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Special Hermitian metrics, complex nilmanifolds and holomorphic deformations

Luis Ugarte

Departamento de Matemáticas - IUMA

Universidad de Zaragoza. 50009 Zaragoza, Spain

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Abstract

We focus on the interaction of several complex invariants of cohomological type and metric properties of compact complex manifolds, as well as their behaviour under holomorphic deformations of the complex structure. Recent results on the complex geometry of nilmanifolds concerning such properties are reviewed. We show that the complex geometry of the 6-dimensional manifold $N \times N$ given by the product of two copies of the Heisenberg nilmanifold N allows to construct a holomorphic deformation with interesting properties in its central limit.

1 Introduction

In this paper we consider compact complex manifolds (M, J) with special Hermitian metrics, mainly balanced and strongly Gauduchon metrics. We focus on the interaction of the existence properties of such metrics and several complex invariants of cohomological type which are related to the $\partial\bar{\partial}$-lemma condition. An important problem is the study of the deformation limits of these properties under holomorphic deformations of the complex structure. We show that the class of complex nilmanifolds provides a very rich source of explicit examples of analytic families of compact complex manifolds with interesting and unusual behaviours in their central fibres.

In Section 2.1 we first recall the definition and the main properties of some complex invariants of cohomological type on a compact complex manifold (M, J) of complex dimension n which are related to the $\partial\bar{\partial}$-lemma condition, namely $\mathbf{f}_k(M, J)$ for $0 \leq k \leq n$

and $\mathbf{k}_r(M,J)$ for $r \geq 1$. Such complex invariants have been introduced in [7, 26, 41] and they are defined by means of the Bott-Chern cohomology $H^{p,q}_{\mathrm{BC}}(M,J)$, the Aeppli cohomology $H^{p,q}_{\mathrm{A}}(M,J)$ and the terms in the Frölicher spectral sequence $\{E_r(M,J)\}_{r\geq 1}$ (see below for details). In Section 2.2 we consider some special Hermitian metrics on compact complex manifolds. It is well known that the existence of a Kähler metric imposes strong topological conditions on the manifold, in particular, (M,J) must satisfy the $\partial\bar\partial$-lemma [16], which in addition implies the degeneration of the Frölicher sequence at E_1 and the formality of the manifold. On the other hand, Gauduchon proved in [21] that in the conformal class of any Hermitian metric on (M,J) there always exists a Hermitian metric F satisfying $\partial\bar\partial F^{n-1} = 0$. Between the Kähler class and the Gauduchon class, other interesting classes of special Hermitian metrics have been considered in relation to several problems in differential and algebraic geometry. A metric F is called *balanced* if $dF^{n-1} = 0$, and a characterization of the existence of balanced metrics in terms of currents was given in [28]. More recently, Popovici has introduced a new class of Hermitian metrics [32], namely the class of *strongly Gauduchon* (*sG* for short) metrics, in relation to the study of central limits of analytic families of projective manifolds [33]. Such metrics are defined by the condition that ∂F^{n-1} is $\bar\partial$-exact. Section 2.3 is devoted to the class of *sGG* manifolds introduced and investigated in [36], which are defined as those compact complex manifolds whose sG cone coincides with the Gauduchon cone. There are several characterizations of the sGG manifolds, for instance, as those compact complex manifolds (M,J) for which every Gauduchon metric is sG, as well as those (M,J) satisfying the following special case of the $\partial\bar\partial$-lemma: if Ω is a d-closed $(n,n-1)$-form that is ∂-exact, then Ω is also $\bar\partial$-exact. Moreover, in [36] two numerical characterizations of the sGG manifolds are obtained, which involve the Bott-Chern number $h^{0,1}_{\mathrm{BC}}(M,J)$, the Hodge number $h^{0,1}_{\bar\partial}(M,J)$ and the first Betti number $b_1(M)$ of the manifold (see Proposition 2.5).

In Section 3 we address the problems of *openness* and *closedness* of the properties considered in the previous sections under holomorphic deformations of the complex structure. For each k such that $0 \leq k \leq n$, we say that a compact complex manifold (M,J) has the property $\mathcal{F}_k$ if the Angella-Tomassini invariant $\mathbf{f}_k(M,J) = \sum_{p+q=k}(h^{p,q}_{\mathrm{BC}}(M,J) + h^{p,q}_{\mathrm{A}}(M,J)) - 2b_k(M)$ vanishes. By [7], a compact complex manifold (M,J) satisfies the $\partial\bar\partial$-lemma if and only if it has the property $\mathcal{F}_k$ for every k. Similarly, we say that (M,J) has the property $\mathcal{K}$ if the Schweitzer invariant $\mathbf{k}_1(M,J) = h^{1,1}_{\mathrm{BC}}(M,J) + 2h^{0,2}_{\bar\partial}(M,J) - b_2(M)$ vanishes. Compact complex manifolds (M,J) satisfying the $\partial\bar\partial$-lemma necessarily have the property $\mathcal{K}$. The properties $\mathcal{F}_k$ and $\mathcal{K}$ are open, i.e. they are stable under holomorphic deformations of the complex structure. Other properties which are open are the degeneration of the Frölicher spectral sequence at E_1 [24], the sG property [32] and the sGG property [36]. However, the balanced property is not open [2].

It is now known that the closedness of all these properties under holomorphic deformations fails. In [17] it was proved that the property of the Frölicher spectral sequence degenerating at E_1 is not closed under holomorphic deformations. The first example of an analytic family of compact complex manifolds $(X_t)_{t\in\Delta}$, Δ being an open disc around the origin in $\mathbb{C}$, such that the complex invariants $\mathbf{k}_1(X_t) = \mathbf{f}_2(X_t) = 0$ for any $t \neq 0$, but $\mathbf{k}_1(X_0) \neq 0$ and $\mathbf{f}_2(X_0) \neq 0$ (i.e. the properties $\mathcal{K}$ and $\mathcal{F}_k$ for $k = 2$ are not closed) was constructed in [26]. We will denote here this concrete analytic family by $\mathcal{X}$, i.e. $\mathcal{X} = (X_t)_{t\in\Delta}$. Its construction was based on an appropriate deformation of an invariant complex structure on a 6-dimensional nilmanifold. The family $\mathcal{X}$ suggested that the $\partial\bar{\partial}$-lemma might be a non-closed property, as it has later been confirmed by Angella and Kasuya in [6] (see also [19]) by constructing certain holomorphic deformation of the Nakamura solvmanifold. Moreover, the Frölicher spectral sequence of any compact complex manifold in the analytic family $\mathcal{X}$ degenerates at E_1 except for the central fibre [10]. This analytic family $\mathcal{X}$ also allows to show that the sGG property is not closed under holomorphic deformations [36]. Furthermore, the fibres X_t have balanced metric for any $t \in \Delta\backslash\{0\}$, but the central fibre X_0 does not admit any sG metric, so the balanced property and the sG property are not closed [10].

Since the compact complex manifolds X_t in the analytic family $\mathcal{X}$ are given by complex nilmanifolds (M, J_t) where J_t is invariant, in Section 4 we focus on the class of 6-dimensional nilmanifolds M endowed with invariant complex structures J; that is, $M = \Gamma\backslash G$ is a compact quotient of a 6-dimensional simply-connected nilpotent Lie group G by a lattice Γ of maximal rank in G, and J stems naturally from a "complex" structure J on the Lie algebra $\mathfrak{g}$ of G. Such nilmanifolds are classified through their underlying Lie algebras [40] (see Theorem 4.2). A crucial result in the theory of nilmanifolds is Nomizu's theorem [31] which asserts that the de Rham cohomology of M is canonically isomorphic to the cohomology of its underlying Lie algebra $\mathfrak{g}$. Many efforts have been made to achieve a Nomizu's type result for the Dolbeault cohomology and other complex invariants of cohomological type on (M, J), and several advances under additional conditions on the invariant complex structure J can be found in [4, 11, 12, 14, 15, 27, 37, 38, 39]. Such results, together with the classification of invariant complex structures J in dimension 6 obtained in [10], allow to compute the Bott-Chern cohomology groups $H^{p,q}_{\mathrm{BC}}(M, J)$ [5, 26]. The Bott-Chern numbers are given in Tables 1–3 below for any invariant complex structure J (up to isomorphism). On the other hand, we collect in Theorems 4.4, 4.5 and 4.8, and Propositions 4.6 and 4.7, the general results about the Frölicher spectral sequence, the existence of balanced and sG metrics, as well as the sGG condition for nilmanifolds in dimension 6 obtained in [10, 36, 44, 46]. These results allow to conclude that, apart from the obvious implications, most of the previous properties of compact complex manifolds are unrelated.

The real nilmanifold in the analytic family of compact complex manifolds $\mathcal{X}$ mentioned above, is not a product since the Lie algebra underlying the nilmanifold is irreducible. In Section 5 we consider the complex geometry of the product $N \times N$ of two copies of the (3-dimensional) Heisenberg nilmanifold N. This geometry turns out to be surprisingly rich as it allows to construct a new holomorphic family of compact complex manifolds $(N \times N, J_t)$ satisfying similar properties to those of the family $\mathcal{X}$ (see Theorem 5.2 for details). To our knowledge this is the first example of a holomorphic deformation having such properties, constructed on a 6-dimensional product manifold.

2 Complex invariants and Hermitian geometry

In this section we recall the definitions and main properties of some complex invariants of cohomological type on a compact complex manifold (M, J) which are related to the $\partial\bar{\partial}$-lemma condition. Some important classes of special Hermitian metrics on (M, J) are also considered, as well as several relations among them.

2.1 Complex invariants related to the $\partial\bar{\partial}$-lemma

Let (M, J) be a compact complex manifold of complex dimension n and consider $\Omega^{p,q}(M)$ the space of forms of bidegree (p, q) with respect to the complex structure J, i.e. $\Omega^k_{\mathbb{C}}(M) = \oplus_{p+q=k}\Omega^{p,q}(M)$ for $0 \leq k \leq 2n$.

It is well known that the Dolbeault cohomology groups $H^{p,q}_{\bar{\partial}}(M, J)$ of (M, J) are defined by

$$H^{p,q}_{\bar{\partial}}(M, J) = \frac{\ker\{\bar{\partial}\colon \Omega^{p,q}(M) \longrightarrow \Omega^{p,q+1}(M)\}}{\operatorname{im}\{\bar{\partial}\colon \Omega^{p,q-1}(M) \longrightarrow \Omega^{p,q}(M)\}}.$$

These groups are complex invariants of the manifold. The Frölicher spectral sequence $\{E_r(M, J)\}_{r\geq 1}$ of a complex manifold (M, J) is the spectral sequence associated to the double complex $(\Omega^{p,q}(M), \partial, \bar{\partial})$, where $\partial + \bar{\partial} = d$ is the decomposition, with respect to J, of the exterior differential d [20]. The first term $E_1(M, J)$ in the sequence is precisely the Dolbeault cohomology of (M, J), that is, $E^{p,q}_1(M, J) \cong H^{p,q}_{\bar{\partial}}(M, J)$, and after a finite number of steps this sequence converges to the de Rham cohomology of M, i.e. $H^k_{\mathrm{dR}}(M, \mathbb{C}) \cong \oplus_{p+q=k}E^{p,q}_\infty(M, J)$, which is a topological invariant of M. More concretely, for each $r \geq 1$ there is a sequence of homomorphisms d_r

$$\cdots \longrightarrow E^{p-r,q+r-1}_r(M, J) \xrightarrow{d_r} E^{p,q}_r(M, J) \xrightarrow{d_r} E^{p+r,q-r+1}_r(M, J) \longrightarrow \cdots$$

such that $d_r \circ d_r = 0$ and $E^{p,q}_{r+1}(M, J) = \ker d_r/\operatorname{im} d_r$. The homomorphisms d_r are induced by ∂. When $r = 1$ the homomorphism $d_1\colon H^{p,q}_{\bar{\partial}}(M, J) \longrightarrow H^{p+1,q}_{\bar{\partial}}(M, J)$ is given by $d_1([\alpha_{p,q}]) = [\partial\alpha_{p,q}]$, for $[\alpha_{p,q}] \in H^{p,q}_{\bar{\partial}}(M, J)$. For $r = 2$ we have

$$E^{p,q}_2(M, J) = \frac{\{\alpha_{p,q} \in \Omega^{p,q}(M) \mid \bar{\partial}\alpha_{p,q} = 0,\ \partial\alpha_{p,q} = -\bar{\partial}\alpha_{p+1,q-1}\}}{\{\bar{\partial}\beta_{p,q-1} + \partial\gamma_{p-1,q} \mid \bar{\partial}\gamma_{p-1,q} = 0\}},$$

and the homomorphism $d_2\colon E_2^{p,q}(M,J) \longrightarrow E_2^{p+2,q-1}(M,J)$ is given by $d_2([\alpha_{p,q}]) = [\partial\alpha_{p+1,q-1}]$, for $[\alpha_{p,q}] \in E_2^{p,q}(M,J)$. We will focus on compact complex manifolds (M,J) of complex dimension 3, so it is sufficient to describe the spectral sequence up to the third step E_3 because in general $E_r(M,J) \cong E_\infty(M,J)$ for any $r \geq \dim_{\mathbb{C}}(M,J)$ (for general descriptions of d_r and $E_r^{p,q}$ see for example [13]).

In addition to the Dolbeault cohomology groups $H_{\bar\partial}^{p,q}(M,J)$ and the Frölicher terms $E_r^{p,q}(M,J)$, the Bott-Chern and Aeppli cohomologies [1, 8] define additional complex invariants of (M,J) given, respectively, by

$$H_{\rm BC}^{p,q}(M,J) = \frac{\ker\{d\colon \Omega^{p,q}(M) \longrightarrow \Omega_{\mathbb{C}}^{p+q+1}(M)\}}{\operatorname{im}\{\partial\bar\partial\colon \Omega^{p-1,q-1}(M) \longrightarrow \Omega^{p,q}(M)\}},$$

and

$$H_{\rm A}^{p,q}(M,J) = \frac{\ker\{\partial\bar\partial\colon \Omega^{p,q}(M) \longrightarrow \Omega^{p+1,q+1}(M)\}}{\operatorname{im}\{\partial\colon \Omega^{p-1,q}(M) \longrightarrow \Omega^{p,q}(M)\} + \operatorname{im}\{\bar\partial\colon \Omega^{p,q-1}(M) \longrightarrow \Omega^{p,q}(M)\}}.$$

By the Hodge theory developed by Schweitzer in [41], all these complex invariants are finite dimensional and one has the isomorphisms $H_{\rm A}^{p,q}(M,J) \cong H_{\rm BC}^{n-q,n-p}(M,J)$. Notice that $H_{\rm BC}^{q,p}(M,J) \cong H_{\rm BC}^{p,q}(M,J)$ by complex conjugation.

From now on we shall denote by $h_{\rm BC}^{p,q}(M,J)$ the dimension of the cohomology group $H_{\rm BC}^{p,q}(M,J)$. The Hodge numbers will be denoted simply by $h_{\bar\partial}^{p,q}(M,J)$ and the Betti numbers by $b_k(M)$.

For any $r \geq 1$ and for any p,q, there are well-defined natural maps

$$H_{\rm BC}^{p,q}(M,J) \longrightarrow E_r^{p,q}(M,J) \qquad \text{and} \qquad E_r^{p,q}(M,J) \longrightarrow H_{\rm A}^{p,q}(M,J).$$

In general these maps are neither injective nor surjective. However, all the maps are isomorphisms if and only if (M,J) satisfies the $\partial\bar\partial$-lemma [16], that is, for any d-closed form α of pure type on (M,J) the following exactness properties are equivalent:

$$\alpha \text{ is } d\text{-exact} \iff \alpha \text{ is } \partial\text{-exact} \iff \alpha \text{ is } \bar\partial\text{-exact} \iff \alpha \text{ is } \partial\bar\partial\text{-exact}.$$

Therefore, if the $\partial\bar\partial$-lemma is satisfied then the previous invariants coincide and in particular one has the Hodge decomposition $H_{\rm dR}^k(M,\mathbb{C}) \cong \oplus_{p+q=k} H_{\bar\partial}^{p,q}(M,J)$ for any k, where in addition $H_{\bar\partial}^{p,q}(M,J) \cong \overline{H_{\bar\partial}^{q,p}(M,J)}$.

Recently, Angella and Tomassini have introduced in [7] new complex invariants that measure how far the compact complex manifold (M,J) is from satisfying the $\partial\bar\partial$-lemma condition.

Theorem 2.1. *[7] On any compact complex manifold (M,J) of complex dimension n the following inequalities are satisfied:*

$$\sum_{p+q=k} \left(h_{\rm BC}^{p,q}(M,J) + h_{\rm BC}^{n-p,n-q}(M,J)\right) \geq 2b_k(M), \qquad 0 \leq k \leq 2n.$$

Moreover, all these inequalities are equalities if and only if (M,J) satisfies the $\partial\bar\partial$-lemma.

Let us denote by $\mathbf{f}_k(M,J)$ the non-negative integer given by

$$\mathbf{f}_k(M,J) = \sum_{p+q=k} \left(h^{p,q}_{\mathrm{BC}}(M,J) + h^{n-p,n-q}_{\mathrm{BC}}(M,J) \right) - 2b_k(M).$$

By the dualities in the Bott-Chern and de Rham cohomologies, it is clear that $\mathbf{f}_{2n-k}(M,J) = \mathbf{f}_k(M,J)$. Now, for each $0 \leq k \leq n$, we consider the property

$$\mathcal{F}_k = \{\text{the compact complex manifold } (M,J) \text{ satisfies } \mathbf{f}_k(M,J) = 0\}.$$

Hence, by Theorem 2.1 a compact complex manifold (M,J) satisfies the $\partial\bar{\partial}$-lemma if and only if it has the property $\mathcal{F}_k$ for every $k \leq n$.

On the other hand, for any compact complex manifold (M,J) Schweitzer proved in [41, Lemma 3.3] that

$$h^{1,1}_{\mathrm{BC}}(M,J) + 2\, h^{0,2}_{\bar{\partial}}(M,J) \geq b_2(M),$$

and moreover, if (M,J) satisfies the $\partial\bar{\partial}$-lemma then the equality holds. More generally, one has

Proposition 2.2. *[26] If (M,J) is a compact complex manifold then for any $r \geq 1$*

$$h^{1,1}_{\mathrm{BC}}(M,J) + 2 \dim E^{0,2}_r(M,J) \geq b_2(M),$$

where $E^{0,2}_r(M,J)$ denotes the r-step $(0,2)$-term of the Frölicher spectral sequence. Furthermore, if (M,J) satisfies the $\partial\bar{\partial}$-lemma then the above inequalities are all equalities.

From now on, we will denote by $\mathbf{k}_r(M,J)$, $r \geq 1$, the non-negative integer given by

$$\mathbf{k}_r(M,J) = h^{1,1}_{\mathrm{BC}}(M,J) + 2 \dim E^{0,2}_r(M,J) - b_2(M).$$

Therefore, $\mathbf{k}_r(M,J)$ are complex invariants which vanish if the manifold (M,J) satisfies the $\partial\bar{\partial}$-lemma. Notice that $\mathbf{k}_1(M,J) \geq \mathbf{k}_2(M,J) \geq \mathbf{k}_3(M,J) = \mathbf{k}_r(M,J) \geq 0$ for any $r \geq 4$.

In general $\mathbf{k}_1(M,J)$, $\mathbf{k}_2(M,J)$ and $\mathbf{k}_3(M,J)$ do not coincide [26], but the vanishing of $\mathbf{k}_1(M,J)$ implies the vanishing of any other $\mathbf{k}_r(M,J)$. This fact justifies to consider the following property:

$$\mathcal{K} = \{\text{the compact complex manifold } (M,J) \text{ satisfies } \mathbf{k}_1(M,J) = 0\}.$$

Obviously, any compact complex manifold satisfying the $\partial\bar{\partial}$-lemma has the property $\mathcal{K}$.

2.2 Special Hermitian metrics

Let (M,J) be a compact complex manifold of complex dimension n. A Hermitian metric g on (M,J) can be described by means of a positive definite smooth form F on

M of bidegree $(1,1)$ with respect to J. In what follows, we will refer to F as a Hermitian structure or as a Hermitian metric without distinction.

A Hermitian structure is *Kähler* if the form F is closed, that is, F is a symplectic form compatible with the complex structure. It is well known that the existence of a Kähler metric imposes strong topological conditions on the manifold. In particular, (M,J) satisfies the $\partial\bar{\partial}$-lemma [16], which in addition implies the formality of the manifold.

On the other hand, Gauduchon proved [21] that in the conformal class of any Hermitian metric there exists a Hermitian metric F satisfying $\partial\bar{\partial}F^{n-1}=0$. We will refer to a metric satisfying this condition as a *Gauduchon* metric.

Between the Kähler class and the Gauduchon class, other interesting classes of special Hermitian metrics have been considered in relation to several problems in differential and algebraic geometry. A metric F is *balanced* if $dF^{n-1}=0$, and the existence of balanced metrics in terms of currents was investigated in [28]. More recently, Popovici has introduced a new class of Hermitian metrics in relation to the study of the central limit of analytic families of projective manifolds: a metric F is called *strongly Gauduchon* (*sG* for short) if ∂F^{n-1} is $\bar{\partial}$-exact [32, 33].

By the definitions, any Kähler metric is balanced, any balanced metric is sG, and any sG metric is a Gauduchon metric, that is:

$$\text{Kähler} \implies \text{balanced} \implies \text{sG} \implies \text{Gauduchon}.$$

The converses to these implications are not true: for instance, one can find examples in the class of nilmanifolds (see Section 4 for details). However, as it is pointed out in [26], if a compact complex manifold (M,J) satisfies that the natural map

$$\zeta\colon H^{n,n-1}_{\bar{\partial}}(M,J) \longrightarrow H^{n,n-1}_{\mathrm{A}}(M,J), \qquad \zeta([\Omega]_{\bar{\partial}}) := [\Omega]_A \tag{1}$$

is injective (in particular, if the $\partial\bar{\partial}$-lemma is satisfied or if $h^{n,n-1}_{\bar{\partial}}(M,J)=0$) then any Gauduchon metric is an sG metric: in fact, if $\partial\bar{\partial}F^{n-1}=0$ then ∂F^{n-1} defines a class in the Dolbeault cohomology group $H^{n,n-1}_{\bar{\partial}}(M,J)$ such that the Aeppli cohomology class $[\partial F^{n-1}]_A=0$ in $H^{n,n-1}_{\mathrm{A}}(M,J)$, so the injectivity of ζ implies the existence of a complex form α of bidegree $(n,n-2)$ such that $\partial F^{n-1}=\bar{\partial}\alpha$. Therefore, if ζ is injective then by Gauduchon's result there exists an sG metric in the conformal class of any Hermitian metric. Notice that by Serre duality and by the dualities between Aeppli and Bott-Chern cohomologies, the injectivity of ζ implies $h^{0,1}_{\bar{\partial}}(M,J)=\dim H^{n,n-1}_{\bar{\partial}}(M,J)\leq \dim H^{n,n-1}_{\mathrm{A}}(M,J)=h^{0,1}_{\mathrm{BC}}(M,J)$.

In the next section we consider in more detail the compact complex manifolds for which any Gauduchon metric is sG, showing that the latter property is actually equivalent to the injectivity of the map (1).

2.3 The strongly Gauduchon cone

Let (M, J) be a compact complex manifold of complex dimension n. The *Gauduchon cone* of (M, J) is defined in [35] as the open convex cone

$$\mathcal{C}_G(M, J) \subset H_A^{n-1,\, n-1}(M, J)$$

consisting of the (real) Aeppli cohomology classes $[F^{n-1}]_A$ which are $(n-1)$-powers of Gauduchon metrics F on (M, J).

