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Special Hermitian metrics, complex nilmanifolds and holomorphic deformations

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Abstract

We focus on the interaction of several complex invariants of cohomological type and metric properties of compact complex manifolds, as well as their behaviour under holomorphic deformations of the complex structure. Recent results on the complex geometry of nilmanifolds concerning such properties are reviewed. We show that the complex geometry of the 6-dimensional manifold $N \times N$ given by the product of two copies of the Heisenberg nilmanifold N allows to construct a holomorphic deformation with interesting properties in its central limit.

1 Introduction

In this paper we consider compact complex manifolds (M, J) with special Hermitian metrics, mainly balanced and strongly Gauduchon metrics. We focus on the interaction of the existence properties of such metrics and several complex invariants of cohomological type which are related to the $\partial\bar{\partial}$-lemma condition. An important problem is the study of the deformation limits of these properties under holomorphic deformations of the complex structure. We show that the class of complex nilmanifolds provides a very rich source of explicit examples of analytic families of compact complex manifolds with interesting and unusual behaviours in their central fibres.

In Section 2.1 we first recall the definition and the main properties of some complex invariants of cohomological type on a compact complex manifold (M, J) of complex dimension n which are related to the $\partial\bar{\partial}$-lemma condition, namely $\mathbf{f}_k(M, J)$ for $0 \leq k \leq n$

and $\mathbf{k}_r(M,J)$ for $r \geq 1$. Such complex invariants have been introduced in [7, 26, 41] and they are defined by means of the Bott-Chern cohomology $H^{p,q}_{\mathrm{BC}}(M,J)$, the Aeppli cohomology $H^{p,q}_{\mathrm{A}}(M,J)$ and the terms in the Frölicher spectral sequence $\{E_r(M,J)\}_{r\geq 1}$ (see below for details). In Section 2.2 we consider some special Hermitian metrics on compact complex manifolds. It is well known that the existence of a Kähler metric imposes strong topological conditions on the manifold, in particular, (M,J) must satisfy the $\partial\bar\partial$-lemma [16], which in addition implies the degeneration of the Frölicher sequence at E_1 and the formality of the manifold. On the other hand, Gauduchon proved in [21] that in the conformal class of any Hermitian metric on (M,J) there always exists a Hermitian metric F satisfying $\partial\bar\partial F^{n-1} = 0$. Between the Kähler class and the Gauduchon class, other interesting classes of special Hermitian metrics have been considered in relation to several problems in differential and algebraic geometry. A metric F is called *balanced* if $dF^{n-1} = 0$, and a characterization of the existence of balanced metrics in terms of currents was given in [28]. More recently, Popovici has introduced a new class of Hermitian metrics [32], namely the class of *strongly Gauduchon* (*sG* for short) metrics, in relation to the study of central limits of analytic families of projective manifolds [33]. Such metrics are defined by the condition that ∂F^{n-1} is $\bar\partial$-exact. Section 2.3 is devoted to the class of *sGG* manifolds introduced and investigated in [36], which are defined as those compact complex manifolds whose sG cone coincides with the Gauduchon cone. There are several characterizations of the sGG manifolds, for instance, as those compact complex manifolds (M,J) for which every Gauduchon metric is sG, as well as those (M,J) satisfying the following special case of the $\partial\bar\partial$-lemma: if Ω is a d-closed $(n,n-1)$-form that is ∂-exact, then Ω is also $\bar\partial$-exact. Moreover, in [36] two numerical characterizations of the sGG manifolds are obtained, which involve the Bott-Chern number $h^{0,1}_{\mathrm{BC}}(M,J)$, the Hodge number $h^{0,1}_{\bar\partial}(M,J)$ and the first Betti number $b_1(M)$ of the manifold (see Proposition 2.5).

