$\widetilde{m}_{ij} = \tfrac{w_i}{w_{i-1}} \tfrac{n-i+1}{i} \tfrac{t_j}{1-t_j}, \quad \text{for } 0 \le j < i \le n, \tag{19}$$

$$p_{ii} = \tfrac{w_i}{W(t_i)} \tbinom{n}{i} \tfrac{(1-t_i)^{n-i}}{\prod_{k=0}^{i-1}(1-t_k)} \prod_{k=0}^{i-1}(t_i - t_k), \quad \text{for } 0 \le i \le n. \tag{20}$$

5.3 HRA with Jacobi-Stirling matrices Matrices

In [50], the Jacobi-Stirling numbers were presented as the coefficients of the integral composite powers of the Jacobi differential operator

$$I_{\alpha,\beta}[y](t) = \frac{1}{(1-t)^{\alpha}(1+t)^{\beta}} \left(-(1-t)^{\alpha+1}(1+t)^{\beta+1}y'(t)\right)', \tag{21}$$

with α, β real numbers greater than -1. The Jacobi-Stirling numbers $JS_n^{(j)}(z)$ of the second kind only depend on the parameter $z = \alpha + \beta + 1 (> -1)$ and satisfy the following recurrence relation

$$JS_n^{(j)}(z) = JS_{n-1}^{(j-1)}(z) + j(j+z)\, JS_{n-1}^{(j)}(z) \quad (n, j \geq 1), \tag{22}$$

$$JS_n^{(0)}(z) = JS_0^{(j)}(z) = 0, \quad JS_0^{(0)}(z) = 1. \tag{23}$$

Again, the Jacobi-Stirling numbers $Jc_n^{(j)}(z)$ of the first kind only depend on the parameter $z = \alpha + \beta + 1$ and satisfy the following recurrence relation

$$Jc_n^{(j)}(z) = Jc_{n-1}^{(j-1)}(z) + (n-1)(n-1+z)\, Jc_{n-1}^{(j)}(z) \quad (n, j \geq 1), \tag{24}$$

$$Jc_n^{(0)}(z) = Jc_0^{(j)}(z) = 0, \quad Jc_0^{(0)}(z) = 1. \tag{25}$$

The Jacobi-Stirling numbers $Jc_n^{(j)}(z)$ of the first kind are a generalization of the Legendre-Stirling numbers because for $z = 1$ we obtain the Legendre-Stirling numbers.

In Theorem 4.2 of [96] the Jacobi-Stirling numbers of the second kind $JS_n^{(j)}$ were defined via the following expansion of the n-th composite power of $I_{\alpha,\beta}[y](t)$:

$$(1-t)^{\alpha}(1+t)^{\beta} I_{\alpha,\beta}[y](t) = \sum_{j=0}^{n} (-1)^j (JS_n^{(j)}(\alpha+\beta+1)(1-t)^{\alpha+j}(1+t)^{\beta+j} y^{(j)}(t))^{(k)},$$

where $I_{\alpha,\beta}[y](t)$ is the Jacobi differential operator (21).

The Jacobi-Stirling numbers $JS_n^{(j)}(z)$ of the second kind satisfy

$$x^n = \sum_{j=0}^{n} JS_n^{(j)}(z) \left\langle x \right\rangle_j (z) \quad (n \in \mathbf{N}),$$

where

$$\left\langle x \right\rangle_j (z) := \prod_{i=0}^{j-1} (x - i(i+z))$$

for all $j \geq 1$ and $\left\langle x \right\rangle_0 (z) := 1$. The (unsigned) Jacobi-Stirling numbers of the first kind $Jc_n^{(j)}(z)$ are defined via

$$\left\langle x \right\rangle_n (z) = \sum_{j=0}^{n} (-1)^{n+j} Jc_n^{(j)}(z) x^j \quad (n \in \mathbf{N}).$$

Here we consider the infinite matrices $JS(z) = (JS_i^{(j)}(z))_{i,j\geq 0}$ and $Jc(z) = (Jc_i^{(j)}(z))_{i,j\geq 0}$ and their corresponding truncated matrices given by the formulas $JS_n(z) = (JS_i^{(j)}(z))_{0\leq i,j\leq n-1}$ and $Jc_n(z) = (Jc_i^{(j)}(z))_{0\leq i,j\leq n-1}$ formed by the Jacobi-Stirling numbers of the first and second kind, respectively. In [40], the HRA calculation of singular values and inverses of the matrices $JS_n(z)$ and $Jc_n(z)$ was presented through their bidiagonal decomposition.