Let us consider the map T, induced by ∂ in cohomology, given by

$$T\,:\, H_A^{n-1,\, n-1}(M, J) \longrightarrow H_{\bar\partial}^{n,\, n-1}(M, J), \qquad T([\Omega]_A) := [\partial\Omega]_{\bar\partial} \tag{2}$$

for any $[\Omega]_A \in H_A^{n-1,\, n-1}(M, J)$. The *strongly Gauduchon cone* (sG cone, for short) was defined in [35] as the intersection of the Gauduchon cone with the kernel of the linear map T, i.e.

$$\mathcal{C}_{sG}(M, J) = \mathcal{C}_G(M, J) \cap \ker T \subset \mathcal{C}_G(M, J) \subset H_A^{n-1,\, n-1}(M, J).$$

Notice that, either all the Gauduchon metrics F for which F^{n-1} belongs to a given Aeppli-Gauduchon class $[F^{n-1}]_A \in \mathcal{C}_G(M, J)$ are sG, or none of them is; that is to say, the sG property is cohomological.

The following class is introduced in [36]: a compact complex manifold (M, J) is said to be an *sGG manifold* if the sG cone of (M, J) coincides with the Gauduchon cone of (M, J), i.e. $\mathcal{C}_{sG}(M, J) = \mathcal{C}_G(M, J)$. Since the kernel of T is a vector subspace of $H_A^{n-1,n-1}(M, J)$, its intersection with $\mathcal{C}_G(M, J)$ leaves the latter unchanged if and only if T vanishes identically.

It is clear that any sGG manifold (M, J) is an sG manifold because every Gauduchon metric F on (M, J) is sG. Also any compact complex manifold satisfying the $\partial\bar\partial$-lemma is sGG because the map ζ given by (1) is injective. Therefore:

$$\partial\bar\partial\text{-manifold} \quad\Longrightarrow\quad \text{sGG manifold} \quad\Longrightarrow\quad \text{sG manifold.}$$

The converses to these implications do not hold in general, and again one can find examples in the class of nilmanifolds (see Section 4).

In [36] two numerical characterizations of the sGG manifolds are obtained. The first one is given in terms of the Bott-Chern number $h_{\rm BC}^{0,1}(M, J)$ and the Hodge number $h_{\bar\partial}^{0,1}(M, J)$.

Theorem 2.3. *[36] On any compact complex manifold (M, J) we have $h_{\rm BC}^{0,1}(M, J) \leq h_{\bar\partial}^{0,1}(M, J)$. Moreover, (M, J) is an sGG manifold if and only if $h_{\rm BC}^{0,1}(M, J) = h_{\bar\partial}^{0,1}(M, J)$.*

The second numerical characterization of sGG manifolds involves the first Betti number $b_1(M)$ and the Hodge number $h_{\bar\partial}^{0,1}(M, J)$.

Theorem 2.4. *[36] On any compact complex manifold* (M,J) *we have* $b_1(M) \leq 2\,h^{0,1}_{\bar\partial}(M,J)$. *Moreover,* (M,J) *is an sGG manifold if and only if* $b_1(M) = 2\,h^{0,1}_{\bar\partial}(M,J)$.

It is well known that a compact complex surface is Kähler if and only if its first Betti number is even (a proof of this fact follows from Kodaira's classification of surfaces, [29] and [42]; see [9] and [25] for a direct proof). Hence, Theorem 2.4 makes the sGG manifolds reminiscent of the compact Kähler surfaces. It is clear that in complex dimension 2, the Kähler and the sGG conditions are equivalent. However, in dimension ≥ 3 the sGG property is much weaker than the Kähler one.

In the proof of Theorem 2.3 (see [36, Theorem 2.1]) it is shown that the map ζ given by (1) is always surjective, and that it is injective if and only if the manifold is sGG. In the following result we sum up the equivalent descriptions of the sGG property discussed above:

Proposition 2.5. *For a compact complex manifold* (M,J), *the following statements are equivalent:*

(*i*) (M,J) *is an sGG manifold;*

(*ii*) *every Gauduchon metric* F *on* (M,J) *is strongly Gauduchon;*

(*iii*) *the map* ζ *given by* (1) *is injective;*

(*iv*) *the map* T *given by* (2) *vanishes identically;*

(*v*) *the following special case of the* $\partial\bar\partial$*-lemma holds: for every d-closed* $(n,n-1)$*-form* Ω *on* (M,J), *if* Ω *is* ∂*-exact, then* Ω *is also* $\bar\partial$*-exact;*

(*vi*) $h^{0,1}_{\rm BC}(M,J) = h^{0,1}_{\bar\partial}(M,J)$;

(*vii*) $b_1(M) = 2\,h^{0,1}_{\bar\partial}(M,J)$.

3 Holomorphic deformations

In this section we address some problems about the behaviour of the properties considered in the previous section under holomorphic deformations of the complex structure.

Let Δ denote an open disc around the origin in $\mathbb{C}$. Following [34, Definition 1.12], a given property $\mathcal{P}$ of a compact complex manifold is said to be *open* under holomorphic deformations if for every holomorphic family of compact complex manifolds $(M,J_t)_{t\in\Delta}$ and for every $t_0 \in \Delta$ the following implication holds:

(M,J_{t_0}) has the property $\mathcal{P} \Longrightarrow (M,J_t)$ has the property $\mathcal{P}$ for all $t\in\Delta$ sufficiently close to t_0.

A given property $\mathcal{P}$ of a compact complex manifold is said to be *closed* under holomorphic deformations if for every holomorphic family of compact complex manifolds $(M, J_t)_{t\in\Delta}$ and for every $t_0 \in \Delta$ the following implication holds:

(M, J_t) has the property $\mathcal{P}$ for all $t \in \Delta\backslash\{t_0\} \Longrightarrow (M, J_{t_0})$ has the property $\mathcal{P}$.

Let us first consider the case when the property $\mathcal{P} = \mathcal{K}$ or $\mathcal{F}_k$ for some fixed k such that $0 \leq k \leq n$. Using the upper-semicontinuity of the Hodge numbers $h^{p,q}_{\bar\partial}(M, J_t)$ and that of the Bott-Chern numbers $h^{p,q}_{\rm BC}(M, J_t)$ as t varies in Δ (proved in [24, Theorem 4] and [41], respectively), it is easy to conclude that such properties are open under holomorphic deformations. In fact, for instance, for $\mathcal{P} = \mathcal{K}$, if $(M, J_t)_{t\in\Delta}$ is such that (M, J_{t_0}) has the property $\mathcal{K}$, then

$$b_2(M) = h^{1,1}_{\rm BC}(M, J_{t_0}) + 2\, h^{0,2}_{\bar\partial}(M, J_{t_0}) \geq h^{1,1}_{\rm BC}(M, J_t) + 2\, h^{0,2}_{\bar\partial}(M, J_t) \geq b_2(M),$$

for all t sufficiently close to t_0. Therefore, $\mathbf{k}_1(M, J_t) = 0$ and (M, J_t) also has the property $\mathcal{K}$.

However, the property $\mathcal{K}$ is not closed because, as proved in [26], there exists a holomorphic family of compact complex manifolds $\mathcal{X} = (M, J_t)_{t\in\Delta}$ such that $\mathbf{k}_1(M, J_0) \neq 0$ but $\mathbf{k}_1(M, J_t) = 0$ for all $t \in \Delta\backslash\{0\}$. The analytic family $\mathcal{X}$ is constructed on a 6-dimensional nilmanifold and the construction also suggests that one cannot expect a single property $\mathcal{F}_k$ to be closed. More concretely:

Theorem 3.1. *[26] The properties $\mathcal{K}$ and $\mathcal{F}_k$ for $k = 2$ are not closed.*

Concerning the $\partial\bar\partial$-lemma, a similar argument as above proves that it is an open property [7]. However, recently Angella and Kasuya have shown in [6] that it is not closed (see Remark 5.4 below). The upper-semicontinuity of the Hodge numbers also implies that the degeneration of the Frölicher spectral sequence at E_1 is an open property. Eastwood and Singer proved in [17] that this property is not closed by constructing a holomorphic family where all the fibres are twistor spaces. The analytic family $\mathcal{X}$ mentioned above provides another example, based on the complex geometry of nilmanifolds, showing the non-closedness of the property of degeneration of the Frölicher sequence at E_1 (see [10]).

The situation about openness and closedness of metric properties is as follows. Kodaira and Spencer [24] proved that the Kähler property is open, and Hironaka showed in [23] that in complex dimension ≥ 3 the Kähler property is not closed. Notice that, since a compact complex surface is Kähler if and only if its first Betti number is even, the Kähler property is closed in complex dimension 2.

Alessandrini and Bassanelli proved in [2] that the balanced property is not open. In contrast to the balanced case, Popovici has shown in [32] that the sG property is always open under holomorphic deformations, and conjectured in [34, Conjectures 1.21 and 1.23] that both the sG and the balanced properties are closed under holomorphic deformation. However, the family $\mathcal{X} = (M, J_t)_{t\in\Delta}$ satisfies that (M, J_t) has a balanced metric for all $t \in \Delta\backslash\{0\}$, but (M, J_0) does not admit any sG metric. In particular:

Theorem 3.2. *[10] The balanced and the sG properties are not closed.*

It follows from Theorem 2.4 and the upper-semicontinuity of the Hodge number $h^{0,1}_{\bar\partial}$ that the sGG property is open. Nevertheless, the analytic family $\mathcal{X}$ allows to show that the sGG property is not closed. Hence:

Theorem 3.3. *[36] The sGG property is open, but not closed.*

On the other hand, the existence of an sG metric in the central limit of a holomorphic deformation is guaranteed under the strong condition of the $\partial\bar\partial$-lemma. Concretely:

Proposition 3.4. *[34] Let $(M, J_t)_{t\in\Delta}$ be an analytic family of compact complex manifolds. If the $\partial\bar\partial$-lemma holds on (M, J_t) for every $t \in \Delta\backslash\{0\}$, then the central limit (M, J_0) has an sG metric.*

An interesting problem is if the conclusion in the previous proposition holds under weaker conditions than the $\partial\bar\partial$-lemma. The following result, which is a direct consequence of the properties of the family $\mathcal{X}$ mentioned above, shows that it is not true under the weaker property $\mathcal{K}$ or $\mathcal{F}_k$ for some particular k. Moreover:

Proposition 3.5. *There exists a holomorphic family of compact complex manifolds $\mathcal{X} = (M, J_t)_{t\in\Delta}$ of complex dimension 3, such that (M, J_t) satisfies the properties $\mathcal{F}_2$ and $\mathcal{K}$, admits balanced metric, is sGG and has degenerate Frölicher sequence for each $t \in \Delta\backslash\{0\}$, but (M, J_0) does not admit sG metrics.*

The result by Alessandrini and Bassanelli mentioned above about the non-openness of the balanced property is based on a holomorphic deformation of the Iwasawa manifold, which is a particular example of a complex manifold belonging to the class of 6-dimensional nilmanifolds endowed with an invariant complex structure. The analytic family $\mathcal{X}$ proving Theorems 3.1 and 3.2, the non-closedness of the sGG property in Theorem 3.3, and Proposition 3.5 is constructed by deforming appropriately an abelian complex structure J_0 on a 6-dimensional nilmanifold. In the next section we review the main results on the invariant complex geometry of 6-dimensional nilmanifolds.

4 Complex geometry of nilmanifolds

In this section we focus on the complex geometry of nilmanifolds and their interesting properties in relation to the problems considered in Sections 2 and 3. Notice that the problems addressed in those sections are nontrivial only for $n \geq 3$; in fact, for compact complex manifolds of complex dimension 2 the Frölicher spectral sequence always degenerate at E_1, the balanced condition is the same as the Kähler condition, and the existence of a Kähler metric is equivalent to the first Betti number be even. Therefore, we will mainly focus on complex nilmanifolds of (real) dimension 6.

In what follows, M will denote a nilmanifold of (real) dimension $2n$ and J an invariant complex structure on M, i.e. $M = \Gamma\backslash G$ is a compact quotient of a simply-connected nilpotent Lie group G by a lattice Γ of maximal rank in G, and J stems naturally from a "complex" structure J on the Lie algebra $\mathfrak{g}$ of G.

A crucial result in the theory of nilmanifolds is Nomizu's theorem [31] which asserts that the de Rham cohomology of M is canonically isomorphic to the cohomology of its underlying Lie algebra $\mathfrak{g}$, i.e. $H^k_{\mathrm{dR}}(M) \cong H^k_{\mathrm{dR}}(\mathfrak{g})$. Using this result, Hasegawa [22] proved that the Chevalley-Eilenberg complex $(\bigwedge^*(\mathfrak{g}^*), d)$ of $\mathfrak{g}$ provides a minimal model of M and that it is formal if and only if the Lie algebra is abelian, that is to say, the nilmanifold M is a torus. Therefore, by [16] a complex nilmanifold never satisfies the $\partial\bar{\partial}$-lemma, unless it is a complex torus.

Concerning a Nomizu type result for the Dolbeault cohomology of (M, J), several advances have been obtained under additional conditions on the invariant complex structure J. First, we recall that Salamon gave in [40] a characterization of the invariant complex structures as those endomorphisms $J\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $J^2 = -\mathrm{Id}$ for which there exists a basis $\{\omega^j\}_{j=1}^n$ of the i-eigenspace $\mathfrak{g}^{1,0}$ of the extension of J to $\mathfrak{g}^*_{\mathbb{C}} = \mathfrak{g}^* \otimes_{\mathbb{R}} \mathbb{C}$ satisfying

$$d\omega^1 = 0, \qquad d\omega^j \in \mathcal{I}(\omega^1, \ldots, \omega^{j-1}), \qquad \text{for } j = 2, \ldots, n,$$

where $\mathcal{I}(\omega^1, \ldots, \omega^{j-1})$ is the ideal in $\bigwedge^* \mathfrak{g}^*_{\mathbb{C}}$ generated by $\{\omega^1, \ldots, \omega^{j-1}\}$.

A generic invariant complex structure J satisfies $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{2,0}(\mathfrak{g}^*) \oplus \bigwedge^{1,1}(\mathfrak{g}^*)$ with respect to the bigraduation induced by J on the exterior algebra $\bigwedge^* \mathfrak{g}^*_{\mathbb{C}}$. When J is *abelian* [3] the Lie algebra differential d satisfies $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{1,1}(\mathfrak{g}^*)$, a condition which is equivalent to the complex subalgebra $\mathfrak{g}_{1,0} = (\mathfrak{g}^{1,0})^*$ being abelian. On the other hand, the complex structures associated to complex Lie algebras satisfy $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{2,0}(\mathfrak{g}^*)$ and we will refer to them as *complex-parallelizable* structures. Both abelian and complex-parallelizable structures are particular classes of *nilpotent* complex structures, introduced and studied in [15], for which there is a basis $\{\omega^j\}_{j=1}^n$ for $\mathfrak{g}^{1,0}$ satisfying

$$d\omega^1 = 0, \qquad d\omega^j \in \bigwedge^2 \langle \omega^1, \ldots, \omega^{j-1}, \omega^{\bar{1}}, \ldots, \omega^{\overline{j-1}} \rangle, \qquad \text{for } j = 2, \ldots, n,$$

where $\omega^{\bar{i}}$ stands for $\overline{\omega^i}$.

When J is complex-parallelizable, Sakane proved in [39] that the natural inclusion

$$\left(\bigwedge\nolimits^{p,q}(\mathfrak{g}^*), \bar{\partial}\right) \hookrightarrow (\Omega^{p,q}(M), \bar{\partial}) \tag{3}$$

induces an isomorphism

$$\iota\colon H^{p,q}_{\bar\partial}(\mathfrak{g},J) \longrightarrow H^{p,q}_{\bar\partial}(M,J) \tag{4}$$

between the Lie-algebra Dolbeault cohomology of $(\mathfrak{g}, J)$ and the Dolbeault cohomology of (M, J). More general conditions under which the inclusion (3) induces an isomorphism (4) can be found in [11, 15, 38]; in particular, it is always true for abelian complex structures on nilmanifolds. We will discuss below the 6-dimensional case in detail.

Concerning the calculation of the Bott-Chern cohomology and the Frölicher spectral sequence of nilmanifolds with invariant complex structure, one has the following Nomizu type result. The first part is proved by Angella in [4, Theorem 2.8] and the second part follows from an inductive argument given in [14, Theorem 4.2].

Theorem 4.1. *If the natural inclusion* (3) *induces an isomorphism* (4) *between the Lie-algebra Dolbeault cohomology of* $(\mathfrak{g}, J)$ *and the Dolbeault cohomology of the complex nilmanifold* (M, J)*, then:*

(*i*) *the natural map*

$$\iota\colon H^{p,q}_{\mathrm{BC}}(\mathfrak{g},J) \longrightarrow H^{p,q}_{\mathrm{BC}}(M,J)$$

between the Lie-algebra Bott-Chern cohomology of $(\mathfrak{g}, J)$ *and the Bott-Chern cohomology of* (M, J) *is also an isomorphism for any* $0 \le p, q \le n$*;*

(*ii*) *the natural map*

$$\iota\colon E^{p,q}_{r}(\mathfrak{g},J) \longrightarrow E^{p,q}_{r}(M,J)$$

between the term in the Lie-algebra Frölicher sequence of $(\mathfrak{g}, J)$ *and the term in the Frölicher spectral sequence of* (M, J) *is also an isomorphism for any* $0 \le p, q \le n$.

The only 4-dimensional nilmanifolds having invariant complex structures are the torus $\mathbb{T}^4$ and the Kodaira-Thurston manifold [43]. The latter was the first known example of a compact symplectic manifold not admitting Kähler metric. In six dimensions, the nilmanifolds admitting invariant complex structures are classified through their underlying Lie algebras. The following result provides a classification of such nilmanifolds in terms of the different types of complex structures that they admit.

Theorem 4.2. *[40, 44] A nilmanifold* M *of (real) dimension* 6 *has an invariant complex structure if and only if its underlying Lie algebra is isomorphic to one in the following* *list:*

$$\begin{aligned}
\mathfrak{h}_1 &= (0,0,0,0,0,0), & \mathfrak{h}_{10} &= (0,0,0,12,13,14),\\
\mathfrak{h}_2 &= (0,0,0,0,12,34), & \mathfrak{h}_{11} &= (0,0,0,12,13,14+23),\\
\mathfrak{h}_3 &= (0,0,0,0,0,12+34), & \mathfrak{h}_{12} &= (0,0,0,12,13,24),\\
\mathfrak{h}_4 &= (0,0,0,0,12,14+23), & \mathfrak{h}_{13} &= (0,0,0,12,13+14,24),\\
\mathfrak{h}_5 &= (0,0,0,0,13+42,14+23), & \mathfrak{h}_{14} &= (0,0,0,12,14,13+42),\\
\mathfrak{h}_6 &= (0,0,0,0,12,13), & \mathfrak{h}_{15} &= (0,0,0,12,13+42,14+23),\\
\mathfrak{h}_7 &= (0,0,0,12,13,23), & \mathfrak{h}_{16} &= (0,0,0,12,14,24),\\
\mathfrak{h}_8 &= (0,0,0,0,0,12), & \mathfrak{h}_{19}^- &= (0,0,0,12,23,14-35),\\
\mathfrak{h}_9 &= (0,0,0,0,12,14+25), & \mathfrak{h}_{26}^+ &= (0,0,12,13,23,14+25).
\end{aligned}$$

Moreover:

(*a*) *For* $\mathfrak{h}_{19}^-$ *and* $\mathfrak{h}_{26}^+$*, any complex structure is non-nilpotent;*

(*b*) *For* $\mathfrak{h}_k$*,* $1 \leq k \leq 16$*, any complex structure is nilpotent;*

(*c*) *For* $\mathfrak{h}_1$*,* $\mathfrak{h}_3$*,* $\mathfrak{h}_8$ *and* $\mathfrak{h}_9$*, any complex structure is abelian;*

(*d*) *For* $\mathfrak{h}_2$*,* $\mathfrak{h}_4$*,* $\mathfrak{h}_5$ *and* $\mathfrak{h}_{15}$*, there exist both abelian and non-abelian nilpotent complex structures;*

(*e*) *For* $\mathfrak{h}_6$*,* $\mathfrak{h}_7$*,* $\mathfrak{h}_{10}$*,* $\mathfrak{h}_{11}$*,* $\mathfrak{h}_{12}$*,* $\mathfrak{h}_{13}$*,* $\mathfrak{h}_{14}$ *and* $\mathfrak{h}_{16}$*, any complex structure is not abelian.*

Here, for instance, the notation $\mathfrak{h}_2 = (0,0,0,0,12,34)$ means that there exists a basis $\{e^i\}_{i=1}^6$ of the dual of the Lie algebra (or equivalently, a basis of invariant real 1-forms on the nilmanifold) such that $de^1 = de^2 = de^3 = de^4 = 0$, $de^5 = e^1 \wedge e^2$ and $de^6 = e^3 \wedge e^4$. Notice that $\mathfrak{h}_2$ is isomorphic to the product of two copies of the 3-dimensional real Heisenberg algebra $(0,0,12)$ (see Section 5 for more details).

By Theorem 4.2, if a 6-dimensional nilmanifold M admits invariant complex structures then all of them are either nilpotent or non-nilpotent. This special property does not hold in higher dimensions [15].

An interesting problem is to obtain a description of the moduli space of invariant complex structures on each nilmanifold. Andrada, Barberis and Dotti classified in [3] the abelian complex structures in dimension 6, whereas the classification of the non-nilpotent complex structures was given in [45] and the general classification was obtained recently in [10]. Let J and J' be two invariant complex structures on a nilmanifold M with underlying Lie algebra $\mathfrak{g}$. Recall that J and J' are said to be *equivalent* if there is an automorphism $F\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ of the Lie algebra such that $J' = F^{-1} \circ J \circ F$. Now, if $\mathfrak{g}_J^{1,0}$ and $\mathfrak{g}_{J'}^{1,0}$ denote the $(1,0)$-subspaces of $\mathfrak{g}_{\mathbb{C}}^*$ associated to J and J', respectively, then the complex structures J and J' are equivalent if and only if there exists a $\mathbb{C}$-linear isomorphism $F^*\colon \mathfrak{g}_J^{1,0} \longrightarrow \mathfrak{g}_{J'}^{1,0}$ such that $d \circ F^* = F^* \circ d$.

It is well known that, up to equivalence, there are only two complex-parallelizable structures, which are defined by the complex equations

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \rho\,\omega^{12},$$

where $\rho = 0$ or 1. The corresponding Lie algebras are $\mathfrak{h}_1$ when $\rho = 0$, and hence the complex nilmanifold is a complex torus $\mathbb{T}^3_{\mathbb{C}}$, and $\mathfrak{h}_5$ when $\rho = 1$, and the complex nilmanifold is the Iwasawa manifold given by a quotient of the *complex* Heisenberg group.

According to [10], the remaining complex structures in dimension 6 can be parametrized by the following three families of complex equations:

Family I: $d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \rho\,\omega^{12} + \omega^{1\bar{1}} + \lambda\,\omega^{1\bar{2}} + D\,\omega^{2\bar{2}},$

where $\rho \in \{0,1\}$, $\lambda \in \mathbb{R}^{\geq 0}$ and $D \in \mathbb{C}$ with $\mathfrak{Im}\,D \geq 0$. The complex structure is abelian if and only if $\rho = 0$. The Lie algebras admitting complex structures in this family are $\mathfrak{h}_2, \ldots, \mathfrak{h}_6$ and $\mathfrak{h}_8$.

Family II: $d\omega^1 = 0, \quad d\omega^2 = \omega^{1\bar{1}}, \quad d\omega^3 = \rho\,\omega^{12} + B\,\omega^{1\bar{2}} + c\,\omega^{2\bar{1}},$

where $\rho \in \{0,1\}$, $B \in \mathbb{C}$ and $c \in \mathbb{R}^{\geq 0}$, with $(\rho, B, c) \neq (0,0,0)$. The complex structure is abelian if and only if $\rho = 0$, and the Lie algebras admitting complex structures in this family are $\mathfrak{h}_7$ and $\mathfrak{h}_9, \ldots, \mathfrak{h}_{16}$.

Family III: $d\omega^1 = 0, \quad d\omega^2 = \omega^{13} + \omega^{1\bar{3}}, \quad d\omega^3 = \varepsilon\, i\,\omega^{1\bar{1}} \pm i(\omega^{1\bar{2}} - \omega^{2\bar{1}}),$

where $\varepsilon = 0$ or 1. The complex structures are non-nilpotent, and the Lie algebras are $\mathfrak{h}^-_{19}$ (for $\varepsilon = 0$) and $\mathfrak{h}^+_{26}$ (for $\varepsilon = 1$).