In Section 3 we address the problems of *openness* and *closedness* of the properties considered in the previous sections under holomorphic deformations of the complex structure. For each k such that $0 \leq k \leq n$, we say that a compact complex manifold (M,J) has the property $\mathcal{F}_k$ if the Angella-Tomassini invariant $\mathbf{f}_k(M,J) = \sum_{p+q=k}(h^{p,q}_{\mathrm{BC}}(M,J) + h^{p,q}_{\mathrm{A}}(M,J)) - 2b_k(M)$ vanishes. By [7], a compact complex manifold (M,J) satisfies the $\partial\bar\partial$-lemma if and only if it has the property $\mathcal{F}_k$ for every k. Similarly, we say that (M,J) has the property $\mathcal{K}$ if the Schweitzer invariant $\mathbf{k}_1(M,J) = h^{1,1}_{\mathrm{BC}}(M,J) + 2h^{0,2}_{\bar\partial}(M,J) - b_2(M)$ vanishes. Compact complex manifolds (M,J) satisfying the $\partial\bar\partial$-lemma necessarily have the property $\mathcal{K}$. The properties $\mathcal{F}_k$ and $\mathcal{K}$ are open, i.e. they are stable under holomorphic deformations of the complex structure. Other properties which are open are the degeneration of the Frölicher spectral sequence at E_1 [24], the sG property [32] and the sGG property [36]. However, the balanced property is not open [2].

It is now known that the closedness of all these properties under holomorphic deformations fails. In [17] it was proved that the property of the Frölicher spectral sequence degenerating at E_1 is not closed under holomorphic deformations. The first example of an analytic family of compact complex manifolds $(X_t)_{t\in\Delta}$, Δ being an open disc around the origin in $\mathbb{C}$, such that the complex invariants $\mathbf{k}_1(X_t) = \mathbf{f}_2(X_t) = 0$ for any $t \neq 0$, but $\mathbf{k}_1(X_0) \neq 0$ and $\mathbf{f}_2(X_0) \neq 0$ (i.e. the properties $\mathcal{K}$ and $\mathcal{F}_k$ for $k = 2$ are not closed) was constructed in [26]. We will denote here this concrete analytic family by $\mathcal{X}$, i.e. $\mathcal{X} = (X_t)_{t\in\Delta}$. Its construction was based on an appropriate deformation of an invariant complex structure on a 6-dimensional nilmanifold. The family $\mathcal{X}$ suggested that the $\partial\bar{\partial}$-lemma might be a non-closed property, as it has later been confirmed by Angella and Kasuya in [6] (see also [19]) by constructing certain holomorphic deformation of the Nakamura solvmanifold. Moreover, the Frölicher spectral sequence of any compact complex manifold in the analytic family $\mathcal{X}$ degenerates at E_1 except for the central fibre [10]. This analytic family $\mathcal{X}$ also allows to show that the sGG property is not closed under holomorphic deformations [36]. Furthermore, the fibres X_t have balanced metric for any $t \in \Delta\backslash\{0\}$, but the central fibre X_0 does not admit any sG metric, so the balanced property and the sG property are not closed [10].

Since the compact complex manifolds X_t in the analytic family $\mathcal{X}$ are given by complex nilmanifolds (M, J_t) where J_t is invariant, in Section 4 we focus on the class of 6-dimensional nilmanifolds M endowed with invariant complex structures J; that is, $M = \Gamma\backslash G$ is a compact quotient of a 6-dimensional simply-connected nilpotent Lie group G by a lattice Γ of maximal rank in G, and J stems naturally from a "complex" structure J on the Lie algebra $\mathfrak{g}$ of G. Such nilmanifolds are classified through their underlying Lie algebras [40] (see Theorem 4.2). A crucial result in the theory of nilmanifolds is Nomizu's theorem [31] which asserts that the de Rham cohomology of M is canonically isomorphic to the cohomology of its underlying Lie algebra $\mathfrak{g}$. Many efforts have been made to achieve a Nomizu's type result for the Dolbeault cohomology and other complex invariants of cohomological type on (M, J), and several advances under additional conditions on the invariant complex structure J can be found in [4, 11, 12, 14, 15, 27, 37, 38, 39]. Such results, together with the classification of invariant complex structures J in dimension 6 obtained in [10], allow to compute the Bott-Chern cohomology groups $H^{p,q}_{\mathrm{BC}}(M, J)$ [5, 26]. The Bott-Chern numbers are given in Tables 1–3 below for any invariant complex structure J (up to isomorphism). On the other hand, we collect in Theorems 4.4, 4.5 and 4.8, and Propositions 4.6 and 4.7, the general results about the Frölicher spectral sequence, the existence of balanced and sG metrics, as well as the sGG condition for nilmanifolds in dimension 6 obtained in [10, 36, 44, 46]. These results allow to conclude that, apart from the obvious implications, most of the previous properties of compact complex manifolds are unrelated.