The following result is a consequence of Proposition 4 of [96] and states the bidiagonal decomposition of the matrices $JS_n(z)$.

Theorem 5.4 *The Jacobi-Stirling matrix $JS_n(z)$, $n \in \mathbf{N}$, admits a factorization of the form*

$$JS_n(z) = \overline{G}_1^2 \cdots \overline{G}_{n-1}^2, \tag{26}$$

where $\overline{G}_i^2$, $i \in \{1, \ldots, n-1\}$, are the $n \times n$ upper bidiagonal triangular matrices given by

$$\overline{G}_i^2 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \ddots & \ddots & & & & & \vdots \\ & & 1 & 0 & & & & \vdots \\ & & & 1 & m_{i+1,1} & & & \vdots \\ & & & & 1 & m_{i+2,2} & & \vdots \\ & & & & & \ddots & \ddots & 0 \\ & & & & & & 1 & m_{n,n-i} \\ & & & & & & & 1 \end{pmatrix}, \tag{27}$$

where $m_{ij} = j(z+j)$ for $1 \leq j < i \leq n$.

The next result is also a consequence of Proposition 4 of [96] and provides the bidiagonal decomposition of the matrices $Jc_n(z)$.

Theorem 5.5 *The Jacobi-Stirling matrix $Jc_n(z)$, $n \in \mathbf{N}$, admits a factorization of the form*

$$Jc_n(z) = \overline{G}_1^1 \cdots \overline{G}_{n-1}^1, \tag{28}$$

where $\overline{G}_i^1$, $i \in \{1, \ldots, n-1\}$, are the $n \times n$ upper bidiagonal triangular matrices given by

$$\overline{G}_i^1 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \ddots & \ddots & & & & & \vdots \\ & & 1 & 0 & & & & \vdots \\ & & & 1 & \overline{m}_{i+1,1} & & & \vdots \\ & & & & 1 & \overline{m}_{i+2,2} & & \vdots \\ & & & & & \ddots & \ddots & 0 \\ & & & & & & 1 & \overline{m}_{n,n-i} \\ & & & & & & & 1 \end{pmatrix}, \tag{29}$$

where $\overline{m}_{ij} = (i-j)(z+i-j)$ for all $1 \leq j < i \leq n$.

5.4 HRA with Laguerre matrices and Lah numbers

Laguerre polynomials form a classical family of orthogonal polynomials (cf. [11]) and present many applications. For instance, they are used for Gaussian quadrature to numerically compute integrals. The larger family of generalized Laguerre polynomials presents important applications in quantum mechanics (see [78]). For $\alpha > -1$, the generalized Laguerre polynomials are given by

$$L_n^{(\alpha)}(t) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{t^k}{k!}, \quad n = 0, 1, 2, \ldots \tag{30}$$

and they are orthogonal polynomials on $[0, \infty)$ with respect to the weight function $x^\alpha e^{-x}$.

Given a real number x and a positive integer k, let us denote the corresponding *falling factorial* by

$$x^{k)} := x(x-1)(x-2)\cdots(x-k+1).$$

Let us also denote $x^{0)} := 1$. Let $M := \left(L_{j-1}^{(\alpha)}(t_{i-1})\right)_{1\le i,j\le n+1}$ be the collocation matrix of the generalized Laguerre polynomials at $(0 >)t_0 > t_1 > \ldots > t_n$, let P_U be the $(n+1)\times(n+1)$ upper triangular Pascal matrix with $\binom{j-1}{i-1}$ as its (i, j)-entry for $j \ge i$ and let S_α and J be the $(n+1)\times(n+1)$ diagonal matrices:

$$S_\alpha := \operatorname{diag}\left((\alpha+i)^{i)}\right)_{0\le i\le n}, \quad J := \operatorname{diag}\left((-1)^i\right)_{0\le i\le n}. \tag{31}$$

The following result, corresponding to Theorem 2 of [31], assures that, given the parameters $(0 >)t_0 > t_1 > \ldots > t_n$, many algebraic computations with these collocation matrices M can be performed with HRA, as well as the strict total positivity and a particular factorization of these matrices.