Tables 1, 2 and 3 below contain the general classification of invariant complex structures on 6-dimensional nilmanifolds in terms of its underlying Lie algebra and the values of the coefficients $(\rho, \lambda, D = x + iy)$ for Family I, (ρ, B, c) for Family II, and ε for Family III. Different values of the parameters in Table 1, resp. Tables 2 and 3, correspond to non-equivalent complex structures in Family I, resp. Families II and III (see [10] for more details).

Theorem 4.1 above asserts that if the natural isomorphism (4) holds then, in addition to the Dolbeault cohomology of $(M = \Gamma\backslash G, J)$, we also know other complex invariants as the Bott-Chern cohomology and the terms in the Frölicher spectral sequence. And moreover, such complex invariants can be obtained directly from the underlying Lie algebra $\mathfrak{g}$ together with the structure J.

In dimension 4 the natural isomorphism (4) holds for any invariant complex structure J. In dimension 6, Rollenske proved in [38, Section 4.2] that if $\mathfrak{g} \not\cong \mathfrak{h}_7$ then the natural inclusion (3) induces an isomorphism (4) between the Lie-algebra Dolbeault cohomology of $(\mathfrak{g}, J)$ and the Dolbeault cohomology of M.

Remark 4.3. Let $(M = \Gamma\backslash G, J)$ be a 6-dimensional nilmanifold endowed with an invariant complex structure J such that $\mathfrak{g} = \mathfrak{h}_7$. In [38, Theorem 4.4] it is proved that there is a dense subset of the space of all invariant complex structures for which the complex nilmanifold has the structure of a principal holomorphic bundle of elliptic curves over a Kodaira surface, but this does not hold for all complex structures. In fact, the invariant complex structure J may not be compatible with the lattice Γ, as [38, Example 1.14] shows, and hence, one cannot ensure the existence of the natural isomorphism (4) for any invariant complex structure on the nilmanifold.

Concerning the Bott-Chern cohomology of nilmanifolds, in [41] Schweitzer computed it for the Iwasawa manifold and in [4] Angella calculated the Bott-Chern cohomology groups of its small deformations. Notice that by [37, Theorem 2.6], if such deformations are sufficiently small then they are again invariant complex structures. Thus, the Bott-Chern cohomology determined in [5] and [26] for any pair $(\mathfrak{g}, J)$ covers that of any invariant complex structure and its sufficiently small deformations on any 6-dimensional nilmanifold with underlying Lie algebra not isomorphic to $\mathfrak{h}_7$, accordingly to Remark 4.3 and Theorem 4.1 (i).

In Tables 1, 2 and 3 we include the Bott-Chern numbers $h^{p,q}_{\mathrm{BC}}(\mathfrak{g}, J)$ for any J in the Families I, II and III above (see [26] for an explicit description of the generators of the Bott-Chern cohomology groups in terms of the complex equations in Families I, II and III). Therefore, the tables cover all the invariant complex geometry of 6-dimensional nilmanifolds, except for the complex torus and the Iwasawa manifold, which are well known and already given in [41] as we reminded above.

It is clear that in all cases $H^{3,0}_{\mathrm{BC}} = \langle[\omega^{123}]\rangle$ and $H^{3,3}_{\mathrm{BC}} = \langle[\omega^{123\bar{1}\bar{2}\bar{3}}]\rangle$, so $h^{3,0}_{\mathrm{BC}} = h^{3,3}_{\mathrm{BC}} = 1$. Notice that by the duality in the Bott-Chern cohomology it suffices to show the dimensions $h^{p,q}_{\mathrm{BC}}$ for $(p,q) = (1,0), (2,0), (1,1), (2,1), (2,2), (3,1)$ and $(3,2)$. In fact, the dimension of any other Bott-Chern cohomology group is obtained by $h^{q,p}_{\mathrm{BC}} = h^{p,q}_{\mathrm{BC}}$.

	Family I				Bott-Chern numbers						
$\mathfrak{g}$	ρ	λ	$D = x + iy$		$h^{1,0}_{\mathrm{BC}}$	$h^{2,0}_{\mathrm{BC}}$	$h^{1,1}_{\mathrm{BC}}$	$h^{2,1}_{\mathrm{BC}}$	$h^{3,1}_{\mathrm{BC}}$	$h^{2,2}_{\mathrm{BC}}$	$h^{3,2}_{\mathrm{BC}}$
$\mathfrak{h}_2$	0	0	$y = 1$	$x = 0$	2	1	4	6	3	7	3
				$x \neq 0$						6	
	1	1	$y > 0$	$x = -1 \pm \sqrt{1 - y^2}$	2	1	5	6	2	6	3
				$x \neq -1 \pm \sqrt{1 - y^2}$, $x \neq 1$			4				
				$x = 1$						7	
$\mathfrak{h}_3$	0	0	± 1		2	1	4	6	3	7	3
$\mathfrak{h}_4$	0	1	$\frac{1}{4}$		2	1	4	6	3	6	3
	1	1	-2		2	1	5	6	2	6	3
			$D \in \mathbb{R} - \{-2, 0, 1\}$				4				
			1							7	
$\mathfrak{h}_5$	0	1	0		2	2	6	6	3	6	3
			$D \in \left(0, \frac{1}{4}\right)$			1	4				
	1	0	$y = 0$	$x = 0$	2	2	4	6	2	7	3
				$x = \frac{1}{2}$		1				8	
				$x \neq 0, \frac{1}{2}, \quad x > -\frac{1}{4}$						7	
			$0 < y^2 < \frac{3}{4}$	$x = \frac{1}{2}$							
			$y > 0$	$x \neq \frac{1}{2}, \quad x > y^2 - \frac{1}{4}$						6	
		$0 < \lambda^2 < \frac{1}{2}$	$x = 0$	$y = 0$		2					
				$0 < y < \frac{\lambda^2}{2}$		1					
		$\frac{1}{2} \leq \lambda^2 < 1$	$x = 0$	$y = 0$		2					
				$0 < y < \frac{1-\lambda^2}{2}$		1					
		$1 < \lambda^2 \leq 5$	$x = 0$	$y = 0$		2					
				$0 < y < \frac{\lambda^2-1}{2}$		1					
		$\lambda^2 > 5$	$x = 0$	$y = 0$		2					
				$0 < y < \frac{\lambda^2-1}{2}, \; y \neq \sqrt{\lambda^2 - 1}$		1					
				$0 < y < \frac{\lambda^2-1}{2}, \; y = \sqrt{\lambda^2 - 1}$			5				
$\mathfrak{h}_6$	1	1	0		2	2	5	6	2	6	3
$\mathfrak{h}_8$	0	0	0		2	2	6	7	3	8	3

Table 1.— Classification of complex structures in Family I and dimensions of their Bott-Chern cohomology groups.

	Family II			Bott-Chern numbers						
$\mathfrak{g}$	ρ	B	c	$h^{1,0}_{\mathrm{BC}}$	$h^{2,0}_{\mathrm{BC}}$	$h^{1,1}_{\mathrm{BC}}$	$h^{2,1}_{\mathrm{BC}}$	$h^{3,1}_{\mathrm{BC}}$	$h^{2,2}_{\mathrm{BC}}$	$h^{3,2}_{\mathrm{BC}}$
$\mathfrak{h}_7$	1	1	0	1	2	5	6	2	5	3
$\mathfrak{h}_9$	0	1	1	1	1	4	5	3	6	3
$\mathfrak{h}_{10}$	1	0	1	1	1	4	5	2	5	3
$\mathfrak{h}_{11}$	1	$B \in \mathbb{R} - \{0, \frac{1}{2}, 1\}$	$\lvert B-1\rvert$	1	1	4	5	2	5	3
		$\frac{1}{2}$	$\frac{1}{2}$						6	
$\mathfrak{h}_{12}$	1	$\mathfrak{Re}B \neq \frac{1}{2},\ \mathfrak{Im}B \neq 0$	$\lvert B-1\rvert$	1	1	4	5	2	5	3
		$\mathfrak{Re}B = \frac{1}{2},\ \mathfrak{Im}B \neq 0$							6	
$\mathfrak{h}_{13}$	1	1	$0 < c < 2,\ c \neq 1$	1	1	5	5	2	5	3
			1						6	
		$B \neq 1,\ c \neq \lvert B\rvert, \lvert B-1\rvert$, $(c, \lvert B\rvert) \neq (0,1)$, $c^4 - 2(\lvert B\rvert^2+1)c^2 + (\lvert B\rvert^2-1)^2 < 0$				4			5	
		$B \neq 1,\ c = \lvert B\rvert > \frac{1}{2}$, $\lvert B\rvert \neq \lvert B-1\rvert$							6	
$\mathfrak{h}_{14}$	1	1	2	1	1	5	5	2	5	3
		$\lvert B\rvert = \frac{1}{2}$	$\frac{1}{2}$			4			6	
		$c \neq \lvert B-1\rvert$, $(c, \lvert B\rvert) \neq (0,1), (\frac{1}{2}, \frac{1}{2}), (2,1)$, $c^4 - 2(\lvert B\rvert^2+1)c^2 + (\lvert B\rvert^2-1)^2 = 0$							5	
$\mathfrak{h}_{15}$	0	0	1	1	1	5	5	3	5	3
		1	$c \neq 0,\ 1$			4				
			0		2					
	1	0	0	1	2	4	5	2	7	3
		$\lvert B\rvert \neq 0, 1$	0						5	
		1	$c > 2$		1	5				
		$\lvert B\rvert = c$	$0 < c < \frac{1}{2}$			4			6	
		$c \neq 0,\ \lvert B-1\rvert$, $B \neq 1,\ \lvert B\rvert \neq c$, $c^4 - 2(\lvert B\rvert^2+1)c^2 + (\lvert B\rvert^2-1)^2 > 0$							5	
$\mathfrak{h}_{16}$	1	$\lvert B\rvert = 1,\ B \neq 1$	0	1	2	4	5	2	5	3

Table 2.— Classification of complex structures in Family II and dimensions of their Bott-Chern cohomology groups.

The next result was proved in [10] and shows the general behaviour of the Frölicher spectral sequence in dimension 6. For that we applied Theorem 4.1 (ii) to ensure that $E_r^{p,q}(M,J) \cong E_r^{p,q}(\mathfrak{g},J)$ for any p,q and any $r \geq 1$, whenever $\mathfrak{g} \ncong \mathfrak{h}_7$.

Theorem 4.4. *[10] Let $M = \Gamma\backslash G$ be a 6-dimensional nilmanifold endowed with an invariant complex structure J such that the underlying Lie algebra $\mathfrak{g} \ncong \mathfrak{h}_7$. Then the Frölicher spectral sequence $\{E_r(M,J)\}_{r\geq 1}$ behaves as follows:*

(a) If $\mathfrak{g} \cong \mathfrak{h}_1, \mathfrak{h}_3, \mathfrak{h}_6, \mathfrak{h}_8, \mathfrak{h}_9, \mathfrak{h}_{10}, \mathfrak{h}_{11}, \mathfrak{h}_{12}$ or $\mathfrak{h}_{19}^-$, then $E_1(M,J) \cong E_\infty(M,J)$ for any J.

(b) If $\mathfrak{g} \cong \mathfrak{h}_2$ or $\mathfrak{h}_4$, then $E_1(M,J) \cong E_\infty(M,J)$ if and only if J is non-abelian; moreover, any abelian complex structure satisfies $E_1(M,J) \ncong E_2(M,J) \cong E_\infty(M,J)$.

(c) If $\mathfrak{g} \cong \mathfrak{h}_5$, then:

	Family III	Bott-Chern numbers						
$\mathfrak{g}$	ε	$h^{1,0}_{\mathrm{BC}}$	$h^{2,0}_{\mathrm{BC}}$	$h^{1,1}_{\mathrm{BC}}$	$h^{2,1}_{\mathrm{BC}}$	$h^{3,1}_{\mathrm{BC}}$	$h^{2,2}_{\mathrm{BC}}$	$h^{3,2}_{\mathrm{BC}}$
$\mathfrak{h}_{19}^{-}$	0	1	1	2	3	2	4	2
$\mathfrak{h}_{26}^{+}$	1	1	1	2	3	2	3	2

Table 3.— Classification of complex structures in Family III and dimensions of their Bott-Chern cohomology groups.

(c.1) $E_1(M,J) \ncong E_2(M,J) \cong E_\infty(M,J)$ *for the complex-parallelizable structure* J*;*

(c.2) $E_1(M,J) \cong E_\infty(M,J)$ *if and only if* J *is not complex-parallelizable and* $\rho D \neq 0$ *in Table* 1*; moreover,* $E_1(M,J) \ncong E_2(M,J) \cong E_\infty(M,J)$ *when* $\rho D = 0$*.*

(d) If $\mathfrak{g} \cong \mathfrak{h}_{16}$ *or* $\mathfrak{h}_{26}^{+}$*, then* $E_1(M,J) \ncong E_2(M,J) \cong E_\infty(M,J)$ *for any* J*.*

(e) If $\mathfrak{g} \cong \mathfrak{h}_{13}$ *or* $\mathfrak{h}_{14}$*, then* $E_1(M,J) \cong E_2(M,J) \ncong E_3(M,J) \cong E_\infty(M,J)$ *for any* J*.*

(f) If $\mathfrak{g} \cong \mathfrak{h}_{15}$ *and* J *is a complex structure on* $\mathfrak{h}_{15}$ *given in Table* 2*, then:*

(f.1) $E_1(M,J) \ncong E_2(M,J) \cong E_\infty(M,J)$*, when* $c = 0$ *and* $|B-\rho| \neq 0$*;*

(f.2) $E_1(M,J) \cong E_2(M,J) \ncong E_3(M,J) \cong E_\infty(M,J)$*, when* $\rho = 1$ *and* $|B-1| \neq c \neq 0$*;*

(f.3) $E_1(M,J) \ncong E_2(M,J) \ncong E_3(M,J) \cong E_\infty(M,J)$*, when* $\rho = 0$ *and* $|B| \neq c \neq 0$*.*

According to Remark 4.3, we cannot ensure the existence of a canonical isomorphism between $E_r^{p,q}(\mathfrak{g},J)$ and $E_r^{p,q}(M,J)$ for any invariant complex structure J on a nilmanifold M with underlying Lie algebra $\mathfrak{g} \cong \mathfrak{h}_7$. However, it is worth noticing that up to equivalence there is only one complex structure J on $\mathfrak{h}_7$ whose sequence degenerates at the first step, that is, $E_1(\mathfrak{h}_7,J) \cong E_\infty(\mathfrak{h}_7,J)$.

Concerning the existence of balanced or sG metrics, if $(M = \Gamma\backslash G, J)$ is a nilmanifold endowed with an invariant complex structure, then it admits a balanced metric if and only if it has an invariant one. Thus, the existence of such a metric can be detected at the level of the underlying Lie algebra. This fact was proved in [18] and uses the so called *symmetrization process*. Moreover, in [10] it is proved that this process can also be applied to the existence of sG metrics on nilmanifolds, that is to say, $(M = \Gamma\backslash G, J)$ has an sG metric if and only if it has an invariant one. In dimension 6 we have:

Theorem 4.5. *[10, 44] Let* $M = \Gamma\backslash G$ *be a* 6*-dimensional nilmanifold admitting invariant complex structures* J*, and let* $\mathfrak{g}$ *be the underlying Lie algebra. Then:*

(*i*) *There exists* J *having balanced or sG metrics if and only if* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{h}_1, \ldots, \mathfrak{h}_6$ *or* $\mathfrak{h}_{19}^-$.

(*ii*) *If the complex structure* J *is abelian, then an invariant Hermitian metric is sG if and only if it is balanced.*

(*iii*) *For* $\mathfrak{g} \cong \mathfrak{h}_2, \mathfrak{h}_4, \mathfrak{h}_5$ *or* $\mathfrak{h}_6$*, if the complex structure* J *is non-abelian then any invariant Hermitian metric is sG.*

The following result implies that abelian complex structures on nilmanifolds with underlying Lie algebra isomorphic to $\mathfrak{h}_2$ or $\mathfrak{h}_4$ do not admit sG metrics.

Proposition 4.6. *[46] Let* $M = \Gamma \backslash G$ *be a* 6*-dimensional nilmanifold with an abelian complex structure* J *admitting an sG metric. Then the underlying Lie algebra* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{h}_3$ *or* $\mathfrak{h}_5$*.*

There exist compact complex manifolds having sG metrics but not admitting any balanced metric [34, Theorem 1.8]. The general situation for nilmanifolds in dimension 6 is as follows:

Proposition 4.7. *[10] Let* $M = \Gamma \backslash G$ *be a* 6*-dimensional nilmanifold with an invariant complex structure* J *such that* $(M = \Gamma \backslash G, J)$ *does not admit balanced metrics. If* $(M = \Gamma \backslash G, J)$ *has sG metric, then* J *is non-abelian nilpotent and* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{h}_2$*,* $\mathfrak{h}_4$ *or* $\mathfrak{h}_5$*. Moreover, according to the classification in Table 1, such a* J *is given by* $\rho = 1$ *and:* $\lambda = 1, x + y^2 \geq \frac{1}{4}$ *on* $\mathfrak{h}_2$*;* $\lambda = 1, x \geq \frac{1}{4}$ *on* $\mathfrak{h}_4$*; and* $\lambda = 0, y \neq 0$ *or* $\lambda = y = 0, x \geq 0$ *on* $\mathfrak{h}_5$*.*

Although by Theorem 4.5 (*i*) the underlying Lie algebras are the same both in the balanced and the sG case, Proposition 4.7 shows that the complex structures admitting such metrics differ.

We finish this section by considering those complex nilmanifolds $(M = \Gamma \backslash G, J)$ having the sGG property, that is, any Gauduchon metric is sG. By Theorem 2.4 the sGG condition is equivalent to $b_1(M) = 2\, h^{0,1}_{\bar\partial}(M, J)$. For instance, let us consider an invariant complex structure J in the Family I. If J is abelian then $\rho = 0$ and

$$H^{0,1}_{\bar\partial}(M, J) = \langle [\omega^{\bar 1}], [\omega^{\bar 2}], [\omega^{\bar 3}] \rangle,$$

whereas for a non-abelian J, i.e. $\rho = 1$, we have

$$H^{0,1}_{\bar\partial}(M, J) = \langle [\omega^{\bar 1}], [\omega^{\bar 2}] \rangle.$$

If M is not a torus then its first Betti number satisfies $b_1(M) \leq 5$, which implies that $(M = \Gamma \backslash G, J)$ in Family I cannot be sGG when J is abelian.

From a more detailed analysis of the non-abelian complex structures in Family I, together with the study of complex nilmanifolds in Families II and III, it follows:

Theorem 4.8. *[36] Let $M = \Gamma\backslash G$ be a 6-dimensional nilmanifold, not a torus, endowed with an invariant complex structure J. Then, (M, J) is sGG if and only if the Lie algebra underlying M is isomorphic to $\mathfrak{h}_2$, $\mathfrak{h}_4$, $\mathfrak{h}_5$ or $\mathfrak{h}_6$, and the complex structure J is not abelian.*

The complex geometry of 6-dimensional nilmanifolds allows to conclude that, apart from the obvious implications, there are no relations among the different properties of compact complex manifolds introduced in Section 2 (see [10] and [36] for more details):

- there exist sG manifolds that are not balanced;
- there are sG manifolds that are not sGG;
- the balanced property and the sGG property are unrelated;
- the degeneration of the Frölicher spectral sequence at E_1 and the sGG property are also unrelated.

In particular, the sGG class introduced and studied in [36] is a new class of compact complex manifolds.

5 On the complex geometry of the product of two Heisenberg manifolds

In this section we consider the product of two (3-dimensional real) Heisenberg (nil)manifolds, and we show that, despite the simplicity of this 6-dimensional nilmanifold, its complex geometry is surprisingly rich in relation to the deformation problems considered in Section 3.

Let us recall that the Heisenberg group H is the nilpotent Lie group constituted by all the matrices of the form

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}. \tag{5}$$

Since $\{\alpha^1 = dx, \alpha^2 = dy, \alpha^3 = xdy - dz\}$ is a basis of left-invariant 1-forms on H, the structure equations are given by $d\alpha^1 = d\alpha^2 = 0$, $d\alpha^3 = \alpha^{12}$, thus the Lie algebra of H is $\mathfrak{h} = (0, 0, 12)$. Let us consider the lattice Γ given by the matrices in (5) with (x, y, z)-entries lying in $\mathbb{Z}$. Hence, Γ is a lattice of maximal rank in H. From now on, we will denote by N the 3-dimensional nilmanifold $N = \Gamma\backslash H$ and we will refer to N as the Heisenberg nilmanifold.

Let us take another copy of N with basis of 1-forms $\{\beta^1, \beta^2, \beta^3\}$ satisfying $d\beta^1 = d\beta^2 = 0$ and $d\beta^3 = \beta^{12}$. Then, the Lie algebra underlying the 6-dimensional nilmanifold $N \times N$ is isomorphic to $\mathfrak{h} \oplus \mathfrak{h}$, i.e. to the Lie algebra $\mathfrak{h}_2 = (0, 0, 0, 0, 12, 34)$. On the product manifold $N \times N$ we consider the almost-complex structure J_0 defined by

$$J_0(\alpha^1) = -\alpha^2, \quad J_0(\beta^1) = -\beta^2, \quad J_0(\alpha^3) = -\beta^3. \tag{6}$$

It is easy to check that J_0 is integrable, i.e. its Nijehuis tensor vanishes identically, and abelian. Therefore, J_0 defines an invariant abelian complex structure on $N \times N$. The aim of this section is to show that the holomorphic deformations of this simple complex structure have very interesting properties in relation to the existence problem of sG metric, the invariants $\mathbf{f}_2, \mathbf{k}_1$ related to the $\partial\bar{\partial}$-lemma condition, the Frölicher spectral sequence and also with respect to the sGG condition.

In order to fit the complex structure (6) in the general frame of Table 1, we will express J_0 in terms of the basis of 1-forms $\{e^1, e^2, e^3, e^4, e^5, e^6\}$ given by

$$e^1 = \alpha^1, \quad e^2 = \alpha^2, \quad e^3 = \beta^1, \quad e^4 = \beta^2, \quad e^5 = \alpha^3, \quad e^6 = \beta^3, \tag{7}$$

which satisfies the equations

$$de^1 = de^2 = de^3 = de^4 = 0, \quad de^5 = e^{12}, \quad de^6 = e^{34}. \tag{8}$$

Using (6) and (7) we have that the complex forms

$$\omega_0^1 = e^1 - iJ_0e^1 = e^1 + i\,e^2, \quad \omega_0^2 = e^3 - iJ_0e^3 = e^3 + i\,e^4, \quad \omega_0^3 = 2e^6 - 2iJ_0e^6 = 2e^6 - 2i\,e^5, \tag{9}$$

constitute a basis of forms of bidegree $(1,0)$ with respect to the abelian complex structure J_0. Now, it follows from (8) that the complex structure equations for $(N \times N, J_0)$ in the basis $\{\omega_0^1, \omega_0^2, \omega_0^3\}$ are

$$d\omega_0^1 = d\omega_0^2 = 0, \quad d\omega_0^3 = \omega_0^{1\bar{1}} + i\,\omega_0^{2\bar{2}}. \tag{10}$$

Notice that these equations correspond to take $\rho = \lambda = 0$ and $D = i$ for $\mathfrak{h}_2$ in Table 1.