The real nilmanifold in the analytic family of compact complex manifolds $\mathcal{X}$ mentioned above, is not a product since the Lie algebra underlying the nilmanifold is irreducible. In Section 5 we consider the complex geometry of the product $N \times N$ of two copies of the (3-dimensional) Heisenberg nilmanifold N. This geometry turns out to be surprisingly rich as it allows to construct a new holomorphic family of compact complex manifolds $(N \times N, J_t)$ satisfying similar properties to those of the family $\mathcal{X}$ (see Theorem 5.2 for details). To our knowledge this is the first example of a holomorphic deformation having such properties, constructed on a 6-dimensional product manifold.

2 Complex invariants and Hermitian geometry

In this section we recall the definitions and main properties of some complex invariants of cohomological type on a compact complex manifold (M, J) which are related to the $\partial\bar{\partial}$-lemma condition. Some important classes of special Hermitian metrics on (M, J) are also considered, as well as several relations among them.

2.1 Complex invariants related to the $\partial\bar{\partial}$-lemma

Let (M, J) be a compact complex manifold of complex dimension n and consider $\Omega^{p,q}(M)$ the space of forms of bidegree (p, q) with respect to the complex structure J, i.e. $\Omega^k_{\mathbb{C}}(M) = \oplus_{p+q=k}\Omega^{p,q}(M)$ for $0 \leq k \leq 2n$.

It is well known that the Dolbeault cohomology groups $H^{p,q}_{\bar{\partial}}(M, J)$ of (M, J) are defined by

$$H^{p,q}_{\bar{\partial}}(M, J) = \frac{\ker\{\bar{\partial}\colon \Omega^{p,q}(M) \longrightarrow \Omega^{p,q+1}(M)\}}{\operatorname{im}\{\bar{\partial}\colon \Omega^{p,q-1}(M) \longrightarrow \Omega^{p,q}(M)\}}.$$

These groups are complex invariants of the manifold. The Frölicher spectral sequence $\{E_r(M, J)\}_{r\geq 1}$ of a complex manifold (M, J) is the spectral sequence associated to the double complex $(\Omega^{p,q}(M), \partial, \bar{\partial})$, where $\partial + \bar{\partial} = d$ is the decomposition, with respect to J, of the exterior differential d [20]. The first term $E_1(M, J)$ in the sequence is precisely the Dolbeault cohomology of (M, J), that is, $E^{p,q}_1(M, J) \cong H^{p,q}_{\bar{\partial}}(M, J)$, and after a finite number of steps this sequence converges to the de Rham cohomology of M, i.e. $H^k_{\mathrm{dR}}(M, \mathbb{C}) \cong \oplus_{p+q=k}E^{p,q}_\infty(M, J)$, which is a topological invariant of M. More concretely, for each $r \geq 1$ there is a sequence of homomorphisms d_r

$$\cdots \longrightarrow E^{p-r,q+r-1}_r(M, J) \xrightarrow{d_r} E^{p,q}_r(M, J) \xrightarrow{d_r} E^{p+r,q-r+1}_r(M, J) \longrightarrow \cdots$$

such that $d_r \circ d_r = 0$ and $E^{p,q}_{r+1}(M, J) = \ker d_r/\operatorname{im} d_r$. The homomorphisms d_r are induced by ∂. When $r = 1$ the homomorphism $d_1\colon H^{p,q}_{\bar{\partial}}(M, J) \longrightarrow H^{p+1,q}_{\bar{\partial}}(M, J)$ is given by $d_1([\alpha_{p,q}]) = [\partial\alpha_{p,q}]$, for $[\alpha_{p,q}] \in H^{p,q}_{\bar{\partial}}(M, J)$. For $r = 2$ we have

$$E^{p,q}_2(M, J) = \frac{\{\alpha_{p,q} \in \Omega^{p,q}(M) \mid \bar{\partial}\alpha_{p,q} = 0,\ \partial\alpha_{p,q} = -\bar{\partial}\alpha_{p+1,q-1}\}}{\{\bar{\partial}\beta_{p,q-1} + \partial\gamma_{p-1,q} \mid \bar{\partial}\gamma_{p-1,q} = 0\}},$$

and the homomorphism $d_2\colon E_2^{p,q}(M,J) \longrightarrow E_2^{p+2,q-1}(M,J)$ is given by $d_2([\alpha_{p,q}]) = [\partial\alpha_{p+1,q-1}]$, for $[\alpha_{p,q}] \in E_2^{p,q}(M,J)$. We will focus on compact complex manifolds (M,J) of complex dimension 3, so it is sufficient to describe the spectral sequence up to the third step E_3 because in general $E_r(M,J) \cong E_\infty(M,J)$ for any $r \geq \dim_{\mathbb{C}}(M,J)$ (for general descriptions of d_r and $E_r^{p,q}$ see for example [13]).