Theorem 5.6 *Let $M := \left(L_{j-1}^{(\alpha)}(t_{i-1})\right)_{1\le i,j\le n+1}$ for $(0 >)t_0 > t_1 > \ldots > t_n$ with $\alpha > -1$, let P_U be the $(n+1)\times(n+1)$ upper triangular Pascal matrix, let S_α and J be the $(n+1)\times(n+1)$ diagonal matrices given by (31) and let $V := (t_{i-1}^{j-1})_{1\le i,j\le n+1}$. Then $M = VJS_\alpha^{-1}P_U S_0^{-1}S_\alpha$ is an STP matrix and, given the parametrization t_i $(0 \le i \le n)$, the following computations can be performed with HRA: all the eigenvalues, all the singular values and the inverse of M, as well as the solution of the linear systems $Mx = b$, where $b = (b_0, \ldots, b_n)^T$ has alternating signs.*

The particular case $\alpha = 0$ corresponds to the classical Laguerre polynomials. Extending (30) to the case $\alpha = -1$, it was obtained in [31] an analogous result to

Theorem 5.6 for the particular set of polynomials:

$$L_0^{(-1)}(t) = 1, \quad L_n^{(-1)}(t) = \sum_{k=1}^{n} (-1)^k \binom{n-1}{n-k} \frac{t^k}{k!}, \; n =, 1, 2, \ldots,$$

The interest of these polynomials arises from the close relationship between their coefficients and the unsigned Lah numbers (cf. [14]), and which will be described below:

$$L_n^{(-1)}(t) = \tfrac{1}{n!} \textstyle\sum_{k=1}^{n} (-1)^k L(n,k) t^k \text{ for } n \geq 1$$
$$\text{with} \quad L(n,k) := \tbinom{n-1}{k-1} \tfrac{n!}{k!}, k \leq n.$$

The unsigned Lah numbers $L(n, k)$ are included as the sequence A105278 in the Online Encyclopedia of Integer Sequences (OEIS). The Lah numbers were introduced by Ivo Lah in 1955 (see [80]) and arise in applications such as combinatorics and analysis (see pages 44–45 of [113]).

5.5 HRA with Bessel matrices

Bessel polynomials arise in many fields such as partial differential equations, number theory, algebra and statistics (see [69]). They form an orthogonal sequence of polynomials and are related to the modified Bessel function of the second kind (cf. pp. 7 and 34 of [69]). They are also closely related to the reverse Bessel polynomials, with many applications in Electrical Engineering, in network analysis of electrical circuits (cf page 145 of [69]). The coefficients of the reverse Bessel polynomials are also known, in Combinatorics, as signless Bessel numbers of the first kind. The Bessel numbers are also closely related to the Stirling numbers [70, 116]. Bessel polynomials also occur naturally in the theory of traveling spherical waves (cf. [79]) and are very important for some problems of static potentials, signal processing and electronics. The zeros of Bessel polynomials and generalized Bessel polynomials also play a crucial role in applications in Electrical Engineering and are related with the length of the parameter domain where cycloidal spaces admit shape preserving representations ([22]).

Let us recall that the *Bessel polynomials* are defined by

$$B_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{2^k (n-k)! k!} x^k, \quad n = 0, 1, 2 \ldots. \tag{32}$$

Given a real positive integer n, let us define the corresponding *semifactorial* by

$$n!! = \prod_{k=0}^{[n/2]-1} (n - 2k).$$

Let $A = (a_{ij})_{1\le i,j\le n}$ be the lower triangular matrix such that

$$(B_0(x), B_1(x), \dots, B_{n-1}(x))^T = A(1, x, \dots, x^{n-1})^T, \tag{33}$$

that is, the lower triangular matrix A is defined by

$$a_{ij} := \begin{cases} \frac{(i+j-2)!}{2^{j-1}(i-j)!(j-1)!} = \frac{(2j-2)!}{2^{j-1}(j-1)!}\binom{i+j-2}{i-j}, & \text{if } i \ge j, \\ 0, & \text{if } i < j. \end{cases} \tag{34}$$

We now recall Theorem 3 of [32], which proves the total positivity of A, and provides $\mathcal{BD}(A)$.