The compact complex manifold $(N \times N, J_0)$ can be described as follows. The Lie group $H \times H$ endowed with the complex structure J_0 can be realized by the complex matrices of the form

$$(H \times H, J_0) = \left\{ \begin{pmatrix} 1 & -z_1 & -iz_2 & z_3 \\ 0 & 1 & 0 & \bar{z}_1 \\ 0 & 0 & 1 & \bar{z}_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid z_1, z_2, z_3 \in \mathbb{C} \right\}.$$

In terms of the complex coordinates (z_1, z_2, z_3), the left translation by an element (a_1, a_2, a_3) of $(H \times H, J_0)$ is given by

$$L^*_{(a_1,a_2,a_3)}(z_1, z_2, z_3) = (z_1 + a_1,\ z_2 + a_2,\ z_3 - a_1\bar{z}_1 - ia_2\bar{z}_2 + a_3).$$

The basis $\{\omega_0^1, \omega_0^2, \omega_0^3\}$ of left-invariant complex (1,0)-forms on $(H \times H, J_0)$ is expressed in these complex coordinates as

$$\omega_0^1 = dz_1, \quad \omega_0^2 = dz_2, \quad \omega_0^3 = dz_3 + z_1 d\bar{z}_1 + iz_2 d\bar{z}_2.$$

Now, the complex nilmanifold $(N \times N, J_0)$ can be realized as the quotient of $(H \times H, J_0)$ by the lattice defined by taking (z_1, z_2, z_3) as Gaussian integers.

As we reminded in Section 4, the Dolbeault cohomology of the compact complex manifold $(N \times N, J_0)$ can be computed explicitly from the pair $(\mathfrak{h}_2, J_0)$, i.e. $H^{p,q}_{\bar\partial}(N \times N, J_0) \cong H^{p,q}_{\bar\partial}(\mathfrak{h}_2, J_0)$ for any $0 \leq p, q \leq 3$. In order to perform an appropriate holomorphic deformation of J_0 we first compute the particular Dolbeault cohomology group

$$H^{0,1}_{\bar\partial}(N \times N, J_0) \cong H^{0,1}_{\bar\partial}(\mathfrak{h}_2, J_0) = \langle [\omega_0^{\bar 1}], [\omega_0^{\bar 2}], [\omega_0^{\bar 3}] \rangle.$$

We consider the small deformation J_t given by

$$t \frac{\partial}{\partial z_2} \otimes \omega_0^{\bar 1} + it \frac{\partial}{\partial z_1} \otimes \omega_0^{\bar 2} \in H^{0,1}(X_0, T^{1,0}X_0),$$

where X_0 denotes the complex manifold $(N \times N, J_0)$. This deformation is defined for any $t \in \Delta = \{t \in \mathbb{C} \mid |t| < 1\}$ and the analytic family of compact complex manifolds $(N \times N, J_t)$ has a complex basis $\{\omega_t^1, \omega_t^2, \omega_t^3\}$ of type (1,0) with respect to J_t given by

$$J_t: \quad \omega_t^1 = \omega_0^1 + it\,\omega_0^{\bar 2}, \qquad \omega_t^2 = \omega_0^2 + t\,\omega_0^{\bar 1}, \qquad \omega_t^3 = \omega_0^3. \tag{11}$$

We can express the complex structures J_t in terms of the real basis of 1-forms $\{e^1, \ldots, e^6\}$ given by (7) as follows. Let us denote by t_1 the real part of t and by t_2 its imaginary part, i.e. $t = t_1 + i\,t_2$. From (9) and (11) we get

$$\omega_t^1 = e^1 - t_2 e^3 + t_1 e^4 + i(e^2 + t_1 e^3 + t_2 e^4), \quad \omega_t^2 = e^3 + t_1 e^1 + t_2 e^2 + i(e^4 + t_2 e^1 - t_1 e^2), \quad \omega_t^3 = 2e^6 - 2i\, e^5.$$

Since the complex form ω_t^k, $1 \leq k \leq 3$, is declared to be of bidegree (1,0) with respect to the complex structure J_t, necessarily

$$e^2 + t_1 e^3 + t_2 e^4 = -J_t(e^1 - t_2 e^3 + t_1 e^4), \quad e^4 + t_2 e^1 - t_1 e^2 = -J_t(e^3 + t_1 e^1 + t_2 e^2), \quad e^5 = J_t(e^6),$$

and we have that the complex structure J_t in the basis $\{e^1, \ldots, e^6\}$ is given by

$$J_t = \frac{-1}{1+|t|^4} \begin{pmatrix} 2|t|^2 & |t|^4 - 1 & 2(t_2 - t_1|t|^2) & -2(t_1 + t_2|t|^2) & 0 & 0 \\ 1 - |t|^4 & 2|t|^2 & -2(t_1 + t_2|t|^2) & -2(t_2 - t_1|t|^2) & 0 & 0 \\ 2(t_1 - t_2|t|^2) & 2(t_2 + t_1|t|^2) & -2|t|^2 & |t|^4 - 1 & 0 & 0 \\ 2(t_2 + t_1|t|^2) & -2(t_1 - t_2|t|^2) & 1 - |t|^4 & -2|t|^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

for each $t \in \Delta$.

In the following result we find complex structure equations for every J_t, except for the central limit $t = 0$, that fit in the classification given in Table 1.

Proposition 5.1. *Let $(N \times N, J_t)$ be the product of two copies of the 3-dimensional Heisenberg nilmanifold N endowed with the complex structure J_t given by (11). For each $t \in \Delta\backslash\{0\}$, there is a (global) basis $\{\eta_t^1, \eta_t^2, \eta_t^3\}$ of complex forms of bidegree $(1,0)$ with respect to J_t satisfying*

$$d\eta_t^1 = d\eta_t^2 = 0, \quad d\eta_t^3 = \eta_t^{12} + \eta_t^{1\bar{1}} + \eta_t^{1\bar{2}} + i\,\frac{1+|t|^4}{4|t|^2}\eta_t^{2\bar{2}}. \tag{12}$$

Proof. By a direct calculation using (10), we get that the (1,0)-basis $\{\omega_t^1, \omega_t^2, \omega_t^3\}$ given in (11) satisfies $d\omega_t^1 = d\omega_t^2 = 0$ and

$$d\omega_t^3 = \frac{2i\bar{t}}{1+|t|^4}\omega_t^{12} + \frac{1-i|t|^2}{1+|t|^4}\omega_t^{1\bar{1}} + \frac{i-|t|^2}{1+|t|^4}\omega_t^{2\bar{2}}.$$

For each $t \in \Delta\backslash\{0\}$, let us consider the new (1,0)-basis $\{\tau_t^1 = \omega_t^1, \tau_t^2 = \frac{2i\bar{t}}{1-i|t|^2}\omega_t^2, \tau_t^3 = \frac{1+|t|^4}{1-i|t|^2}\omega_t^3\}$. Hence, the complex structure equations for J_t, $t \neq 0$, in this basis are expressed as

$$d\tau_t^1 = d\tau_t^2 = 0, \quad d\tau_t^3 = \tau_t^{12} + \tau_t^{1\bar{1}} + D'\,\tau_t^{2\bar{2}}, \quad 0 < |t| < 1,$$

where $D' = -\frac{1}{2} + i\frac{1-|t|^4}{4|t|^2}$.

Now, we consider another basis $\{\eta_t^1, \eta_t^2, \eta_t^3\}$ of bidegree $(1,0)$ with respect to J_t given by

$$\tau_t^1 = \eta_t^1 - D'\bar{\sigma}\,\eta_t^2, \quad \tau_t^2 = \sigma\,\eta_t^1 + \eta_t^2, \quad \tau_t^3 = (1 + D'|\sigma|^2)\eta_t^3,$$

where $\sigma = 2(1+|t|^2 i)/(1+|t|^4)$. By a long but direct calculation we arrive at

$$d\eta_t^1 = d\eta_t^2 = 0, \quad d\eta_t^3 = \eta_t^{12} + \eta_t^{1\bar{1}} + \eta_t^{1\bar{2}} + D''\,\eta_t^{2\bar{2}},$$

where $D'' = -i\,\frac{1+|t|^4}{4|t|^2}$. Finally, by [10, Proposition 2.4] there exists a (1,0)-basis with respect to which we can take the complex conjugate of D'' as the coefficient of $\eta_t^{2\bar{2}}$, that is, we arrive at the equations (12), and the proof is complete. □

The analytic family $\mathcal{X}$ mentioned in Section 3 was the first example of an analytic family of compact complex manifolds $(X_t)_{t\in\Delta}$ such that the complex invariants $\mathbf{f}_2(X_t) = \mathbf{k}_1(X_t) = 0$ for any $t \neq 0$, but $\mathbf{f}_2(X_0) \neq 0$ and $\mathbf{k}_1(X_0) \neq 0$ (Theorem 3.1), and its construction was based on an appropriate deformation of the abelian complex structure on the nilmanifold with underlying Lie algebra $\mathfrak{h}_4$ [10, 26]. Moreover, the Frölicher spectral sequence of any X_t degenerates at E_1 except for $t = 0$ [10]. The family $\mathcal{X}$ also allows to show that the sGG property is not closed under holomorphic deformations (Theorem 3.3) and, furthermore, the fibres X_t have balanced metric for any $t \in \Delta\backslash\{0\}$, but the central limit X_0 does not admit any strongly Gauduchon metric (Theorem 3.2).

The real nilmanifold underlying the compact complex manifolds X_t is not a product since the Lie algebra $\mathfrak{h}_4$ is irreducible. The following result is a bit of a surprise because it provides an example of a product manifold with a holomorphic family of complex structures satisfying similar properties to those of the analytic family $\mathcal{X}$.

Theorem 5.2. *Let $(N\times N, J_t)_{t\in\Delta}$ be the analytic family given by the product of two copies of the 3-dimensional Heisenberg nilmanifold N endowed with the complex structures J_t given by* (11)*. Then:*

(i) The invariants $\mathbf{f}_2(N\times N, J_t)$ and $\mathbf{k}_1(N\times N, J_t)$ vanish for each $t\in\Delta\backslash\{0\}$, however $\mathbf{f}_2(N\times N, J_0)=3$ and $\mathbf{k}_1(N\times N, J_0)=2$.

(ii) The Frölicher spectral sequence of $(N\times N, J_t)$ degenerates at the first step for each $t\in\Delta\backslash\{0\}$, however the central limit satisfies $E_1(N\times N, J_0)\ncong E_2(N\times N, J_0)\cong E_\infty(N\times N, J_0)$.

(iii) The complex manifold $(N\times N, J_t)$ is sGG for each $t\in\Delta\backslash\{0\}$, but $(N\times N, J_0)$ does not admit strongly Gauduchon metrics.

Proof. We will apply the general result on nilmanifolds given in Section 4 to the complex structures J_t on $N\times N$ given by (11). For the proof of (i) we first notice that the second Betti number of $N\times N$ is equal to 8. From Table 1 we have that $h^{1,1}_{\rm BC}(N\times N, J_t)=4$ for any $t\in\Delta$, but the Hodge number $\dim E_1^{0,2}(N\times N, J_t)=h^{0,2}_{\bar\partial}(N\times N, J_t)$ depends on the complex structure J_t. In fact, for $t\neq 0$ it follows from (12) that the Dolbeault class $[\eta_t^{\bar1\bar2}]$ vanishes because $\eta_t^{\bar1\bar2}=\bar\partial\eta_t^{\bar3}$, hence $H^{0,2}_{\bar\partial}(N\times N, J_t)=\langle[\eta_t^{\bar1\bar3}],[\eta_t^{\bar2\bar3}]\rangle$ and $h^{0,2}_{\bar\partial}(N\times N, J_t)=2$, for any $t\neq 0$. For the central limit, by (10) we get $H^{0,2}_{\bar\partial}(N\times N, J_0)=\langle[\omega_0^{\bar1\bar2}],[\omega_0^{\bar1\bar3}],[\omega_0^{\bar2\bar3}]\rangle$, so $h^{0,2}_{\bar\partial}(N\times N, J_0)=3$. In conclusion, the complex invariant

$$\mathbf{k}_1(N\times N, J_t)=h^{1,1}_{\rm BC}(N\times N, J_t)+2\dim E_1^{0,2}(N\times N, J_t)-b_2(N\times N)$$

vanishes on $\Delta\backslash\{0\}$, but it equals 2 for the central limit.

For the invariant $\mathbf{f}_2$ we recall that

$$\mathbf{f}_2(N\times N, J_t)=2\,h^{2,0}_{\rm BC}(N\times N, J_t)+h^{1,1}_{\rm BC}(N\times N, J_t)+2\,h^{3,1}_{\rm BC}(N\times N, J_t)+h^{2,2}_{\rm BC}(N\times N, J_t)-2b_2(N\times N).$$

By Table 1 we have that $h^{2,0}_{\rm BC}(N\times N, J_t)$ is constant and equal to 1, whereas $h^{3,1}_{\rm BC}(N\times N, J_t)=2$ and $h^{2,2}_{\rm BC}(N\times N, J_t)=6$ for any $t\neq 0$, but $h^{3,1}_{\rm BC}(N\times N, J_0)=3$ and $h^{2,2}_{\rm BC}(N\times N, J_0)=7$. Hence, $\mathbf{f}_2(N\times N, J_t)=0$ for any $t\neq 0$, but $\mathbf{f}_2(N\times N, J_0)=3$.

Property (ii) follows directly from the general study given in Theorem 4.4 (b). In fact, the nilpotent complex structure J_t is abelian if and only if $t=0$, hence the Frölicher spectral sequence of $(N\times N, J_t)$ degenerates at the first step if and only if $t\neq 0$. Moreover, the Frölicher sequence of the central limit collapses at the second step.

For the proof of (iii), since J_t is non-abelian for $t\neq 0$, by applying Theorem 4.8 to the $\mathfrak{h}_2$ case we have that $(N\times N, J_t)$ is sGG for any $t\neq 0$. Nevertheless the central limit is not sG because J_0 is abelian and we can apply Proposition 4.6. □

Remark 5.3. Notice that complex manifold $(N \times N, J_t)_{t\in\Delta}$ does not admit balanced metric for any value of t. In fact, for any invariant non-abelian complex structure J on $N \times N$, there exists a (1,0)-basis $\{\omega^1, \omega^2, \omega^3\}$ satisfying

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}} + D\,\omega^{2\bar{2}},$$

where $D = x + iy$ with $y > 0$, and by Proposition 4.7 we know that $(N \times N, J)$ admits a balanced metric if and only if $x + y^2 < \frac{1}{4}$. In our case, for J_t $(t \neq 0)$ it follows from Proposition 5.1 that $x = 0$ and $y = \frac{1+|t|^4}{4|t|^2}$, thus

$$x + y^2 - \frac{1}{4} = \left(\frac{1-|t|^4}{4|t|^2}\right)^2 > 0$$

and there is no balanced metric on $(N \times N, J_t)$.

Remark 5.4. Angella and Kasuya found in [6] a holomorphic deformation that shows that the $\partial\bar{\partial}$-lemma property is not closed. More recently, in [19] it is proved the existence of an analytic family of compact complex manifolds $(X_t)_{t\in\Delta}$ such that X_t satisfies the $\partial\bar{\partial}$-lemma and admits balanced metric for any $t \in \Delta\backslash\{0\}$, but the central limit X_0 neither satisfies the $\partial\bar{\partial}$-lemma nor admits balanced metrics. Both constructions are based on the complex geometry of the real solvmanifold underlying the Nakamura manifold [30], which is not diffeomorphic to a product manifold.

We finish with the following related question:

Question 5.5. *Does there exist a holomorphic deformation of* $(N \times N, J_0)$ *admitting balanced metrics?*

To answer this question, a more detailed study of the Kuranishi space of deformations of J_0 would be required.

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Hunting the Axion

Igor G. Irastorza

Departamento de Física Teórica,

Área de Física Atómica, Molecular y Nuclear.

Universidad de Zaragoza, 50009 Zaragoza

Premio a la Investigación de la Academia 2014. Sección de Físicas

Abstract

Probably the most important challenge of Particle Physics for the coming century is to understand the fundamental nature of the Dark Universe. More than 95% of the Universe is composed by two components, Dark Matter and Dark Energy, for which Cosmology provides growing evidence, but of which we lack a proper fundamental (Particle Physics) interpretation. The recent discovery of the Higgs boson at CERN seems to complete a very successful chapter of Particle Physics, which embodies the current understanding of the known particles and their interactions in the so-called Standard Model. However, a number of theoretical considerations point to the necessity of Physics beyond the Standard Model. With no experimental indication of such Physics in accelerators (whose data is on the contrary outstandingly well explained with just the Standard Model), is the existence of Dark Matter (and to some extent Dark Energy) viewed as the most powerful observational motivation for the study of extensions of the Standard Model. Some of these extensions predict extremely light and neutral particles that interact very weakly with standard model particles. The *axion* is the most popular and the prototype of this class of particles. Motivated by theory, but maybe also by some intriguing astrophysical observations, they at the same time constitute perfect Dark Matter candidates. Highly complementary to the exploration of the high energy frontier at the Large Hadron Collider (LHC) at CERN, the searches for axions at the low energy frontier are recently attracting an increasing attention. In this article I will review the motivation of the axion hypothesis, and the reason for the current interest, the

status of the experimental techniques of these searches and their prospects to eventually detect the axion in the near future. I will finally focus on a recent initiative, the International Axion Observatory (IAXO), the most ambitious project to test the existence of axions. IAXO will be built on the experience acquired so far in the field and will highly surpass current experimental sensitivity to venture deep into uncharted axion parameter space. If the axion exists, IAXO will have good chances of discovering it. It could be the next breakthrough discovery in our race to understand the Universe, and how it works at its most fundamental level.

Contents

1 Introduction

The currently accepted picture of the Universe shows that the conventional baryonic matter (planets, stars, galaxies, interstellar dust and gas, etc.) comprises only less than 5% of the total matter-energy of the Universe. The remaining majority –the dark side of the Universe– is made of Dark Energy (DE) 68.3% and Dark Matter (DM) 26.8%[1]. DE is commonly seen as a sort of vacuum energy that would be responsible for the accelerated expansion of the Universe, however whether and how we can understand it at the fundamental level (e.g. as a quantum field) is still totally unclear. On the other hand, DM seems to be composed by a kind of matter that is invisible, collision-less, and whose existence is now clearly evidenced through its gravitational effects. The particle nature of DM however is currently a mystery, and its determination is one of the hottest issues in modern fundamental physics.

From the particle physics perspective, DM particles must have mass, be electrically neutral and interact very weakly with the rest of matter. They have to provide a viable mechanism for being produced in sufficient quantities in the early Universe. Within the Standard Model (SM) of particle physics, the neutrinos have been long ago considered as possible DM candidates, however, they fail to reproduce the large scale structure of the Universe, and their contribution must be restricted to a very small fraction of relativistic (hot) DM. One has, therefore, to resort to extensions of the SM in order to find a candidate that can solve the puzzle. The problem of DM is thus linked to the very interesting topic of physics beyond the SM. Although there are numerous ways to postulate DM candidates in beyond-SM theories, the attention is focused on SM extensions that are well motivated *per se* (i.e. they are proposed to solve one of the theoretical shortcomings of the SM) and in addition provide a good DM candidate. A very popular example, attracting most of the experimental attention, are the Weakly Interacting Massive Particles (WIMPs) that appear in supersymmetric extensions of the SM.

Other extensions of the SM predict particles that could lie hidden at the low energy frontier, and could also be good candidates for the DM. The axion appearing as part of the Peccei-Quinn mechanism solving the strong CP problem is the prototype of this kind of particles. The fact that Supersymmetry has not yet been observed at the Large Hadron Collider (LHC) at CERN and that no clear signal of WIMPs has appeared in Dark Matter experiments (despite an enormous advance in sensitivity of these experiments in the last decade) has increased the community's interest for searching for axions. However axions are independently and powerfully motivated, and a Dark Matter composed by both WIMPs and axions is perfectly viable implying that they should not be considered as alternative and exclusive solutions to the same problem.

The search for axions has been a relatively minor, but continuously growing, field of experimental particle physics since the axion hypothesis was first proposed in the late 70's. Recently the field is going through a transition. In this article I will briefly review the theoretical motivation for axions, the latest advances in our understanding of their role in Cosmology and some astrophysical scenarios. I will review the status of the experimental searches for axions and finally I will focus on the International Axion Observatory (IAXO), a recent initiative of larger scale that any previous axion experiment, that aims to explore a large fraction of the still allowed parameter space for the axion.

2 The strong CP problem and axions

The recent discovery of the Higgs boson would complete the experimental confirmation of the particle content of the very successful Standard Model (SM) of particle physics. However, it is known that the SM is incomplete, as it does not explain some basic features of our Universe, like e.g. the nature of DM and DE, or the matter-antimatter unbalance, and does not provide satisfactory explanations for a number of features of the SM itself. One of the latter is the so-called *strong-CP problem*, or why the strong interactions seem not to violate the charge-parity (CP) symmetry [2]. Indeed, the lagrangian of Quantum Chromodynamics (QCD) includes a CP-violating term:

$$\mathcal{L}_{\bar{\theta}} = \bar{\theta}\frac{\alpha_s}{8\pi}G^{\mu\nu a}\tilde{G}_{\mu\nu a} \tag{1}$$

where G is the gluon field and $\tilde{G}$ its dual, and α_s the strong coupling constant.

One of the experimental consequences of the term $\mathcal{L}_{\bar{\theta}}$ would be the existence of electric dipole moments (EDMs) for protons and neutrons. The non-observation of the neutron EDM [3] puts a very strong limit on the magnitude of the $\bar{\theta}$ parameter in (1) of about $|\bar{\theta}| \lesssim 10^{-10}$. But the problem is more than the unexplained smallness of an arbitrary SM parameter. The $\bar{\theta}$-angle is actually the sum of two contributions, which are in principle unrelated. The first is the angle characterizing the QCD vacuum and the second is the common phase of the matrix of SM quarks –coming from the Higgs Yukawa couplings which are known to violate CP sizeably. Thus, the smallness of $\bar{\theta}$ requires that two completely unrelated terms of the SM (from two different sectors) cancel each other with a precision of at least 10^{-10}. The strong CP problem constitutes a very serious fine-tuning issue that remains unexplained in the SM.

The most convincing solution of the strong CP problem is the Peccei-Quinn (PQ) mechanism, proposed in 1977 [4, 5]. It introduces a new U(1) global symmetry (the PQ symmetry) that is spontaneously broken at a high scale f_a. This implies the existence of a new field a which appears as the pseudo-Nambu-Goldstone boson of the new symmetry. The term $\mathcal{L}_{\bar{\theta}}$ ends up absorbed in a new term of the type $G^{\mu\nu}\tilde{G}_{\mu\nu}a/f_a$ where a is now a dynamical variable, which can relax to a CP-conserving minimum. This solves the fine-tuning problem dynamically, for any value of f_a. The main observational consequence of the PQ mechanism, as was first pointed out by Weinberg and Wilczek [6, 7], is that the quantum excitations of this field –albeit very weakly coupled– are potentially observable as new particles: axions.

The PQ mechanism fixes some of the properties of the axion [8, 9] like, e.g., its mass m_a, acquired via mixing with the pseudoscalar mesons,

$$m_a \simeq \frac{m_\pi f_\pi}{f+a} \frac{\sqrt{m_u m_d}}{m_u + m_d} \simeq 6\,\text{meV} \frac{10^9\,\text{GeV}}{f_a} \tag{2}$$

where $m_\pi = 135$ MeV is the pion mass, $f_\pi \approx 92$ MeV the pion decay constant and $m_{u,d}$ the light quark masses. This same mixing makes the axions interact with hadrons and photons. All the axion couplings are suppressed by the PQ symmetry scale f_a, which is not determined by theory.

More concrete axion properties depend on the specific implementation of the PQ symmetry in the SM. Originally f_a was identified with the weak scale, but accelerator data quickly ruled out this possibility constraining it to be higher than about 10^5 GeV. If the fermions of the SM do not have PQ charge, axions do not couple with them at tree level. These are called "hadronic axions", of which the KSVZ [10, 11] model is an often quoted example. Other models, like the DSFZ [12, 13], feature tree-level coupling with SM fermions, e.g., the axion electron coupling g_{ae}.