In addition to the Dolbeault cohomology groups $H_{\bar\partial}^{p,q}(M,J)$ and the Frölicher terms $E_r^{p,q}(M,J)$, the Bott-Chern and Aeppli cohomologies [1, 8] define additional complex invariants of (M,J) given, respectively, by

$$H_{\mathrm{BC}}^{p,q}(M,J) = \frac{\ker\{d\colon \Omega^{p,q}(M) \longrightarrow \Omega_{\mathbb{C}}^{p+q+1}(M)\}}{\mathrm{im}\,\{\partial\bar\partial\colon \Omega^{p-1,q-1}(M) \longrightarrow \Omega^{p,q}(M)\}},$$

and

$$H_{\mathrm{A}}^{p,q}(M,J) = \frac{\ker\{\partial\bar\partial\colon \Omega^{p,q}(M) \longrightarrow \Omega^{p+1,q+1}(M)\}}{\mathrm{im}\,\{\partial\colon \Omega^{p-1,q}(M) \longrightarrow \Omega^{p,q}(M)\} + \mathrm{im}\,\{\bar\partial\colon \Omega^{p,q-1}(M) \longrightarrow \Omega^{p,q}(M)\}}.$$

By the Hodge theory developed by Schweitzer in [41], all these complex invariants are finite dimensional and one has the isomorphisms $H_{\mathrm{A}}^{p,q}(M,J) \cong H_{\mathrm{BC}}^{n-q,n-p}(M,J)$. Notice that $H_{\mathrm{BC}}^{q,p}(M,J) \cong H_{\mathrm{BC}}^{p,q}(M,J)$ by complex conjugation.

From now on we shall denote by $h_{\mathrm{BC}}^{p,q}(M,J)$ the dimension of the cohomology group $H_{\mathrm{BC}}^{p,q}(M,J)$. The Hodge numbers will be denoted simply by $h_{\bar\partial}^{p,q}(M,J)$ and the Betti numbers by $b_k(M)$.

For any $r \geq 1$ and for any p,q, there are well-defined natural maps

$$H_{\mathrm{BC}}^{p,q}(M,J) \longrightarrow E_r^{p,q}(M,J) \qquad \text{and} \qquad E_r^{p,q}(M,J) \longrightarrow H_{\mathrm{A}}^{p,q}(M,J).$$

In general these maps are neither injective nor surjective. However, all the maps are isomorphisms if and only if (M,J) satisfies the $\partial\bar\partial$-lemma [16], that is, for any d-closed form α of pure type on (M,J) the following exactness properties are equivalent:

$$\alpha \text{ is } d\text{-exact} \iff \alpha \text{ is } \partial\text{-exact} \iff \alpha \text{ is } \bar\partial\text{-exact} \iff \alpha \text{ is } \partial\bar\partial\text{-exact}.$$

Therefore, if the $\partial\bar\partial$-lemma is satisfied then the previous invariants coincide and in particular one has the Hodge decomposition $H_{\mathrm{dR}}^k(M,\mathbb{C}) \cong \oplus_{p+q=k} H_{\bar\partial}^{p,q}(M,J)$ for any k, where in addition $H_{\bar\partial}^{p,q}(M,J) \cong \overline{H_{\bar\partial}^{q,p}(M,J)}$.

Recently, Angella and Tomassini have introduced in [7] new complex invariants that measure how far the compact complex manifold (M,J) is from satisfying the $\partial\bar\partial$-lemma condition.

Theorem 2.1. *[7] On any compact complex manifold (M,J) of complex dimension n the following inequalities are satisfied:*

$$\sum_{p+q=k} \left(h_{\mathrm{BC}}^{p,q}(M,J) + h_{\mathrm{BC}}^{n-p,n-q}(M,J)\right) \geq 2b_k(M), \qquad 0 \leq k \leq 2n.$$

Moreover, all these inequalities are equalities if and only if (M,J) satisfies the $\partial\bar\partial$-lemma.