Theorem 5.7 *Let $A = (a_{ij})_{1\le i,j\le n}$ be the lower triangular matrix in (33) defined by (34). Then we have that*

(i) the pivots of the NE of A are given by

$$p_{ij} = \frac{1}{2^{j-1}}\frac{(i-1)!}{(i-j)!}\prod_{r=1}^{j-1}\frac{(2i-r-1)}{(i-j+r)}, \quad 1 \le j \le i \le n, \tag{35}$$

and the multipliers by

$$m_{ij} = \frac{(2i-2)(2i-3)}{(2i-j-1)(2i-j-2)}, \quad 1 \le j < i \le n, \tag{36}$$

(ii) A is a nonsingular TP matrix

(iii) and the bidiagonal factorization of A is given by

$$\mathcal{BD}(A)_{ij} = \begin{cases} \frac{(2i-2)(2i-3)}{(2i-j-1)(2i-j-2)}, & \text{if } i > j, \\ 1, & \text{if } i = j = 1, \\ (2i-3)!!, & \text{if } i = j > 1, \\ 0, & \text{if } i < j, \end{cases} \tag{37}$$

and can be computed to HRA.

Let us now introduce the collocation matrices of the Bessel polynomials. Given a sequence of parameters $0 < t_0 < t_1 < \cdots < t_{n-1}$, we call the collocation matrix of the Bessel polynomials $(B_0, \dots, B_{n-1})$ at this sequence of parameters,

$$M = M\begin{pmatrix} B_0, \dots, B_{n-1} \\ t_0, \dots, t_{n-1} \end{pmatrix} = (B_{j-1}(t_{i-1}))_{1\le i,j\le n},$$

a *Bessel matrix*.

The following result corresponds to Theorem 4 of [32] and shows that the Bessel matrices are STP and that some usual algebraic problems with these matrices can be solved to HRA.

Theorem 5.8 *Given a sequence of parameters* $0 < t_0 < t_1 < \cdots < t_{n-1}$*, the corresponding Bessel matrix* M *is an STP matrix and given the parametrization* t_i *(*$0 \le i \le n-1$*), the following computations can be performed with HRA: all the eigenvalues, all the singular values, the inverse of the Bessel matrix* M*, and the solution of the linear systems* $Mx = b$*, where* $b = (b_1, \ldots, b_n)^T$ *has alternating signs.*

By reversing the order of the coefficients of $B_n(x)$ in (32), we can introduce the *reverse Bessel polynomials*:

$$B_n^r(x) = \sum_{k=0}^{n} \frac{(n+k)!}{2^k (n-k)!k!} x^{n-k}, \quad n = 0, 1, 2 \ldots, \tag{38}$$

Given a sequence of parameters $0 < t_0 < t_1 < \cdots < t_{n-1}$ we call the collocation matrix of the reverse Bessel polynomials $(B_0^r, \ldots, B_{n-1}^r)$ at that sequence,

$$M_r = M \begin{pmatrix} B_0^r, \ldots, B_{n-1}^r \\ t_0, \ldots, t_{n-1} \end{pmatrix} = (B_{j-1}^r(t_{i-1}))_{1 \le i,j \le n}$$

a *reverse Bessel matrix*.

The following result, which corresponds to Theorem 6 of [32], shows that the reverse Bessel matrices are STP and that some usual algebraic problems with these matrices can be solved to HRA.

Theorem 5.9 *Given a sequence of parameters* $0 < t_0 < t_1 < \cdots < t_{n-1}$*, the corresponding reverse Bessel matrix* M_r *is an STP matrix and given the parametrization* t_i *(*$0 \le i \le n-1$*), the following computations can be performed with HRA: all the eigenvalues, all the singular values, the inverse of the reverse Bessel matrix* M_r*, and the solution of the linear systems* $M_r x = b$*, where* $b = (b_1, \ldots, b_n)^T$ *has alternating signs.*

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