The property of axions most relevant for experiment is the axion-two-photon coupling $g_{a\gamma}$

$$\mathcal{L}_{a\gamma} \equiv -\frac{g_{a\gamma}}{4} a F^{\mu\nu} \tilde{F}_{\mu\nu} = g_{a\gamma}\, \mathbf{E} \cdot \mathbf{B}\, a\,, \tag{3}$$

where F is the electromagnetic field-strength tensor and $\tilde{F}$ its dual, while $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields. The coupling $g_{a\gamma}$ can be expressed as

$$g_{a\gamma} = \frac{\alpha}{2\pi f_a} C_\gamma \quad ; \quad C_\gamma \equiv \frac{E}{N} - \frac{2}{3}\frac{4m_d + m_u}{m_d + m_u} \simeq \frac{E}{N} - 1.92 \tag{4}$$

where the loop factor $\alpha/2\pi f_a$ reflects the fact that this is a coupling generated from the electromagnetic anomaly. C_γ is a coefficient of order 1 with two contributions: a model-independent one due to the axion mixing with pseudoscalar mesons and a model dependent one E/N which arises if the PQ symmetry is not only colour-anomalous but also has a non-zero electromagnetic anomaly (E and N are the electromagnetic and colour anomalies of the PQ symmetry). In general, a broad range of values for E/N is possible, depending on the axion model (e.g. for DFSZ $E/N = 8/3$ and $C_\gamma \simeq 0.75$, whereas for KSVZ $E/N = 0$ and $C_\gamma \simeq -1.92$, if the new heavy quarks are taken without electric charge).

Because the axion-photon interaction is generic and because photons offer many experimental options, most axion search strategies are based on this interaction. Axions mix with photons in the presence of external magnetic fields, leading to axion-photon oscillations [14, 15], similar to the well known neutrino oscillations, and to changes in the polarization state of photons propagating in a magnetic field [15, 16]. The $a\gamma\gamma$ coupling also leads to the Primakoff conversion of plasma photons into axions within stellar cores, the main axion emission channel of the Sun. The Primakoff conversion is also behind the detection principle of axion helioscopes, haloscopes [14] as well as LSW experiments, as discussed later on. The results of these searches are therefore represented in the parameter space ($g_{a\gamma}$,m_a) that is shown in Fig. 1. Because $g_{a\gamma}$ and m_a are linked for a specific axion model (both are inversely proportional to f_a), an axion model is represented by a straight diagonal line in such plot (the green line in Fig. 1 correspond to the KSVZ model). The overall spread of axion models resulting from the possible values of E/N in (4) is represented by the width of the yellow band of Fig. 1.

In some particular implementations of the PQ mechanism, the axion may couple to leptons at tree level. This is most importantly the case of models in which the SM is embedded in a Grand Unified Theory, where colour and electroweak interactions are unified in a larger non-abelian symmetry. The coupling to electrons can be written in two forms, equivalent for our purposes,

$$\mathcal{L}_{ae} = C_{ae}\frac{\partial_\mu a}{f_a}\bar{\psi}_e\gamma^\mu\gamma^5\psi_e \leftrightarrow g_{ae}a\bar{\psi}_e\gamma^5\psi_e \tag{5}$$

where C_{ae} is a coefficient of order 1 given by specifics of the model. The equivalent Yukawa coupling is $g_{ae} = C_{ae}m_e/f_a$ where m_e is the electron mass. For instance in the DSFZ model [12, 13] $C_{ae} = \frac{1}{3}\cos^2\beta$ where $\tan\beta$ is the ratio of the v.e.vs of the two Higgses present in the theory. When C_{ae} is zero at tree-level, a non-zero value is generated by radiative corrections, but being loop-suppressed is typically irrelevant. When the axion couples to electrons with $C_{ae} \sim \mathcal{O}(1)$, this coupling drives the most efficient axion-production reactions in stars like the Sun, low-mass red giants and white dwarf stars.

2.1 Main constraints on the axion properties

Since it was first proposed, the axion has been thoroughly studied for its implications in astrophysics, cosmology and particle physics. The most relevant limits on its properties have been drawn from astrophysical considerations [29]. The emission of axions from the Sun is nowadays best constrained from the increase they imply in the solar neutrino flux with respect to the standard one without axions [30]. The axion flux originated from the Primakoff effect constrains the coupling $g_{a\gamma} \lesssim 0.7 \times 10^{-9}$ GeV^{-1} while the axio-Bremsstrahlung in electron collisions (and other reactions involving electrons) constrains

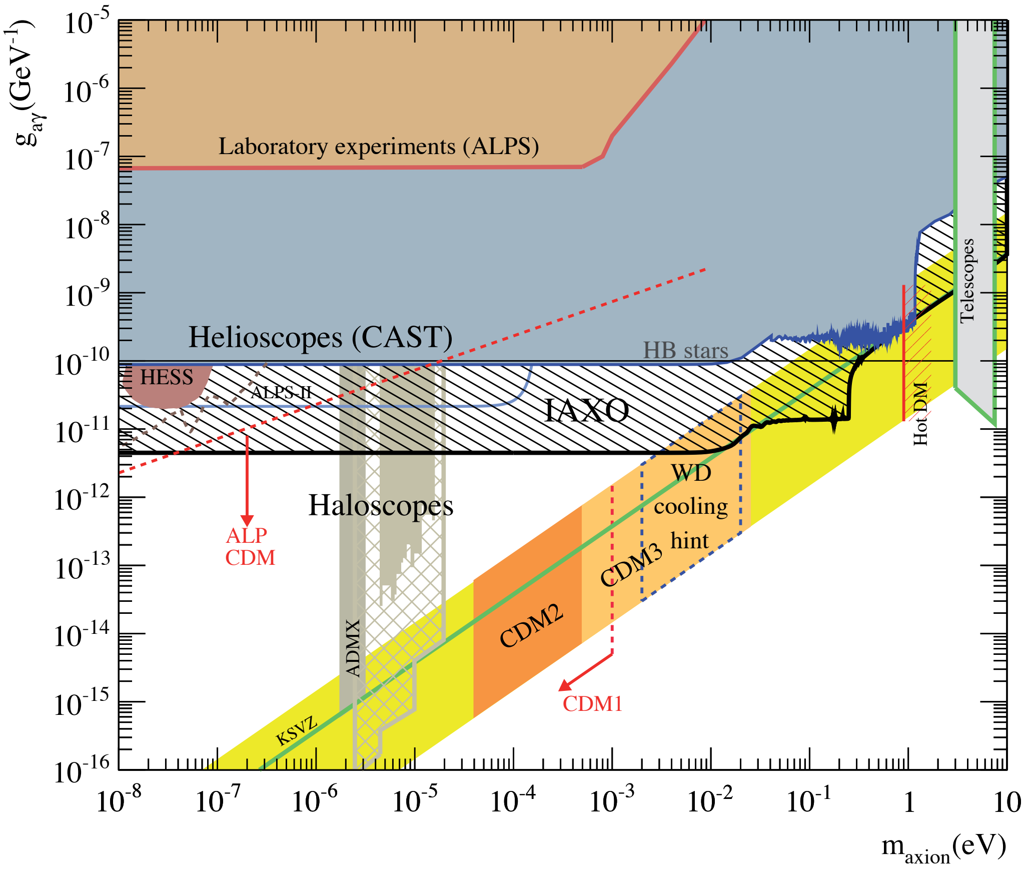

Figure 1.— Comprehensive axion/ALP parameter space, highlighting the three main front lines of direct detection experiments: LSW experiments (ALPS [17]), helioscopes and haloscopes. The blue line corresponds to the current helioscope limits, dominated by CAST [18, 19, 20, 21] for practically all axion masses. Also shown are the constraints from horizontal branch (HB) stars, and hot dark matter (HDM) and the ones from searches of decay lines in telescopes [22, 23, 24]. The yellow "axion band" is defined roughly by $m_a f_a \sim m_\pi f_\pi$ with a somewhat arbitrary width representing the range of realistic axion models. The green line refers to the KSVZ model. The orange parts of the band correspond to cosmologically interesting axion models: models in the "classical axion window" possibly composing the totality of DM (labelled "CDM2") or a fraction of it ("CDM3"). The anthropic window ("CDM1") corresponds to a range unbound on the left and up to ∼1 meV. For more generic ALPs, practically all the allowed space up to the red dash line may contain valid ALP CDM models [25]. The region of axion masses invoked in the WD cooling anomaly is shown by the blue dash line. The region at low m_a above the dashed grey line is the one invoked in the context of the transparency of the universe [26] (note that it extends to masses lower than the ones in the plot), while the solid brown region is excluded by HESS data [27]. The labeled hatched region represents the expected sensitivity of IAXO in the baseline helioscope configuration as explained in the text. Also future prospects of ADMX (hatched brown region) and ALPS-II [28] (light blue line) are shown.

$g_{ae} < 2.5 \times 10^{-11}$. The population of low-mass horizontal-branch (HB) stars and red-giants (RG) in globular clusters gets decreased and increased respectively when axions are freely emitted from their interiors. Fitting the observed population to numerical simulations one derives the limits $g_{a\gamma} \lesssim 10^{-10}$ GeV^{-1} (mostly from the impact on HBs) [29, 31] and $g_{ae} < 4.7 \times 10^{-13}$ at 95% C.L. (from the tip of the RG branch in the cluster M5) [32]. A more recent and detailed revision of this result lowers the bound to $g_{a\gamma} \lesssim 0.66 \times 10^{-10}$ GeV^{-1} [33]. However, at the same time, the data seem to slightly prefer some level of axion emission. Recently, it has been argued that the Primakoff flux of axions from $g_{a\gamma} > 0.8 \times 10^{-10}$ GeV^{-1} will shorten so much the helium-burning phase (so called blue loop) of *massive stars* that Cepheids could not be observed [34], and thus it is excluded. Observations and theory of white dwarf cooling fit to the extent that they can exclude values of the electron coupling $g_{ae} > 3 \times 10^{-13}$ [35, 36, 37, 38]. However, some observable such as the luminosity function and the period decrease of the variable ZZ Ceti star G117-B15A seem to prefer some slight extra cooling.

Due to their coupling of protons and nucleons, axions can be efficiently emitted from the core of a Type-II supernova shortening the neutrino pulse. From the observation of the $\sim$10 s duration neutrino burst of SN1987A, which fits the expectations without extra axion cooling, one can derive the limit to the axion-proton Yukawa-like coupling $g_{ap} \lesssim 10^{-9}$ [39]. In general one has $g_{ap} = C_{ap} m_p / f_a$. For hadronic axions $C_{ap} \sim 0.4$ and thus $f_a > 4.8 \times 10^8$ GeV or, equivalently, $m_a < 16$ meV. For DFSZ axions C_{ap} tends to be smaller and the constraints weaker, but not much because then the axion-neutron coupling becomes relevant. Similar bounds for $f_a \gtrsim 10^9$ GeV arise from neutron star cooling [40, 41]. These limits remain fairly rough estimates, due to the uncertainty in the axion emission rate, supernova modeling and the observations.

Other astrophysical, cosmological and experimental bounds, some of them commented on later, further constrain the allowed axion parameter space. However, not only have these bounds not rejected the axion, but the motivation for the existence of axions, beyond the strong CP problem, has grown on several fronts. The axion is a candidate for the dark matter of the Universe, and several tantalizing hints in astrophysics could be the result of axion-like particles at play. More recent theoretical advances are defining a more generic category of light fundamental particles, the weakly interacting slim particles (WISPs), of which the axion is the most outstanding prototype, appearing in other well-motivated extensions of the SM, like string theory. After 35 years, the axion not only remains associated with the the most compelling solution to the strong CP problem, but it is recognized as one of the best motivated experimental portals to physics beyond the SM.

3 The axion as a dark matter candidate

As mentioned in the introduction, the particle DM requires new fields beyond the SM. A popular example is the "weakly interacting massive particle" (WIMP) typically appearing in supersymmetric extensions of the SM, and actively searched for in underground experiments. However, for the time being there is no hint of supersymmetry at the LHC and also no clear signature for WIMPs in direct-detection experiments whose sensitivity to the WIMP-nucleon cross-section has advanced by an amazing four orders of magnitude over the last decade.

It has been known since the early 80s that the PQ mechanism provides a very compelling scenario for relic axion production. As shown below, axions are just as attractive a solution to the DM problem as WIMPs. Like the latter, they appear in extensions of the SM that are independently motivated and also provide a valid DM candidate (i.e. they are not conceived *ad hoc* for that purpose). Moreover, the possibility of a mixed WIMP-axion DM is not only not excluded, but theoretically appealing [42, 43]. Conventionally, both axion and WIMP cold DM are thought to behave identically at cosmological and astrophysical scales, so there is no hint from cosmology to prefer one or the other (or both). However, although still speculative, some potentially discriminating signatures have been proposed. It has been recently suggested [44] that cold axions form a Bose-Einstein condensate, and this would produce a peculiar structure in DM galactic halos (caustic rings) for which some observational evidence seems to exist [45]. This fact would be applicable to any WISP cold DM population, but not to WIMPs.

Relic axions can be produced thermally by collisions of particles in the primordial plasma, just like WIMPs. However, being quite light particles, this axion population contributes –like neutrinos– to the hot DM component. This production mechanism is more important for larger m_a. Cosmological observations constraining the amount of hot DM, can be translated into an upper bound on the axion mass of $m_a \lesssim 0.9$ eV [46, 47].

Most interesting from the cosmological point of view is the non-thermal production of axions: the *vacuum-realignment* mechanism and the decay of topological defects (axion strings and domain-walls), both producing non-relativistic axions and therefore contributing to the cold DM [48, 49].

In the very early universe, when the temperature drops below $\sim f_a$, i.e. at the PQ phase transition, the axion field appears in the theory and sets its initial value differently in different causally-connected regions. Later, at the QCD phase transition, the axion potential rises and only then does the axion acquire its mass m_a. Then the axion field relaxes to its CP conserving minimum, around which it oscillates with decreasing amplitude (thus solving the strong CP problem dynamically). These oscillations represent a population of non-relativistic axions, with a density that depends on the unknown initial value of the field before the start of the oscillations (initial misalignment angle $\theta_0 \equiv a_0/f_a \in (-\pi, \pi)$). Moreover, because a/f_a is an angle variable, discrete domains, differing in 2π naturally form after QCD transition and at their borders topological defects, i.e., strings and walls, form too. These defects soon decay radiating a large amount of non-relativistic axions which add up to the realignment population. While the realignment population density is relatively easily calculable (although dependent on the value of θ_0), the population from the decay of topological defects suffers from significant uncertainties.

In general, two main cosmological scenarios can be considered, depending on whether inflation happens after (*pre-inflation* scenario) or before (*post-inflation* scenario) the PQ phase transition (or if the PQ symmetry is restored by reheating after inflation). In the pre-inflation case, the axion field is homogenized by inflation, the value of θ_0 is thus unique in all the observable Universe, and the topological defects are diluted away. In that case the axion cold DM density is easily determined, as *only* the realignment mechanism contributes to it. Expressed as the ratio of axion DM to the observed value $\Omega_{\rm DM,obs}h^2 = 0.111(6)$ is [50, 51]

$$\frac{\Omega_a}{\Omega_{\rm DM,obs}} \sim \theta_0^2 F \left(\frac{f_a}{5\times 10^{11}\,{\rm GeV}}\right)^{1.184} \simeq \theta_0^2 F \left(\frac{12\,\mu{\rm eV}}{m_a}\right)^{1.184} . \tag{6}$$

where $F = F(\theta_0, f_a)$ is a correction factor accounting for anharmonicities, the delay of the oscillations when θ_0 is large and any effects of non-standard cosmologies (the above expression is computed assuming radiation domination during the QCD phase transition).

Contrary to thermal production, this mechanism leads to larger relic axion density for lower m_a. For typical values of $\theta_0 \sim \mathcal{O}(1)$, m_a should exceed $\sim$10 μeV to have a relic axion density not exceeding the known CDM density. Much smaller masses could still give the correct amount of DM if by some means θ_0 is accidentally small, something that could be justified for instance by anthropic reasons [52]. Axion masses up to $m_a \sim$ meV can still give the adequate relic density for large θ_0.

In the post-inflation scenario, the value of θ_0 is randomly distributed in different causally-connected parts of the universe at the time of the PQ phase transition. One then has to average the above result for $\theta_0 \in (-\pi, \pi)$ and obtain a robust estimate of the DM contribution due to vacuum-realignment:

$$\frac{\Omega_{a,\rm VR}}{\Omega_{\rm DM,obs}} \sim \left(\frac{40\,\mu{\rm eV}}{m_a}\right)^{1.184} . \tag{7}$$

However, in this case the contribution of axion strings and domain-wall decays to axion DM must be taken into account, but its computation is rather uncertain and a matter of a longstanding debate. Some authors argue that the contribution is of the same order as $\Omega_{a,\rm VR}$ [53], while others [49, 54] find it considerably larger:

$$\frac{\Omega_{a,\text{string+wall}}}{\Omega_{\text{DM,obs}}} \sim \left(\frac{400\,\mu\text{eV}}{m_a}\right)^{1.184} \tag{8}$$

In any case, axions could easily account for the totality of cold DM needed by current cosmological models. A prediction of the axion mass leading to this situation is not possible with precision due to the above uncertainties, but it is clear that this can happen for a wide range of feasible axion models well beyond current limits. The "classic axion window" [49], $m_a \sim 10^{-5} - 10^{-3}$ eV, associated with the post-inflation scenario above, is often quoted as the preferred m_a range for axion cold DM, although much lower masses are still possible in the fine-tuned models of the pre-inflation scenario, sometimes also called "anthropic axion window" [55, 56].

Recently, the BICEP2 experiment announced the detection of primordial gravitational waves in the CMB [57]. If true, this would point to a very high energy scale for inflation. If PQ transition had happened before that scale, the axion field would have imprinted isocurvature perturbations in the CMB which are not observed. The BICEP2 observation and interpretation have been questioned and independent experimental confirmation is needed. If confirmed, the pre-inflation scenario for the axion DM would be ruled out, and the mass of the axion would be constrained to values above $\sim 10^{-5}$ eV. QCD axions with masses above the classic window can still solve the DM problem if non-standard cosmological scenarios are invoked [58], or they can be a subdominant DM component. Mixed axion-WIMP DM is a possibility that may even be theoretically appealing [42, 43]. Moreover, axions are not the only WISPs allowing for a solution to the dark matter question. The nonthermal production mechanisms attributed to axions are indeed generic to bosonic WISPs such as axion-like particles or hidden photons (see next section). As recently shown [25], a wide range of $g_{a\gamma} - m_a$ space can generically contain models with adequate DM density.

To summarize, axions are as attractive a solution to the DM problem as WIMPs. In the current situation, with no hint from supersymmetry at the LHC and without a clear signature in WIMP direct-detection experiments, the hypothesis of axion DM stands out as increasingly interesting and deserves serious attention. The cosmological implications of the axion are well founded and represent a powerful motivation to push experimental searches well beyond current limits.

4 Other axion-like particles (ALPs)

Although the axion is the best motivated and most studied prototype, a whole category of particles called axion-like particles (ALPs) or, more generically, weakly interacting sub-eV particles, WISPs, are often invoked in several scenarios, both theoretically and observationally motivated, at the low energy frontier of particle physics [59]. Not necessarily related to the axion, ALPs share part of its phenomenology, and therefore would be searchable by similar experiments. ALPs are light (pseudo)scalar particles that weakly couple to two photons, but not to two gluons like the axion [60, 61]. As such, the ALPs parameters $g_{a\gamma}$ and m_a are now to be viewed as completely independent, and the full parameter space of Fig. 1 is potentially populated by ALPs (not constrained to the yellow band as the axion models are).

ALPs can appear in extensions of the SM as pseudo Nambu-Goldstone bosons of new symmetries broken at high energy. Moreover, it is now known that string theory also predicts a rich spectrum of ALPs (including the axion itself) [62, 63, 64]. Remarkably, the region of the ALP parameter space at reach for future experiments is theoretically favoured as they correspond to string scales contributing to the natural explanation of several hierarchy problems in the SM. It is intriguing that the possible detection of ALPs could become key to the much sought experimental test of string theory.

Beyond ALPs, other important examples of WISPs are hidden photons and minicharged particles [65, 66, 67]. They appear in extensions of the SM including hidden sectors, i.e., sectors that interact with SM particles through the interchange of very heavy particles (e.g., a hidden sector is commonly employed for supersymmetry breaking). Hidden photons have kinetic mixing with normal photons, and therefore show a phenomenology similar to axion-photon oscillations (but this time without the external magnetic field), leading to the dissapearance and regeneration of photons as they propagate in vacuum [66]. Minicharged particles are particles with fractional electric charge, arising naturally in theories with hidden sectors.

As previously mentioned, under some circumstances, WISPs can also provide the right DM density. The nonthermal production mechanisms described in the previous sections are indeed generic to other bosonic WISPs such as ALPs or hidden photons. This WISPy DM has recently been studied [25], and in both cases a wide range of parameter space (in the case of ALPs $g_{a\gamma}$–m_a space) can generically contain models with adequate DM density, part of it at reach of current or future experiments.

It is remarkable that light scalars are also invoked in attempts to find a particle physics interpretation of Dark Energy, the so-called "quintessence" fields. This possibility is very much constrained from the non observation of new long-range forces, unless more sophisticated mechanisms are implemented, mechanisms that lead sometimes to ALP phenomenology [68]. More recently, fields with an environment-dependent mass or couplings, chameleons [69] or galileons, are being studied in this same context. Despite the early stage of development of these concepts, the possibility that detection techniques originally conceived to search for axions or ALPs could evolve into the first particle physics experiments directly testing Dark Energy is truly exciting.

All these families of models compose together a growing field of theoretical research.

It is now acknowledged that, complementary to the conventional research performed at colliders of increasing energy (the high energy frontier), new physics can be hidden at very low energies too (the intensity frontier) for which different experimental tools, based rather on high precision and high source intensity, are required.

5 Astrophysical hints for axions and ALPs

The existence of axions or ALPs may have important consequences for some astrophysical phenomena. Since the early days of axions, well understood stellar physics has been used to constrain axion couplings [29] and derive limits, the most relevant of which have been presented previously. More intriguing are the cases where unexplained astrophysical observations may indicate the effects of an ALP. These situations must be treated with caution because usually an alternative explanation using standard physics or an uncontrolled systematic effect cannot be ruled out. At the same time, such models can further strengthen the physics case for exploring favored regions of parameter space, when other motivations already exist. Two such cases can be considered specially relevant: the excessive transparency of the intergalactic medium to very high energy (VHE) photons, and the anomalous cooling rate of white dwarfs.

VHE photons (i.e., with energies $\gtrsim$ 100 GeV) have a non-negligible probability to interact via e^+e^- pair production with the background photons permeating the Universe – the extragalactic background light (EBL) – when long intergalactic distances are involved. That is, the Universe should be opaque to distant VHE emitters like active galactic nuclei (AGN). EBL density is measured by its imprint in blazar spectra by both HESS[70] and Fermi [71], and found in agreement with models. However, several independent observations seem to indicate that the degree of transparency of the Universe at VHE is too high, even for the lowest density EBL models developed [72, 73]. Current imaging atmospheric Cherenkov telescopes (both HESS [74] and MAGIC [75, 76]) have reported the observation of VHE photons with arrival directions clearly correlated with AGNs, some of them as distant as a $\sim$Gpc, with spectra that require either a too low density EBL, or anomalously hard spectra at origin. Alternatively, these photons could be secondaries produced in electromagnetic cascades [77], but this is in conflict with the sometimes fast time-variability of these sources [26]. Independent additional evidence might come from the observation of ultra high energy cosmic rays (UHECR) of energies $E > 10^{18}$ GeV correlated with very distant blazars [78, 79].

These observations could be easily explained by scenarios invoking photon-ALP oscillations triggered by intervening cosmic magnetic fields. These fields can be the intergalactic magnetic field, or the local magnetic fields at origin (at the AGN itself, or in the case of objects belonging to galactic clusters, the cluster magnetic field) and in the Milky Way.