Let us denote by $\mathbf{f}_k(M,J)$ the non-negative integer given by

$$\mathbf{f}_k(M,J) = \sum_{p+q=k} \left(h^{p,q}_{\mathrm{BC}}(M,J) + h^{n-p,n-q}_{\mathrm{BC}}(M,J) \right) - 2b_k(M).$$

By the dualities in the Bott-Chern and de Rham cohomologies, it is clear that $\mathbf{f}_{2n-k}(M,J) = \mathbf{f}_k(M,J)$. Now, for each $0 \le k \le n$, we consider the property

$$\mathcal{F}_k = \{\text{the compact complex manifold } (M,J) \text{ satisfies } \mathbf{f}_k(M,J) = 0\}.$$

Hence, by Theorem 2.1 a compact complex manifold (M,J) satisfies the $\partial\bar\partial$-lemma if and only if it has the property $\mathcal{F}_k$ for every $k \le n$.

On the other hand, for any compact complex manifold (M,J) Schweitzer proved in [41, Lemma 3.3] that

$$h^{1,1}_{\mathrm{BC}}(M,J) + 2\, h^{0,2}_{\bar\partial}(M,J) \ge b_2(M),$$

and moreover, if (M,J) satisfies the $\partial\bar\partial$-lemma then the equality holds. More generally, one has

Proposition 2.2. *[26] If (M,J) is a compact complex manifold then for any $r \ge 1$*

$$h^{1,1}_{\mathrm{BC}}(M,J) + 2 \dim E_r^{0,2}(M,J) \ge b_2(M),$$

where $E_r^{0,2}(M,J)$ denotes the r-step $(0,2)$-term of the Frölicher spectral sequence. Furthermore, if (M,J) satisfies the $\partial\bar\partial$-lemma then the above inequalities are all equalities.

From now on, we will denote by $\mathbf{k}_r(M,J)$, $r \ge 1$, the non-negative integer given by

$$\mathbf{k}_r(M,J) = h^{1,1}_{\mathrm{BC}}(M,J) + 2 \dim E_r^{0,2}(M,J) - b_2(M).$$

Therefore, $\mathbf{k}_r(M,J)$ are complex invariants which vanish if the manifold (M,J) satisfies the $\partial\bar\partial$-lemma. Notice that $\mathbf{k}_1(M,J) \ge \mathbf{k}_2(M,J) \ge \mathbf{k}_3(M,J) = \mathbf{k}_r(M,J) \ge 0$ for any $r \ge 4$.

In general $\mathbf{k}_1(M,J)$, $\mathbf{k}_2(M,J)$ and $\mathbf{k}_3(M,J)$ do not coincide [26], but the vanishing of $\mathbf{k}_1(M,J)$ implies the vanishing of any other $\mathbf{k}_r(M,J)$. This fact justifies to consider the following property:

$$\mathcal{K} = \{\text{the compact complex manifold } (M,J) \text{ satisfies } \mathbf{k}_1(M,J) = 0\}.$$

Obviously, any compact complex manifold satisfying the $\partial\bar\partial$-lemma has the property $\mathcal{K}$.

2.2 Special Hermitian metrics

Let (M,J) be a compact complex manifold of complex dimension n. A Hermitian metric g on (M,J) can be described by means of a positive definite smooth form F on

M of bidegree $(1,1)$ with respect to J. In what follows, we will refer to F as a Hermitian structure or as a Hermitian metric without distinction.

A Hermitian structure is *Kähler* if the form F is closed, that is, F is a symplectic form compatible with the complex structure. It is well known that the existence of a Kähler metric imposes strong topological conditions on the manifold. In particular, (M,J) satisfies the $\partial\bar{\partial}$-lemma [16], which in addition implies the formality of the manifold.

On the other hand, Gauduchon proved [21] that in the conformal class of any Hermitian metric there exists a Hermitian metric F satisfying $\partial\bar{\partial}F^{n-1}=0$. We will refer to a metric satisfying this condition as a *Gauduchon* metric.

Between the Kähler class and the Gauduchon class, other interesting classes of special Hermitian metrics have been considered in relation to several problems in differential and algebraic geometry. A metric F is *balanced* if $dF^{n-1}=0$, and the existence of balanced metrics in terms of currents was investigated in [28]. More recently, Popovici has introduced a new class of Hermitian metrics in relation to the study of the central limit of analytic families of projective manifolds: a metric F is called *strongly Gauduchon* (*sG* for short) if ∂F^{n-1} is $\bar{\partial}$-exact [32, 33].