Thus, the ALP component can travel unimpeded through the intergalactic medium, and as a result the effective mean free path of the photon increases. Several authors have invoked one of these scenarios [80, 81, 82, 83, 84, 85, 73, 26] to account for the unexplained observations. For some of these cases, approximate required ALP parameters $g_{a\gamma}$ and m_a are drawn. Interestingly, most of them coincide roughly in requiring very small ALP mass $m_a \lesssim 10^{-(10-7)}$ eV (to maintain coherence over sufficiently large magnetic lengths) and a $g_{a\gamma}$ coupling in the ballpark of $g_{a\gamma} \sim 10^{-12}$–$10^{-10}$ GeV^{-1}. A more definite region -shown in Fig. 1- is extracted in [26] from a large sample of VHE gamma-ray spectra. Note that it extends to lower m_a values than the ones shown on the plot. Although these parameters are far from the standard QCD axions, as more generic ALP models they lie just beyond the best current experimental limits on $g_{a\gamma}$ from CAST (see next section). As commented later on (and shown in Fig. 1) most of this region could be explored by IAXO.

The random character of astrophysical magnetic fields produces a particular scattering of the photon arrival probability that complicates the test of the ALP hypothesis [86]. In turn, this randomness can be used to constraints the ALP parameters, as it should imprint irregularities in high-energy source spectra [87]. This effect is used by the HESS collaboration with blazar observations to exclude couplings of the order of a few 10^{-11} GeV^{-1} for masses of $10^{-8} - 10^{-7}$ eV (1), as shown in Fig. 1. The same method is used with X-ray data from the Hydra galaxy cluster [88] constraining $g_{a\gamma} < 8 \times 10^{-12}$ GeV^{-1} for ALP masses $< 10^{-11}$ eV. Still in the X-ray band, some luminosity relations of active galactic nuclei were recently shown to have precisely this particular scatter [89] although this claim is still controversial [90]. Finally, photon-ALP mixing is polarization dependent, a fact that could explain long-distance correlations of quasar polarization [91] and offers further testing opportunities [92]. This possibility is nonetheless challenged by the absence of significant circular polarisation [93].

A different astrophysical scenario for which axion-related hypothesis have been invoked are the interior of white dwarf (WD) stars. The evolution of these objects follows just a gravothermal process of cooling, therefore their luminosity function (number of stars per luminosity interval) is predicted with accuracy by stellar models. The presence of extra cooling via axion emission speeds up the cooling thus suppressing the luminosity function at certain values of the WD luminosity. This is most relevant for non-hadronic axions with coupling to electrons g_{ae}, because axio-bremstrahlung would be very efficient in WDs. These arguments constrain $g_{ae} < 3 \times 10^{-13}$ [35] and they have been cross-checked and improved over the years [36, 37, 38, 94]. However, recent works are based on such a well populated luminosity function and well-studied WD cooling models that are able to claim that a small amount of axion energy loss is actually favored by data [36, 37]. This claim corresponds to $g_{ae} \sim 1 - 2 \times 10^{-13}$. Further evidence for extra cooling in WDs comes from independent observations. The period decrease of certain pulsating WDs provides a direct measurement of their cooling and thus can be used to assess the necessity of non-standard cooling mechanisms. Two pulsating WDs have been studied and shown a preference for axion cooling: the ZZ Ceti star G117-B15A [95] and R548 [96]. Both fit better the expectations for $g_{ae} \simeq 5^{+1.2}_{-1.6} \times 10^{-13}$ and $g_{ae} \simeq 5^{+1.7}_{-4.9} \times 10^{-13}$, respectively ($2\sigma$ intervals quoted). Given the scatter of the preferred values of g_{ae}, the tension with other limits and the possibility of unaccounted systematics or forgotten standard effects it is certainly premature to conclude the existence of axion energy loss in WDs. However, it is intriguing that all theses observables seem to improve with some extra cooling, which could be attributed to axions (or any pseudoscalar with coupling to electrons) with $g_{ae} \sim 1 - 5 \times 10^{-13}$.

These g_{ae} values imply axion decay constants in the range $f_a \in (2{-}5) \times C_{ae} 10^9$ GeV, corresponding to an axion mass $m_a \in (1{-}4)$ meV$/C_{ae}$. For DSFZ axions $C_{ae} < 1/3$ and this value corresponds to axion masses $m_a > 3$ meV (see "WD cooling hint" in Fig. 1). As shown later, IAXO may reach sensitivity to these models through the direct observation of solar axions from g_{ae}-reactions. Finally, generic ALPs appearing in field and string theory extensions of the SM can just, as DFSZ axions, feature a coupling of electrons and photons. In this context, the WD favored region is sometimes expressed as a $g_{a\gamma}$ range [97] of typically $g^{\rm ALP}_{a\gamma} \sim (C^{\rm ALP}_{\gamma}/C^{\rm ALP}_{e}) 2 \times 10^{-13}$ GeV^{-1}, where the model dependence $C^{\rm ALP}_{\gamma}/C^{\rm ALP}_{e}$ can be significant.

Once more, although alternative explanations for these observations cannot be ruled out, it is intriguing that they together point to relatively well defined axion parameters, that are compatible with feasible QCD axion parameters, and that are not excluded by previous bounds. Moreover, axions at the meV scale are very close to the DM favoured window (see previous section), have interesting phenomenological implications [98] and constitute a region especially difficult to explore experimentally. As shown later, IAXO constitutes probably the only realistic experimental technique able to explore (part of) these models.

6 Searches for axions

In spite of their weak interactions, axions could be directly detected in a number of realistic experimental scenarios. Three main categories of experimental approaches can be distinguished depending on the source of axions employed: *haloscopes* look for the relic axions potentially composing our dark matter galactic halo, *helioscopes* look for axions potentially emitted at the core of the sun, and *light-shining-through-wall* (LSW) experiments look for axion-related phenomena generated entirely in the laboratory. All three strategies invoke the generic axion-photon interaction, and thus rely on the use of powerful magnetic fields to trigger the conversion of the axions into photons that can be subsequently detected.

Haloscopes [14] use high-Q microwave cavities inside a magnetic field to detect photons from the conversion of relic axions. Being non relativistic, these axions convert into monochromatic photons of energy equal to m_a. For a cavity resonant frequency matching m_a, the conversion is substantially enhanced. The cavity must therefore be tunable and the data taking is performed by scanning very thin m_a-slices of parameter space. The experimental implementation of this idea was pioneered in Brookhaven [99, 100], and later on continued by the CARRACK [101] and ADMX collaborations. As shown in Fig. 1, only ADMX [102, 103], have reached sensitivities in $g_{a\gamma}$ enough to probe QCD axion models for m_a in the $2-3$ μeV range, under the assumption that axions are the main cold DM component. ADMX is carrying out an active program [104] to improve the background noise of the present experiment, as well as to extend the sensitivity to higher masses (the ADMX-HF setup). In the light of the considerations exposed above, it turns out that current haloscope efforts are focused in the low mass part of the region motivated by cosmology. To apply the haloscope technique to higher axion masses is problematic for a number of reasons. First of all, given that the cavity must resonate at the axion mass, higher masses imply the use of smaller cavities, and therefore lower expected signals. Moreover, smaller cavities usually have poorer quality factors, and the noise figure of the microwave sensors usually increase with frequency. New ideas are recently being put forward to overcome these problems and access the very motivated region of $m_a \sim 10^{-5} - 10^{-3}$ eV. Not exhaustively, they invoke the use of long thin cavities [105, 106] (waveguides or tubes), resonators with a specific dielectric or wire structure to make them resonate to higher frequencies, active resonators, among others[1]. A significant departure from the original haloscope idea is the dish antenna concept [107, 108], proposing to use a spherical dish in a magnetic field to convert the DM axions into photons and and focus them into a single spot. This concept trades off the resonant enhancement of cavities for the potentially large area of a dish antenna.

A different type of relic axion search strategy has been recently proposed (the so-called CASPEr experiment [109]), aiming at detecting the precession of nuclear spin in a material sample that, in the presence of an electric field, should appear by virtue of the time varying nuclear moment induced by the interaction with the background axion DM. Using precision magnetometry this effect should be detectable if the axion has very low masses $m_a \lesssim 10^{-9}$ eV (potentially improvable to $m_a \lesssim 10^{-6}$ eV).

Helioscopes [14] look for axions emitted by the Sun, and therefore do not rely on the assumption of axions being the DM. Axion emission by the solar core is a robust prediction involving well known solar physics and the Primakoff conversion of plasma

[1]For a display of recent ideas see talks at the last Patras Workshop on Axions, WIMPs and WISPs. `http://axion-wimp2014.desy.de/`

photons into axions. Solar axions have ∼keV energies and in strong laboratory magnetic fields can convert back into detectable x-ray photons. Contrary to haloscopes, the signal in helioscopes is independent on the axion mass up to relatively large values (e.g. 0.02 eV for CAST). Using the technique of the buffer gas [110] the sensitivity can be further extended to masses up to ∼eV [19, 20, 21]. The technique of the axion helioscope was first experimentally applied in Brookhaven [111] and later on by the SUMICO helioscope at Tokyo University [112, 113]. Currently, the same basic concept is being used by the CERN Axion Solar Telescope (CAST experiment) [114, 115, 19, 20, 21] with some original additions that provide a considerable step forward in sensitivity. Its latest results, shown in Fig. 1, have surpassed the astrophysical limit $g_{a\gamma} \sim 10^{-10}$ GeV^{-1} in a wide m_a range. This means that for the highest mass values, close to the ∼1 eV, the sensitivity allows tests of some QCD axion models. Together with haloscopes, helioscopes are the only experimental technique with sensitivity to explore realistic QCD axion models. These two techniques are complementary, exploring the lower- (with haloscopes) and the higher-mass (with helioscopes) regions of the axion phase space not yet searched nor excluded. An advantage of helioscopes is that there is a clear scaling strategy to substantially push the present sensitivity frontline to lower values of $g_{a\gamma}$ and m_a, a strategy that is implemented in the IAXO proposal, in which I focus below.

It is worth mentioning that a similar helioscope-like scheme can be invoked in solid crystalline detectors, and used to detect solar axions. In this case, the local conversion into photons is triggered by the periodic electromagnetic field of the crystal [116, 117, 118], giving rise to very characteristic Bragg patterns that have been searched for as by-products of a number of underground WIMP experiments [119, 120, 121, 122, 123]. However, the prospects of this technique have proven limited [124, 125] and do not compete with dedicated helioscope experiments.

LSW [126] experiments use high intensity light sources (e.g., lasers) and strong magnetic fields to produce ALPs in the laboratory. These ALPs can reconvert back into detectable photons after an opaque wall. This technique therefore does not rely on any astrophysical or cosmological assumption for the ALPs. A number of experiments have already used this technique to search for ALPs, with a sensitivity, however, still a few orders of magnitude behind helioscopes (see the ALPS limit in Fig. 1 from [17]). The prospects for future scaled-up setups [127], in particular ALPS-II [128], could surpass current helioscope limits for low m_a values and reach some unexplored ALP parameter space, although in any case, still without enough sensitivity to reach the QCD axion band.

All the searches mentioned up to now rely solely on the axion-photon phenomenology and therefore can be represented in the ALP parameter space of Fig. 1. Other small scale searches have been performed with a less generic scope, sometimes as by-products of other experiments. They are relevant for specific subsets of axion or WISP models. For example, axions have been searched for via more specific phenomenology (e.g. through the axioelectric effect [129, 130, 131, 132], or axion-emitting nuclear transitions [133, 134, 135, 136, 137, 138], among others). Specific non-axion WISPs have been (or are being) searched for in dedicated setups (e.g. hidden photons [139, 140]) or as by-products of axion and ALPs searches (e.g. chameleons [141]) or WIMP searches (e.g. [142]). For an updated review of all initiatives going on, I refer to the community documents prepared in recent roadmapping events, both in US [97, 143] and Europe [144, 145].

Below I will focus on the helioscope frontier. I will argue that the decade-long operation of CAST has not only led to one of the most competitive set of bounds on the axion and ALPs, but also to the establishment of a relevant community and the specific operational experience required to design a scaled-up version (forth generation) of the axion helioscope concept. IAXO is based on these ideas and aims to substantially push the helioscope envelope well into unexplored regions of the axion and ALP parameter space motivated by the arguments detailed in the previous section.

6.1 Solar axions and the axion helioscope frontier

Axions can be produced in the solar interior by a number of reactions. The most relevant channel is the Primakoff conversion of plasma photons into axions in the Coulomb field of charged particles via the generic $a\gamma\gamma$ vertex. The Primakoff solar axion flux, shown on the left of Fig. 2 peaks at 4.2 keV and exponentially decreases for higher energies. This spectral shape is a robust prediction depending only on well known solar physics, while the only unknown axion parameter is $g_{a\gamma}$ and enters the flux as an overall multiplicative factor $\propto g_{a\gamma}^2$. For the particular case of non-hadronic axions having tree-level interactions with electrons, other productions channels (e.g., brehmstrahlung, compton or axion recombination) should be taken into account, as their contribution can be greater than that of the Primakoff mechanism. The calculation of these channels have been recently updated [146] and the corresponding solar spectrum is shown on the right of Fig. 2. However, the usual procedure in helioscopes considers only the Primakoff component because: 1) it maintains the broadest generality and covers a larger fraction of ALPs and 2) astrophysical limits on g_{ae} are quite restrictive and largely disfavour the values that could be reached by helioscopes looking at the non-hadronic solar axion flux. With IAXO, it will be possible for the first time to supersede even astrophysical limits on g_{ae}, opening the possibility to probe an interesting set of models of non-hadronic axions.

By means again of the $a\gamma\gamma$ vertex, solar axions can be efficiently converted back into photons in the presence of an electromagnetic field. The energy of the reconverted photon is equal to the incoming axion, so a flux of detectable x-rays of few keV energies is expected. The probability that an axion going through the transverse magnetic field B

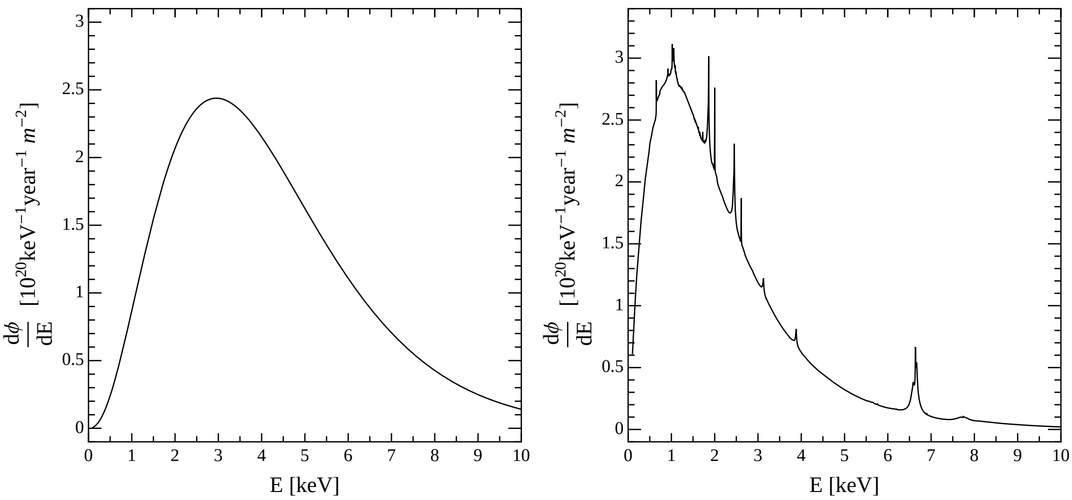

Figure 2.— Solar axion flux spectra at Earth by different production mechanisms. On the left, the most generic situation in which only the Primakoff conversion of plasma photons into axions is assumed. On the right the spectrum originating from processes involving electrons, bremsstrahlung, Compton and axio-recombination [146, 147]. The illustrative values of the coupling constants chosen are $g_{a\gamma} = 10^{-12}$ GeV^{-1} and $g_{ae} = 10^{-13}$. Plots from [148].

over a length L will convert to a photon is given by [14, 115, 18]:

$$P_{a\gamma} = 2.6 \times 10^{-17} \left(\frac{B}{10 \text{ T}}\right)^2 \left(\frac{L}{10 \text{ m}}\right)^2 \left(g_{a\gamma} \times 10^{10} \text{ GeV}\right)^2 \mathcal{F}$$

where the form factor $\mathcal{F}$ accounts for the coherence of the process:

$$\mathcal{F} = \frac{2(1 - \cos qL)}{(qL)^2} \tag{9}$$

and q is the momentum transfer. The fact that the axion is not massless, puts the axion and photon waves out of phase after a certain length. The coherence is preserved ($\mathcal{F} \simeq 1$) as long as $qL \ll 1$, which for solar axion energies and a magnet length of $\sim$10 m happens at axion masses up to $\sim 10^{-2}$ eV, while for higher masses $\mathcal{F}$ begins to decrease, and so does the sensitivity of the experiment. To mitigate the loss of coherence, a buffer gas can be introduced into the magnet beam pipes [110, 19] to impart an effective mass to the photons $m_\gamma = \omega_\text{p}$ (where ω_p is the plasma frequency of the gas, $\omega_\text{p}^2 = 4\pi\alpha n_e/m_e$). For axion masses that match the photon mass, $q = 0$ and the coherence is restored. By changing the pressure of the gas inside the pipe in a controlled manner, the photon mass can be systematically increased and the sensitivity of the experiment can be extended to higher axion masses.

The basic layout of an axion helioscope thus requires a powerful magnet coupled to one or more x-ray detectors. When the magnet is aligned with the Sun, an excess of x-rays at the exit of the magnet is expected, over the background measured at non-alignment periods. This detection concept was first experimentally realized at Brookhaven National Laboratory (BNL) in 1992. A stationary dipole magnet with a field of $B = 2.2$ T and a length of $L = 1.8$ m was oriented towards the setting Sun [111]. The experiment derived an upper limit on $g_{a\gamma}$ (99% CL) $< 3.6 \times 10^{-9}$ GeV^{-1} for $m_a < 0.03$ eV. At the University of Tokyo, a second-generation experiment was built: the SUMICO axion heliscope. Not only did this experiment implement a dynamic tracking of the Sun but it also used a more powerful magnet ($B = 4$ T, $L = 2.3$ m) than the BNL predecessor. The bore, located between the two coils of the magnet, was evacuated and higher-performance detectors were installed [149, 112, 113]. This new setup resulted in an improved upper limit in the mass range up to 0.03 eV of $g_{a\gamma}$(95% CL) $< 6.0 \times 10^{-10}$ GeV^{-1}. Later experimental improvements included the additional use of a buffer gas to enhance sensitivity to higher-mass axions.

A third-generation experiment, the CERN Axion Solar Telescope (CAST), began data collection in 2003. The experiment uses a Large Hadron Collider (LHC) dipole prototype magnet with a magnetic field of up to 9 T over a length of 9.3 m [114]. CAST is able to follow the Sun for several hours per day using a sophisticated elevation and azimuth drive. This CERN experiment is the first helioscope to employ x-ray focusing optics for one of its four detector lines [150], as well as low background techniques from detectors in underground laboratories [151]. During its observational program from 2003 to 2011, CAST operated first with the magnet bores in vacuum (2003–2004) to probe masses $m_a <$ 0.02 eV. No significant signal above background was observed. Thus, an upper limit on the axion-to-photon coupling of $g_{a\gamma}$ (95% CL) $< 8.8 \times 10^{-11}$ GeV^{-1} was obtained [115, 18]. The experiment was then upgraded to be operated with ^{4}He (2005–2006) and ^{3}He gas (2008–2011) to obtain continuous, high sensitivity up to an axion mass of $m_a = 1.17$ eV. Data released up to now provide an average limit of $g_{a\gamma}$ (95% CL) $\lesssim 2.3 \times 10^{-10}$ GeV^{-1}, for the higher mass range of 0.02 eV $< m_a <$ 0.64 eV [19, 20] and of about $g_{a\gamma}$ (95% CL) $\lesssim 3.3 \times 10^{-10}$ GeV^{-1} for 0.64 eV $< m_a <$ 1.17 eV [21], with the exact value depending on the pressure setting.

So far each subsequent generation of axion helioscopes has resulted in an improvement in sensitivity to the axion-photon coupling constant of about a factor 6 over its predecessors. CAST has been the first axion helioscope to surpass the stringent limits from astrophysics $g_{a\gamma} \lesssim 10^{-10}$ GeV^{-1} over a large mass range and to probe previously unexplored ALP parameter space. As shown in Fig. 1, in the region of higher axion masses ($m_a \gtrsim 0.1$ eV), the experiment has entered the band of QCD axion models for the first time and excluded KSVZ axions of specific mass values. CAST is the largest collaboration in axion physics with ~ 70 physicists from about 16 different institutions in Europe and the USA, and one of the first astroparticle experiments at CERN. CAST has demonstrated

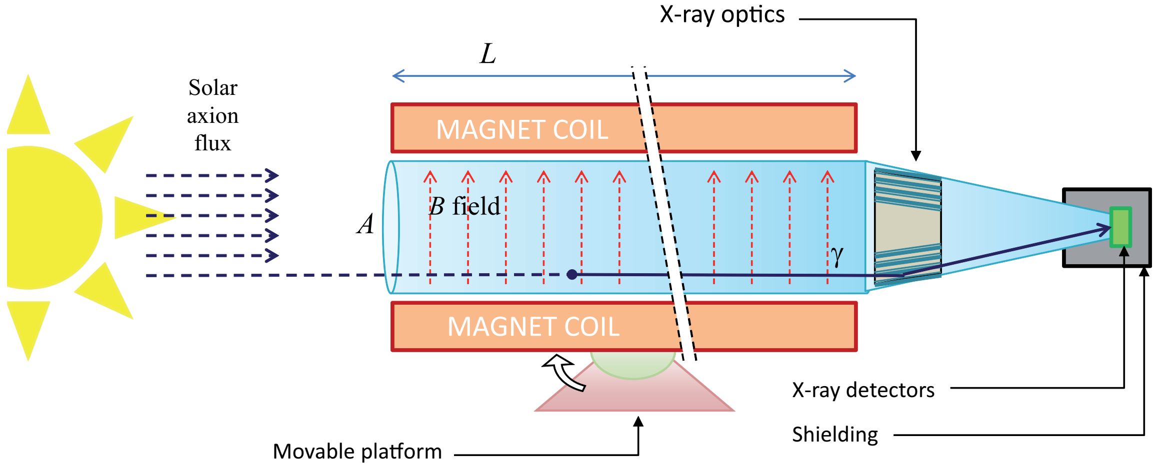

Figure 3.— Conceptual arrangement of an enhanced axion helioscope with x-ray focalization. Solar axions are converted into photons by the transverse magnetic field inside the bore of a powerful magnet. The resulting quasi-parallel beam of photons of cross sectional area A is concentrated by an appropriate x-ray optics onto a small spot area a in a low background detector. The envisaged design for IAXO, shown in Figure 4, includes eight such magnet bores, with their respective optics and detectors.

that the helioscope is the most technologically mature technique for axion detection and that is ready to be scaled-up in size. A further, substantial step beyond the current state-of-the-art represented by CAST is possible [152] with a new fourth-generation axion helioscope as the proposed International Axion Observatory (IAXO).

7 The International Axion Observatory (IAXO)

All axion helioscopes to date have made use of "recycled" magnets that were originally built for other experimental purposes. The CAST success has relied, to a large extent, on the availability of the first-class LHC test magnet. Going substantially beyond CAST sensitivity is possible only by going to a new magnet, designed and built maximizing the helioscope magnet's figure of merit $f_M = B^2\,L^2\,A$, where B, L and A are the magnet's field strength, length and cross sectional area, respectively. f_M is defined proportional to the photon signal from converted axions. Improving CAST f_M can only be achieved [152] by a completely different magnet configuration with a much larger magnet aperture A, which in the case of the CAST magnet is only 3×10^{-3} m^2. However, for this figure of merit to directly translate into signal-to-noise ratio of the overall experiment, the entire cross sectional area of the magnet must be equipped with x-ray focusing optics. The layout of this *enhanced axion helioscope*, sketched in Figure 3, was proposed in [152] as the basis for IAXO.

Thus the central component of IAXO is a new superconducting magnet. Contrary

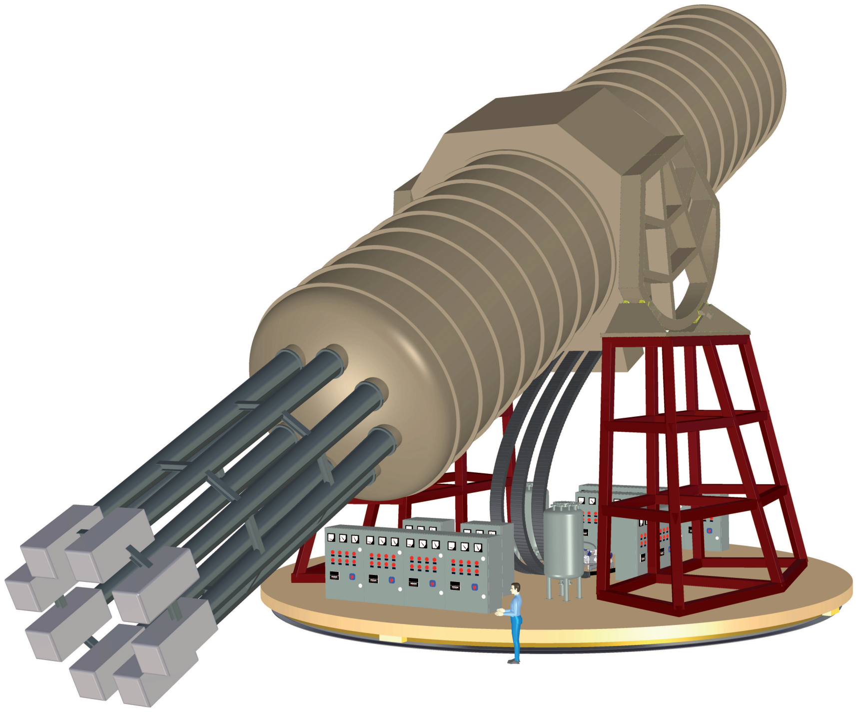

Figure 4.— Schematic view of IAXO. Shown are the cryostat, eight telescopes, the flexible lines guiding services into the magnet, cryogenics and powering services units, inclination system and the rotating platform for horizontal movement. The dimensions of the system can be appreciated by a comparison to the human figure positioned by the rotating table [156].

to previous helioscopes, IAXO's magnet will follow a toroidal configuration [153], to efficiently produce an intense magnetic field over a large volume. Dimensions are fixed by maximizing the figure of merit within realistic limits of the different technologies in play. This consideration leads to a 25 m long and 5.2 m diameter toroid assembled from 8 coils, and generating effectively 2.5 tesla in 8 bores of 600 mm diameter. This represents a 300 times better f_M than CAST magnet. The toroid's stored energy is 500 MJ. The design is inspired by the ATLAS barrel and end-cap toroids[154, 155], the largest superconducting toroids built and presently in operation at CERN. The superconductor used is a NbTi/Cu based Rutherford cable co-extruded with Aluminum, a successful technology common to most modern detector magnets. Figure 1 shows the conceptual design of the overall infrastructure [156]. IAXO needs to track the Sun for the longest possible period. For the rotation around the two axes to happen, the 250 tons magnet is supported at the centre of mass by a system also used for very large telescopes. The necessary magnet services for vacuum, helium supply, current and controls are rotating along with the magnet.

Another area for improvement will be the x-ray optics. Although CAST has proven the concept, only one of the four CAST magnet bores is equipped with optics. Each of the eight IAXO magnet bores will be equipped with x-ray focusing optics that rely on x-ray reflection on surfaces at grazing angles. By working at shallow incident angles, it is possible to make mirrors with high reflectivity. Here the challenge is not so much achieving exquisite focusing or near-unity reflectivity, but the availability of cost-effective x-ray optics of the required size. For nearly 50 years, the x-ray astronomy and astrophysics community has been building telescopes following the design principle of Hans Wolter, employing two conic-shaped mirrors to provide true-imaging optics. This class of optics allows "nesting", that is, placing concentric co-focal x-ray mirrors inside one another to achieve high throughput. The IAXO collaboration envisions using optics similar to those used on NASA's NuSTAR [157], an x-ray astrophysics satellite with two focusing telescopes that operate in the 3 - 79 keV band. The NuSTAR's optics, shown in Figure 5, consists of thousands of thermally-formed glass substrates deposited with multilayer coatings to enhance the reflectivity above 10 keV (figure 2). For IAXO, the multilayer coatings will be designed to match the solar axion spectrum [158].

At the focal plane in each of the optics, IAXO will have small gaseous chambers read by pixelised planes of Micromegas. CAST has enjoyed the sustained development of its detectors towards lower backgrounds during its lifetime. The latest generation of Micromegas detectors in CAST are achieving backgrounds below $\sim 10^{-6}$ counts $\mathrm{keV}^{-1}\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1}$ [159, 160]. This value is already a factor of more than 100 better than the background levels obtained during the first data-taking periods of CAST. Prospects for reducing this level to 10^{-7} counts $\mathrm{keV}^{-1}\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1}$ or even lower appear feasible after the active R&D going on, in particular that being carried out at the University of Zaragoza, under the T-REX ERC-funded project [161, 162, 163, 164]. As part of this work, a replica of the CAST Micromegas detectors is taking data in an underground test bench at the Laboratorio Subterráneo de Canfranc, already reaching a background level of $\sim 10^{-7}$ counts $\mathrm{keV}^{-1}\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1}$ [159]. These background levels are achieved by the use of radiopure detector components, appropriate shielding, and offline discrimination-algorithms on the 3D event topology in the gas registered by the pixelised readout.

The components described so far compose the baseline configuration of IAXO as an enhanced axion helioscope. Beyond this baseline, additional enhancements are being considered to explore extensions of the physics case for IAXO. For some of the additional physics cases of IAXO, lowering the theshold well below 1 keV is interesting. For this reason other types of x-ray detection technologies are also under consideration, like GridPix detectors, Transition Edge Sensors (TES) or low noise CCDs to be placed at the optics focal points. Beyond that, because a high magnetic field in a large volume is an essential component in any axion experiment, IAXO could evolve to a generic "axion facility" and host different detection techniques. The most relevant of these possibilities, at the moment actively explored, is to use microwave cavities and antennas to search for dark-matter axions. Some of the developments being carried out in the community (briefly mentioned above in section 6) could profit from the availability of the large magnetic field of IAXO, and reach sensitivity to axions in mass ranges complementary to those in previous searches.

The IAXO collaboration has recently finished the conceptual design of the experiment [156], and last year a Letter of Intent [148] was submitted to the SPS and PS experiments Committee (SPSC) of CERN. The committee acknowledged the physics goals of IAXO and recommended proceeding with the next stage, the creation of the technical design report (TDR), a necessary step before facing construction. As part of the TDR, prototyping activities are foreseen, including magnet (the construction and test of a single shorter superconducting coil), optics and detector aspects. At the same moment of preparation of this review, a "IAXO optics+detector pathfinder system" has just been assembled in CAST at CERN. It consist of a small x-ray optics (of CAST bore dimensions) coupled to a low-background Micromegas detector. This is the first time that the two techniques pioneered by CAST –x-ray focusing and low background detectors–, previously used in different lines of the experiment, are now coupled together in the same line. It is also the first time that an x-ray optics is manufactured specifically for axion physics. This system will tests the two technologies proposed for IAXO coupled together. The first data is already being taken: the experience with this system will be precious for the preparation of IAXO.

8 Physics potential of IAXO

The sensitivity regions in the $(g_{a\gamma}, m_a)-$plane have been computed from the standard channel of Primakoff solar axions [148]. Figure 1 shows the attainable region in the wider context of axion searches and motivated regions. Figure 6 focuses on the high mass region.

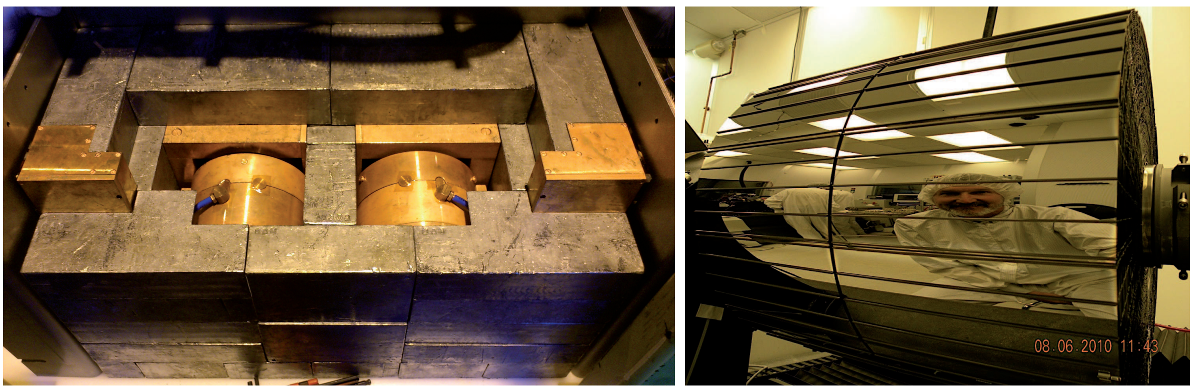

Figure 5.— Left: two lead- and copper-shielded, ultra-low background Micromegas x-ray detectors currently in use at CAST. Right: the NuSTAR x-ray telescope, with optics very similar to that proposed for IAXO.

The sensitivity is calculated assuming realistic experimental parameters (see [148] for details), and two 3-year data taking periods, the first with vacuum in the magnet bores and the second one with variable-pressure gas. The first period determines the sensitivity of IAXO for axion masses below $m_a \lesssim 0.01$ eV, while the second one above that value. The range of densities used (number of gas density steps) will determine how far in m_a to go. The current sensitivity curves are calculated assuming that the gas density is continuously changed from 0 to 1 bar of ^{4}He at room temperature. This program would allow IAXO to reach an axion mass of 0.25 eV. The shape of the sensitivity region in the range $m_a \sim 0.01-0.25$ eV depends on the actual distribution of the exposure time in density, which is for now assumed flat, i.e. equal time is spent at each gas density. This distribution may be redefined in the future according to the evolution of bounds on the axion mass or other eventual results/interest favouring a particular m_a value or range.

As seen, IAXO will be a factor of $\sim 15-20$ more sensitive than CAST in terms of the axion-photon coupling constant $g_{a\gamma}$, which translates in about **5 orders of magnitude more sensitive** in terms of signal intensity. That is, IAXO could be sensitive to $g_{a\gamma}$ values as low as, or even surpassing,

$$g_{a\gamma} \sim 5 \times 10^{-12}\ \text{GeV}^{-1} \tag{10}$$

for a wide range of axion masses up to about 0.01 eV and around $g_{a\gamma} \sim 10^{-11}$ GeV^{-1} up to about 0.25 eV.

While CAST was the first experimental search reaching, and slightly surpassing, the limit $g_{a\gamma} \lesssim 10^{-10}$ GeV^{-1} in this mass range, and therefore started probing ALP parameter space allowed by astrophysics, IAXO will deeply enter into completely unexplored ALP and axion parameter space, as indicated by Fig. 1. At a minimum, IAXO will exclude a large region of the QCD axion phase space that has yet to be explored. If IAXO does discover a new pseudoscalar fundamental, it would be a groundbreaking result for particle physics.

Fig. 6 focuses on the sensitivity for the high mass region $m_a > 1$ meV. At these masses this experiment would explore a broad range of realistic axion models that accompany the Peccei-Quinn solution of the strong CP problem. Its sensitivity would cover axion models with masses down to the few meV range, superseding the SN 1987A energy loss limits on the axion mass. Axion models in this region are of high cosmological interest. As explained in previous sections, they are favoured dark matter candidates and could compose all or part of the cold dark matter of the Universe. In non-standard cosmological scenarios, or in more generic ALP frameworks [25], the range of ALP parameters of interest as DM is enlarged and most of the region at reach by IAXO contains possible dark matter candidates. At the higher part of the range (0.1 - 1 eV) axions are good candidates

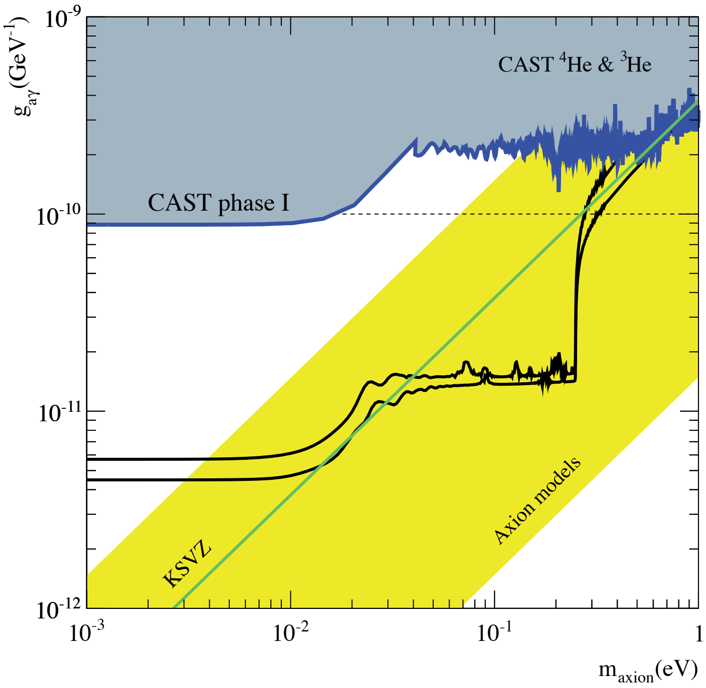

Figure 6.— Close-up of the high mass part of parameter space of Fig. 1 (1 meV $< m_a <$ 1 eV). The two lines correspond to two different set of assumptions (more or less conservative) to compute IAXO sensitivity. Plot from [148].

to the hot DM or additional *dark radiation* that is recently invoked to solve tension in cosmological parameters. At much lower masses, below $\sim 10^{-7}$ eV, the region attainable by IAXO includes ALP parameters invoked repeatedly to explain the anomalies in light propagation over astronomical distances commented in section 5. IAXO could provide a definitive test of this hypothesis.

8.1 Additional IAXO physics cases

In addition to the standard helioscope result exposed above, a number of additional physics cases are being studied to extend the reach of IAXO. Particularly interesting in the sensitivity of IAXO to axions with an axion-electron coupling g_{ae} as have been invoked to solve the anomalous cooling observed in white dwarfs (see above section 5). Figure 7 summarizes the expected sensitivity to axions produced by BCA reactions (see right of Figure 7) in the Sun, with couplings $g_{ae} \sim \mathcal{O}(10^{-13})$ (see [148] for details). In summary, IAXO could directly measure the solar flux of axions produced by the BCA processes, for the first time with sensitivity to values of g_{ae} not previously excluded and relevant to test the hypothesis that the cooling of WD is enhanced by axion emission (via BCA procesees, the same mechanisms that IAXO would be testing in the Sun), and for values of m_a for which QCD axion models can give the needed g_{ae} values.

IAXO can be sensitive to models of other proposed particles at the low energy frontier of particle physics. Some examples are hidden photons or chameleons. They could also be produced in the Sun, and give specific signatures in axion helioscope data. Hidden photons emitted from the Sun have been studied in the context of specific searches [139] or as by-products of axion helioscopes like SUMICO [140] or CAST [165]. Chameleons are scalars with an environment-dependent mass that are proposed in the context of dark energy models. Recent calculations [166, 28] show that resonant production in the magnetic regions of the solar atmosphere might allow for the propagation of these chameleons into a solar helioscope where they can be regenerated into soft x-rays through the inverse Primakoff-effect. For these searches, sensitivity to energies lower than the baseline Primakoff axion spectrum (sub-keV and lower) is needed, something that IAXO could obtain by one or more of the additional equipment described in section [148].

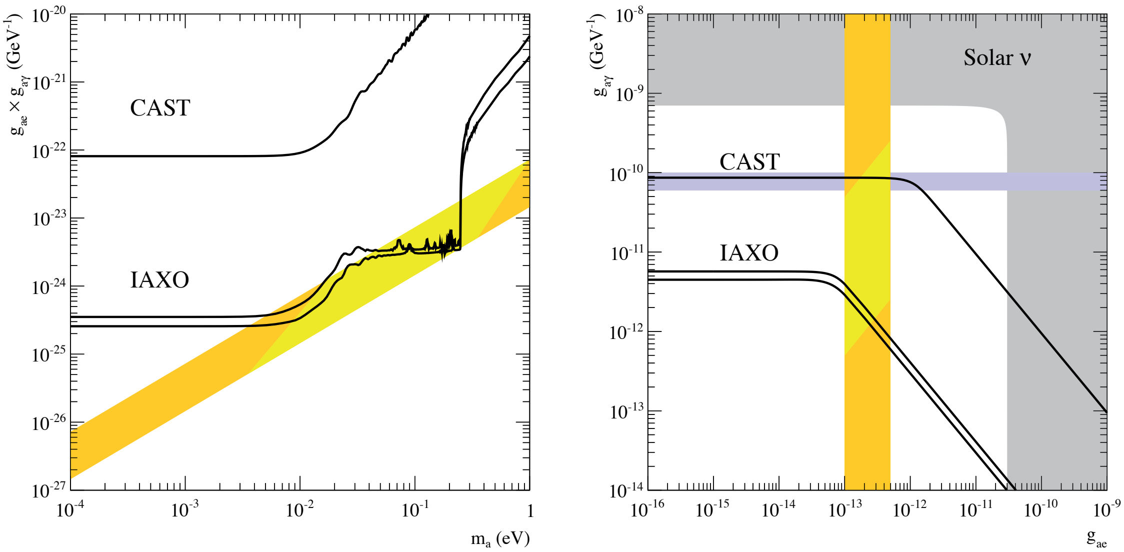

Figure 7.— Left: IAXO sensitivity line for $g_{ae}g_{a\gamma}$ as a function of m_a, assuming the solar emission is dominated by the BCA reactions which involve only the electron coupling g_{ae}. The orange band corresponds to values of $g_{ae} \sim 1 - 5 \times 10^{-13}$ and $g_{a\gamma}$ related to m_a by the DSFZ model with $C_\gamma = 0.75$. The part of the band highlighted in yellow corresponds to those models for which the relation of g_{ae} with m_a is also considered (taking a reasonable range $\cos^2\beta$ =0.01−1). The recent limit of CAST on g_{ae} is also shown. Right: IAXO sensitivity on g_{ae} and $g_{a\gamma}$ for $m_a \lesssim 10$ meV. The gray region is excluded by solar neutrino measurements. The orange band corresponds to values of $g_{ae} \sim 1 - 5 \times 10^{-13}$. The part of the band highlighted in yellow corresponds to models that satisfy the model-dependent relation between $g_{a\gamma}$ and g_{ae} (taking again $C_\gamma = 0.75$ and $\cos^2\beta$ =0.01−1). The recent limit of CAST on g_{ae} is also shown. In the orange bands, axion emission affects white dwarf cooling and the evolution of low-mass red giants; couplings stronger than in these bands are firmly excluded. Likewise, helium-burning stars would be perceptibly affected in the blue band of the right plot and parameters above it are excluded. Plots from [148]

More intriguing would be the possibility to detect relativistic axions or ALPs from other sources in the sky, using IAXO as a true axion telescope. Although most potential astrophysical axion sources will probably be too faint to be detected by IAXO, some relic populations of ALPs could provide detectable signals. If the dark radiation that is recently invoked to relieve the tension in cosmological parameters [1] is composed by relativistic axions or ALPs (from, e.g., primordial decays of heavy fields [167]) they would still linger today as a Cosmic Axion Background with energies of $\sim \mathcal{O}(100)$ eV. With appropriate low energy detectors, the predicted fluxes could be within reach of IAXO for some ALP parameters.

Another experimental configuration of IAXO could be to equip two of the bores with microwave cavities, one of them with a strong emitter, and the second one with a low-noise receiver. This would be an analogous LSW experiment with microwaves, conceptually similar to the one performed in [168]. Given the size of the IAXO magnet bores, the operating frequency of such a configuration would be around 200 Mhz. The potential of this configuration is currently under study.

Yet the most promising option for a IAXO extension seems to be the search for the non-relativistic axions potentially composing the galactic dark matter halo. This could be accomplished by using microwave cavities (and turning IAXO into a haloscope kind of detector), or dish antennas [107, 108] inside the large IAXO magnetic volume. The size and strength of the IAXO magnet make this possibility very appealing and deserves serious consideration. Although many technical questions need to be addressed before considering this as a realistic possibilities, preliminary studies [169] point to very promising prospects. IAXO could easily convert its magnet bores into resonant cavities, and easily gain competitive sensitivity to m_a around and below 1 μeV. A more challenging –but more interesting– option is to fill part of IAXO magnetic volume with a set of power-combined parallelepipedic long thin cavities, that could provide sensitivity right in the mass range around $\sim 10^{-4}$ eV, most favoured by cosmology. Finally, a setup based on the dish antenna might provide access to a wider and higher mass range. In general, there seems to be potential to extend the sensitivity of IAXO well beyond the level of the baseline configuration of Figure 1 in mass ranges complementary to previous searches.

9 Conclusions

After more than three decades, the axion hypothesis remains the most compelling solution to the strong CP problem, one of the most serious blemish of the Standard Model of particle physics. In addition, the issue of the nature of the dark matter of the Universe, one of the biggest mystery that modern fundamental science is facing, could also be solved by the axion. The fact that Supersymmetry does not appear at the LHC, nor do WIMPs in underground experiments, is increasing the interest in axions. In addition, recent work in theory and phenomenology is sharpening their physics case. Some intriguing astrophysical observations could already be hinting at the axion (or an axion-like particle). The experimental efforts to search for axions, still a relatively minor field, are steadily increasing. Dark matter axions at the few μeV could be detected by current haloscopes like ADMX. Pushing haloscope sensitivity to higher masses is technically very challenging and it is the object of a number of recent new ingenuous ideas. At the helioscope frontier, the situation is technically more mature, and the CAST experiment has been running for the last decade with sensitivity to axions at the sub-eV scale. CAST is the largest axion experiment, and has been the seed for the newly proposed International Axion Obvservatory (IAXO), aiming at a 5 orders of magnitude of improvement in signal-to-noise ratio over CAST. IAXO will use CERN's expertise efficiently to venture deep into unexplored axion parameter space. If the axion exists, there is a reasonable chance for it to be seen by IAXO. We may be living through the emergence of a new field in the interface of particle physics and cosmology, with potential groundbreaking consequences for both of them.

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Construcción de los Poliedros Regulares

Luis J. Boya y Cristian Rivera

Departamento de Física Teórica
Universidad de Zaragoza
5009 Zaragoza, Spain

Abstract

Se construyen directamente los cinco poliedros regulares, también llamados *sólidos platónicos* y se recuerdan y prueban algunas de sus propiedades.

1 Construcción

Los cinco sólidos platónicos son el Tetraedro T, el Octaedro O, el Hexaedro H, el Icosaedro I y el Dodecaedro D. Son conocidos en Matemáticas desde los tiempos de los griegos. Aquí damos unas recetas para poder construirlos explícitamente de un modo sencillo, y recordamos y probamos algunas de sus propiedades, ver [1, 2, 3].

Los poliedros regulares viven en tres dimensiones, y tienen, por definición, caras que son polígonos planos regulares de q lados, con el mismo número de caras por vértice (de hecho, veremos que ese número, p, es $p \geq 3$); veremos también que las caras, todas iguales por supuesto, sólo pueden ser triángulos, cuadrados o pentágonos, ($q = 3, 4, 5$) habiendo TRES poliedros regulares con caras triangulares (T, O e I), y DOS con caras cuadrada y pentagonal (H y D). Procedamos directamente a su construcción.

1.1 Tetraedro

Para el *tetraedro* T, partimos de un triángulo regular en el plano, con su centro P. Trazamos desde P la recta perpendicular al plano del triángulo. Un punto arbitrario de esa recta equidista de los tres vértices del triángulo. Por eso, uniendo esos tres vértices con un punto de esa recta, con una arista de longitud igual a la longitud de los lados del triángulo regular de la base, tenemos la construcción del Tetraedro, con sus seis (3+3) aristas iguales.

Resumimos el Tetraedro así: (Vértices $V = 4$, Aristas $A = 6$, Caras $C = 4$). Nótese que $V + C - A = 2$, igualdad que luego comentaremos. Nótese también la *agudeza* de los vértices, en el sentido que las tres caras triangulares de un vértice suman $60° \times 3 = 180°$, la mitad para que las caras "llenen" un plano ($360° = 180° \times 2$). La *agudeza* es, por definición, la diferencia de la suma de los ángulos adyacentes con $360°$ ($-180° + 360° = -180°$); para el tetraedro es, pues, $-180°$ y será la *mínima* entre los cinco poliedros regulares. Tal como la hemos definido, la *agudeza* será siempre *negativa* para los posibles poliedros.

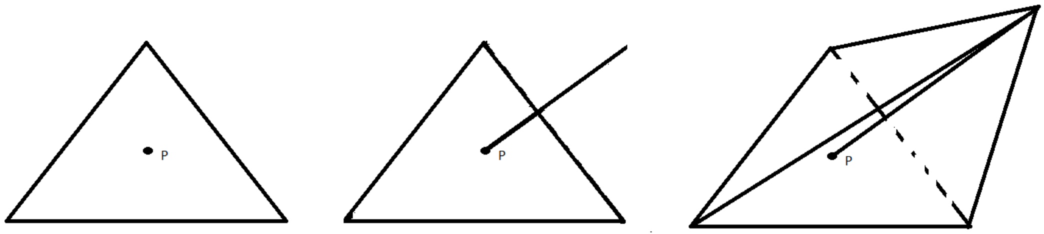

Fig. 1A
El Triángulo de partida, con su centro P

Fig. 1B
La Recta perpendicular desde P

Fig. 1C
El Tetraedro T

1.2 Octaedro

A continuación construimos el Octaedro O. Partimos ahora de un *cuadrado* regular, en el plano (e.g. del papel) y procederemos como antes, construyendo la recta perpendicular desde el centro P del cuadrado, pero ahora en los dos sentidos (por arriba y por debajo de su plano), y tomando a continuación aristas (4+4=8) de longitud igual a las del cuadrado, desde los cuatro vértices del cuadrado a esas dos rectas perpendiculares, por arriba y por abajo, aparecen así los dos nuevos vértices y se completa el octaedro O, que tiene por tanto caras triangulares y aparecen claramente dos nuevos vértices sobre esa recta, uno por arriba y otro por abajo del plano del cuadrado de partida. Figs. $2A$, $2B$ y $2C$.

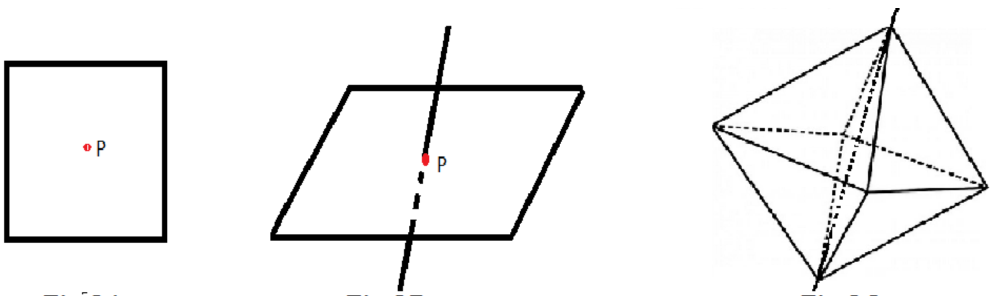

Fig 2A
El cuadrado,

Fig 2B
El cuadrado y su perpendicular

Fig 2C
El octaedro O

En vértices V, aristas A y caras C tenemos ahora, claramente ($V : 4 + 2 = 6$, $A : 4 + 4 + 4 = 12$ y $C : 4 + 4 = 8$, de nuevo $V + C - A = 2$). Nótese que del cuadrado original, quedan los cuatro vértices y las cuatro aristas, pero la cara ha desaparecido. . .: No se trata, pues, de dos pirámides opuestas de base cuadrangular. Nótese también la *agudeza* de los vértices, que es ahora *mayor* que antes ($-240° + 360° = -120°$: cuatro caras triangulares por vértice, así que $60 \times 4 = 240° < 360°$).

1.3 Hexaedro

El *Hexaedro* o cubo H es, en nuestra opinión, el poliedro más fácil de construir. Partimos de un cuadrado regular en el plano e.g. del papel, y en otro plano paralelo al original, digamos por detrás (ver Fig 3B), tomamos otro cuadrado idéntico al anterior. El cubo se completa trazando las cuatro aristas que unen los vértices correspondientes de los dos cuadrados, de longitud igual a la de los lados de los cuadrados. Esas cuatro aristas nuevas son perpendiculares a los planos de los dos cuadrados, y de longitud igual a un lado de cada cuadrado. Y así queda el cubo convencional, o hexaedro, que tiene obviamente $V = 8$, $A = 12$ y $C = 6$, de modo que, como antes $V + C - A = 2$. La *agudeza* del cubo es ahora, también algo *mayor* que antes ($90 \times 3 = 270°$; $270° - 360° = -90°$).

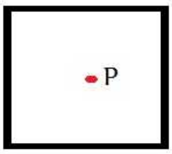

Fig 3A

El cuadrado,
con su centro P

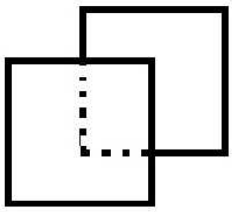

Fig 3B

El segundo cuadrado
detrás del primero.

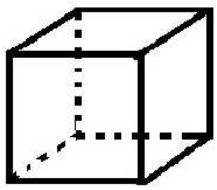

Fig 3C

El Hexaedro, ya
completo, uniendo
los vértices respectivos
de los dos cuadrados.

1.4 Icosaedro

Para la construcción del *Icosaedro* I, partimos de un *pentágono* regular que llamaremos N (Norte), puesto que tomaremos otro pentágono idéntico que llamaremos S (Sur). Tomamos ahora el centro P de un pentágono (digamos el N), y lo dividimos en cinco triágulos isósceles, uniendo ese centro con los cinco vértices del pentágono. Ahora imaginemos que el centro "se levanta" un poco (que es calculable), puesto que los cinco triángulos equivalen a $60° \times 5 = 300° < 360°$. La *agudeza* es ahora $-360° + 300° = -60°$, la *mayor* hasta ahora. Procedemos igualmente a dividir en triángulos isósceles el pentágono S, que ahora mirará un poco "hacia abajo", con la misma agudeza; puestos los dos pentágonos en situación antisimétrica, unimos finalmente con dos aristas (del pentágono N) por vértice (del pentágono S) los dos pentágonos, con longitud igual a los lados del pentágono, resultando las figuras $4A$, $4B$ y $4C$. Los pentágonos S y N están en situación antisimétrica, pues así se garantiza que la "unión" entre los dos se hace con dos aristas (e.g. del pentágono N) con un vértices del pentágono S.

Queda así la construcción $(V, A, C) = (6+6 = 12,\ 10+10+10 = 30,\ 5+5+10 = 20)$, con caras *triangulares*, con cinco por vértice, que definen el Icosaedro regular I.

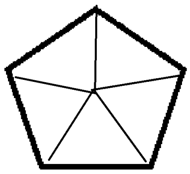

Fig. 4A
El pentágono N,
dividido en 5 triángulos.

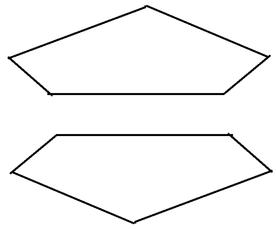

Fig. 4B
Otro pentágono (Sur).

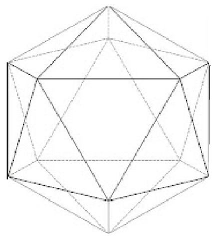

Fig. 4C
Nuevas aristas, desde
los dos casquetes
triangulares.

1.5 Dodecaedro

El poliedro regular más difícil de construir es, de nuevo en nuestra opinión, el *Dodecaedro D*. También partimos de dos pentágonos regulares, N y S. Para cada uno, se prolongan sus cinco vértices con nuevas aristas, hacia afuera de longitud la del lado del pentágono. Se completan cinco pentágonos más rodeando el original (Fig $5A$ y $5B$; pentágono N, luego el S en situación antisimétrica, como en el Icosaedro. Se agregan *dos* lados por pentágono a construir. Nótese que, de momento, hay dos o tres aristas por vértice, por lo que se completa la construcción lanzando una tercera arista desde el pentágono inferior a los pentágonos que rodean el N. Queda así la figura $5C$. La "esfericidad" está asegurada, pues (el ángulo entre dos lados de un pentágono regular es de 108°) $108° \times 3 = 324°$, todavía $< 360°$, por eso el Dodecaedro D recuerda el que más a una esfera (o un balón de fútbol). Para este Dodecaedro D tenemos ahora V, A, C = $(20, 30, 12)$. La *agudeza* es la máxima $324° - 360° = -36°$).

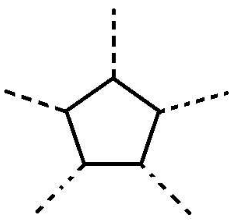

Fig. 5A
El pentágono regular,
con 3 vértices por
arista.

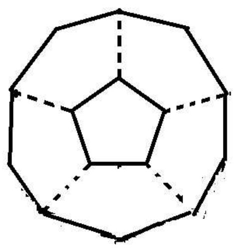

Fig. 5B
Los pentágonos nuevos, N

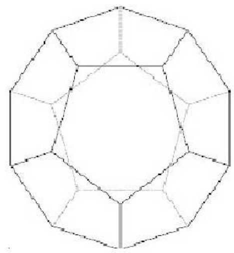

Fig. 5C
Dodecaedro D,
visto desde S .

Como resumen, la construcción explícita de los cinco sólidos platónicos es fácil y directa. El más sencillo parece ser el cubo H, el más difícil el dodecaedro D. En las referencias pueden verse muchas otras construcciones, así como en “Google”.

2 Propiedades

En esta sección justificaremos algunos de los resultados anteriores, y expondremos algunos más; como referencias generales citemos [2] y [4].

En primer lugar, tiene que haber tres o más caras por vértice, para formar una figura tridimensional: con solo dos caras con una arista común NO se puede.

La *agudeza* de un poliedro podemos definirla como la suma de los ángulos en un vértice menos 360°, como hemos indicado en el texto. Si ϕ es el ángulo entre dos aristas contiguas de un polígono regular, para que eventualmente varios polígonos del mismo tipo cierren en una figura de casquete esférico, es decir, NO en un plano, la *agudeza* debe ser negativa; si hay m aristas por vértice, la condición es, obviamente

$$m\,\phi < 360^\circ$$

Para triángulos, cuadrados, pentágonos y hexágonos regulares tenemos ϕ=(60°, 90°, 108° y 120°): se sigue por tanto que sólo triángulos (con tres, cuatro o cinco por vértice), cuadrados o pentágonos (con tres sólo por vértice) son posibles. En particular, los hexágonos están excluidos, aunque ellos pueden “teselar” (cubrir de modo regular) el plano, que también puede ser cubierto por cuadrados (de cuatro en cuatro) y por triángulos (de seis en seis), como es bien conocido. Como resultado, resultan precisamente los cinco poliedros regulares, con sus miles de años de antigüedad... No deja de ser notable que la condición de existencia sea necesaria y suficiente... los griegos asociaban los cuatro primeros poliedros a los “elementos” aire, agua, fuego y tierra; al quinto o Dodecaedro D corresponde a la quinta-esencia de los alquimistas medievales ([1, p. 149]).

Una idea que no hemos tocado aun es la de *dualidad*, un concepto importante. Imaginemos que tomamos, en un cubo, los centros de las seis caras, como vértices y los unimos con dos aristas por (nuevo) vértice: sigue quedando un poliedro regular Π, pero con caras y vértices intercambiado respecto del cubo original: es el llamado “poliedro” dual $\Pi^\star$. Tenemos que el tetraedro $T(\,V, A, C = 4, 6, 4)$ es autodual ($T = T^\star$), pero los otro cuatro poliedros no, en particular ($H^\star = O$) y ($I^\star = D$).

¿Por qué, en todos los casos, $V + C - A = 2$? Es el número de Euler-Poincaré de la esfera, concepto que nos llevaría tiempo desarrollar en detalle; véase, por ejemplo [5].

“Teselar” es cubrir un espacio con politopos: si el espacio es $\mathbb{R}$, sólo cabe un “politopo” unidimensional, con vértices equidistantes. Si es $\mathbb{R}^2$, o sea el plano, éste se puede cubrir con triángulos, cuadrados o hexágonos, como hemos señalado y es bien conocido.

Existe una notación *standard* para politopos, debida al suizo L. Schläfli; para polígonos regulares, basta $\{m\}$, el número de lados; para poliedros, $\{p, q\}$ donde p es el número de lados del polígono y q el número de polígonos por vértice: para los cinco sólidos platónicos, la notación de Schläfli es $T\{3, 3\}$; $O\{3, 4\}$ $H\{4, 3\}$; $I\{3, 5\}$ y $D\{5, 3\}$. Se ve inmediatamente que el dual de $\{p, q\}$ es $\{q, p\}$; eventualmente, la notación de Schläfli no se limita a politopos regulares...

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ACTIVIDADES DE LA REAL ACADEMIA DE CIENCIAS EXACTAS, FÍSICAS, QUÍMICAS Y NATURALES DE ZARAGOZA EN EL AÑO 2014

Sesiones:

En el año 2014 la Real Academia de Ciencias de Zaragoza celebró seis sesiones. En las dos primeras, celebradas los días 12 de marzo y 24 de abril de 2014, para proceder a la lectura de uno de los trabajos que recibieron los Premios de Investigación de la Academia 2013, actos que, por razones de agenda, habían quedado aplazados para el año 2014.

Las restantes sesiones tuvieron lugar los días 7 mayo, 11 de junio, 20 de octubre y 26 de noviembre. En la de 20 de octubre, el Académico Electo D. Manuel Silva Suárez, presentó su discurso de ingreso, respondiéndole en nombre de la Academia el Ilmo. Sr. D. Alberto Elduque Palomo.

También se colaboró con la Facultad de Ciencias y el Departamento de Física Aplicada en la organización una sesión obituario por el fallecido Profesor Manuel Quintanilla Montón, quien fue durante muchos años Tesorero de la Academia.

La Real Academia de Ciencias de Zaragoza participó el miércoles 5 de noviembre en la Sesión Solemne de Apertura Conjunta de curso de las Academias de Aragón en el Paraninfo de la Universidad, que fue organizada por la Academia Aragonesa de Jurisprudencia y Legislación. La lección inaugural fue impartida por el Excmo. Sr. Dr. D. Agustín Luna Serrano y tuvo por título "Acerca de las verdades oficiales del Derecho: el caso de las verdades fiduciarias". La contestación corrió a cargo del Presidente de la Academia Aragonesa de Jurisprudencia y Legislación, Excmo. Sr. Dr. D. Eduardo Montull Lavilla.

Altas y bajas de Académicos Numerarios:

- En la sesión de 20 de octubre, el Académico Electo D. Manuel Silva Suárez, presentó su discurso de ingreso con el título: "De discretos y fluidos, entre fidelidad y complejidad", recibiendo la medalla Nº 19.

- El Ilmo. Sr. D. Javier Sesma Bienzobas, medalla Nº 3, causa baja a petición propia como Académico (Sección de Físicas), siendo aceptada su solicitud en la sesión celebrada el 7 de mayo.

- En la sesión de 11 de junio fueron elegidos miembros de la Academia los Profesores D. Ricardo Ibarra García por la Sección de Físicas, quien recibirá la medalla Nº 20 y D. Fernando J. Lahoz Díaz, por la Sección de Químicas, quién recibirá la medalla Nº 2.

Altas y bajas de Académicos Correspondientes:

- D. Javier Sesma Bienzobas, que ha causado baja como Académico Numerario de la sección de Físicas, ha pasado a ser Académico Correspondiente por la misma sección.

Publicaciones de la Academia:

La Academia ha publicado el volumen 69 de la Revista de la Academia de Ciencias de Zaragoza.

Organización de Congresos y Conferencias:

La Academia ha organizado en 2014, entre otros, los siguientes eventos:

- El día 21 de mayo la Academia convocó y desarrolló la *Jornada de Homenaje al ilustrado aragonés Ignacio Jordán de Asso en el bicentenario de su muerte.* Se sumó así a otros actos de homenaje promovidos por la Universidad de Zaragoza, la Sociedad Económica Aragonesa de Amigos del País y el Ateneo de Zaragoza. La asistencia fue muy reducida, pero altamente valiosa, y las ponencias serán publicadas en una Monografía de la Academia de Ciencias.

  El Vicepresidente D. Juan P. Martínez-Rica fue el responsable de la organización de la mencionada Jornada.

- Colaboración en la organización conjunta con la Facultad de Ciencias de los Ciclos de Conferencias *Cita con la Ciencia* y *Espacio Facultad 2013-2014*, así como en la del *IX Premio de divulgación científica José María Savirón.*

Por último, dentro de la habitual participación de Académicos en numerosos congresos nacionales e internacionales, y en conferencias en el ámbito de la difusión de la ciencia, cabe destacar las siguientes actuaciones:

- El Académico Alberto Elduque ha sido conferenciante invitado en los congresos internacionales *Enveloping Algebras and Representation Theory (EART2014)*, en St. Johns (Canadá), *Geometric Methods in Representation Theory*, en Lancaster (Inglaterra) y en *XII Jornadas de Álgebra No Conmutativa*, en Málaga, siendo miembro del Comité Científico de dicho Congreso.

- El Académico Enrique Artal presentó ponencias invitadas en los Congresos Internacionales *Workshop on Singularities in geometry, topology, foliations and dynamics*, en Mérida (Méjico) y en el *Symposium on Singularities and their Topology* en Hannover, así como en *International Meeting of the American Mathematical Society and the Romanian Mathematical Society* en Alba Iulia (Rumanía) y en *Algebra and Geometry and Topology of Singularities. On the occasion of the 60th birthday of I. Luengo* en Miraflores de la Sierra.

- El Académico Luis Oro ha participado en actividades de las Academias Nacionales de Ciencias de Alemania y Francia, de las que es miembro. Ha sido conferenciante invitado en la *40th International Conference on Coordimnation Chemistry* (Singapur, Julio de 2014) y en el *Europe-Japan Joint Forum in Chemistry* (Estrasburgo, Octubre 2014) así como en el *23th Annual Saudi-Japan Symposium* (Dhahran, Arabia Saudita, Diciembre de 2014). Impartió la Conferencia de clausura del ciclo *75 Aniversario CSIC*, en Zaragoza (diciembre 2014).

- La Académica María Teresa Lozano Imízcoz impartió la conferencia inaugural del *Tercer Encuentro Conjunto RSME-SMM* que se celebró en Zacatacas, (México).

- El Académico Antonio Elipe ha sido presidente del Comité Científico de las *XIV Jornadas de Mecánica Celeste* que tuvieron lugar en Ribadeo en el mes de julio, y presidente del Comité Organizador del *II Congreso Nacional de i+d en Defensa y Seguridad* celebrado en Zaragoza en noviembre.

- El Académico Manuel Silva presentó una conferencia invitada en el *Colloquium UPMC-Sorbonne*, en París.

- El Secretario José F. Cariñena formó parte del Comité Organizador de los *XVI Encuentros de Invierno Mecánica, Geometría y Teoría de control*, 31 de enero, y 1 de febrero y en el *Thematic Day: Discrete Mechanics and Geometric Integrators*, el 30 de enero, en Zaragoza. Presentó conferencias invitadas en los Congresos internacionales *10th AIMS Conf. on Dynamical Systems, Differential Equations and Applications* en Madrid, *II Meeting on Lie systems, generalisations, and applications*, en el Institute of Mathematics, Polish Academy of Sciences en Varsovia (Polonia) y en el Congreso *Symmetries, special functions and superintegrability*, en Valladolid, en el que fue también miembro del Comité Científico del Congreso.

Varios Académicos han colaborado en cursos propios en la Universidad de la Experiencia que organiza la Universidad de Zaragoza.

Premios de investigación 2014

Se concedieron los Premios de Investigación 2014 de la Academia correspondientes a las Secciones de Exactas y Físicas. En la sección de Exactas el Premio fue para el Dr. Luis Ugarte Vilumbrales por su trabajo *Special Hermitian metrics, complex nilmanifolds and holomorphic deformations*, y en la de Físicas el Premio fue para el Dr. Igor García Irastorza por el trabajo titulado *Hunting the axion*. Dichos trabajos se publicarán en el volumen 70 de la Revista de la Academia.

Se ha iniciado el proceso para los Premios 2015 en las Secciones de Químicas y Naturales.

Distinciones y Nombramientos a Académicos.

El Académico Luis Oro ha sido distinguido con el *EuCheMS Award* y la Medalla *Félix Serratosa*.

Otros datos.

La Real Academia de Ciencias de Zaragoza, como el pasado año y en contra de lo sucedido hasta ese momento, no ha recibido subvencióna para su funcionamiento este año del Ministerio de Educación y Ciencia, a través del programa de apoyo a las Reales Academias asociadas al Instituto de España.

Se ha continuado con la actualización de la página web de la Academia, cuya nueva dirección es `http://acz.unizar.es`

Zaragoza, diciembre de 2014

REVISTA DE LA REAL ACADEMIA DE CIENCIAS EXACTAS, FÍSICAS, QUÍMICAS Y NATURALES DE ZARAGOZA

Abstract

La Revista de la Real Academia de Ciencias publishes original research contributions in the fields of Mathematics, Physics, Chemistry and Natural Sciences. All the manuscripts are peer reviewed in order to assess the quality of the work. On the basis of the referee's report, the Editors will take the decision either to publish the work (directly or with modifications), or to reject the manuscript.

1 Normas generales de publicación

1.1 Envío de los manuscritos.

Para su publicación en esta Revista, los trabajos deberán remitirse a

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o bien electrónicamente a la cuenta `elipe@unizar.es`.

La Revista utiliza el sistema de *offset* de edición, empleando el texto electrónico facilitado por los autores, que deberán cuidar al máximo su confección, siguiendo las normas que aquí aparecen.

Los autores emplearán un procesador de texto. Se recomienda el uso de LaTeX, para el que se han diseñado los estilos `academia.sty` y `academia.cls` que pueden obtenerse directamente por internet en `http://acz.unizar.es` o por petición a la cuenta de correo electrónico: `elipe@unizar.es`.

1.2 Dimensiones

El texto de los trabajos, redactados en español, inglés o francés, no deberá exceder de 16 páginas, aunque se recomienda una extensión de 6 a 10 páginas como promedio. El texto de cada página ocupará una caja de 16×25 cm., con espacio y medio entre líneas.

2 Presentación del trabajo.

Los trabajos se presentarán con arreglo al siguiente orden: En la primera página se incluirán los siguientes datos:

a) Título del trabajo: Conciso, pero ilustrativo, con mayúsculas.

b) Autor: Nombre y apellidos del autor o autores, con minúscula.

c) Centro: Centro donde se ha realizado, con su dirección postal.

d) Abstract: En inglés y con una extensión máxima de 200 palabras.

e) Texto

A) Los encabezamientos de cada sección, numerados correlativamente, serán escritos con letras **minúsculas** en negrita. Los encabezamientos de subsecciones, numerados en la forma 1.1, 1.2, ..., 2.1, 2.2, ..., se escribirán en *cursiva*.

B) Las fórmulas estarán centradas y numeradas correlativamente.

C) Las referencias bibliogáficas intercaladas en el texto, deben ser fácilmente identificables en la lista de refencias que aparecerá al final del artículo, bien mediante un número, bien mediante el nombre del autor y año de publicación.

D) Las figuras y tablas, numeradas correlativamente, se intercalarán en el texto. Las figuras se enviarán en formato EPS, o que se pueda convertir a éste con facilidad. Los apéndices, si los hay, se incluirán al final del texto, antes de la bibliografía.

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Por cada trabajo publicado, se entregarán al autor o autores un total de 25 separatas. La Revista permite la inclusión de fotografías o figuras en color, con un coste adicional que correrá a cargo de los autores.

Antonio Elipe
Académico Editor

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