By the definitions, any Kähler metric is balanced, any balanced metric is sG, and any sG metric is a Gauduchon metric, that is:

$$\text{Kähler} \implies \text{balanced} \implies \text{sG} \implies \text{Gauduchon.}$$

The converses to these implications are not true: for instance, one can find examples in the class of nilmanifolds (see Section 4 for details). However, as it is pointed out in [26], if a compact complex manifold (M,J) satisfies that the natural map

$$\zeta\colon H^{n,n-1}_{\bar{\partial}}(M,J) \longrightarrow H^{n,n-1}_{\mathrm{A}}(M,J), \qquad \zeta([\Omega]_{\bar{\partial}}) := [\Omega]_A \tag{1}$$

is injective (in particular, if the $\partial\bar{\partial}$-lemma is satisfied or if $h^{n,n-1}_{\bar{\partial}}(M,J)=0$) then any Gauduchon metric is an sG metric: in fact, if $\partial\bar{\partial}F^{n-1}=0$ then ∂F^{n-1} defines a class in the Dolbeault cohomology group $H^{n,n-1}_{\bar{\partial}}(M,J)$ such that the Aeppli cohomology class $[\partial F^{n-1}]_A=0$ in $H^{n,n-1}_{\mathrm{A}}(M,J)$, so the injectivity of ζ implies the existence of a complex form α of bidegree $(n,n-2)$ such that $\partial F^{n-1}=\bar{\partial}\alpha$. Therefore, if ζ is injective then by Gauduchon's result there exists an sG metric in the conformal class of any Hermitian metric. Notice that by Serre duality and by the dualities between Aeppli and Bott-Chern cohomologies, the injectivity of ζ implies $h^{0,1}_{\bar{\partial}}(M,J)=\dim H^{n,n-1}_{\bar{\partial}}(M,J)\leq \dim H^{n,n-1}_{\mathrm{A}}(M,J)=h^{0,1}_{\mathrm{BC}}(M,J)$.

In the next section we consider in more detail the compact complex manifolds for which any Gauduchon metric is sG, showing that the latter property is actually equivalent to the injectivity of the map (1).

2.3 The strongly Gauduchon cone

Let (M, J) be a compact complex manifold of complex dimension n. The *Gauduchon cone* of (M, J) is defined in [35] as the open convex cone

$$\mathcal{C}_G(M, J) \subset H_A^{n-1,\,n-1}(M, J)$$

consisting of the (real) Aeppli cohomology classes $[F^{n-1}]_A$ which are $(n-1)$-powers of Gauduchon metrics F on (M, J).

Let us consider the map T, induced by ∂ in cohomology, given by

$$T\,:\,H_A^{n-1,\,n-1}(M, J) \longrightarrow H_{\bar\partial}^{n,\,n-1}(M, J), \qquad T([\Omega]_A) := [\partial\Omega]_{\bar\partial} \tag{2}$$

for any $[\Omega]_A \in H_A^{n-1,\,n-1}(M, J)$. The *strongly Gauduchon cone* (sG cone, for short) was defined in [35] as the intersection of the Gauduchon cone with the kernel of the linear map T, i.e.

$$\mathcal{C}_{sG}(M, J) = \mathcal{C}_G(M, J) \cap \ker T \subset \mathcal{C}_G(M, J) \subset H_A^{n-1,\,n-1}(M, J).$$

Notice that, either all the Gauduchon metrics F for which F^{n-1} belongs to a given Aeppli-Gauduchon class $[F^{n-1}]_A \in \mathcal{C}_G(M, J)$ are sG, or none of them is; that is to say, the sG property is cohomological.

The following class is introduced in [36]: a compact complex manifold (M, J) is said to be an *sGG manifold* if the sG cone of (M, J) coincides with the Gauduchon cone of (M, J), i.e. $\mathcal{C}_{sG}(M, J) = \mathcal{C}_G(M, J)$. Since the kernel of T is a vector subspace of $H_A^{n-1,n-1}(M, J)$, its intersection with $\mathcal{C}_G(M, J)$ leaves the latter unchanged if and only if T vanishes identically.

It is clear that any sGG manifold (M, J) is an sG manifold because every Gauduchon metric F on (M, J) is sG. Also any compact complex manifold satisfying the $\partial\bar\partial$-lemma is sGG because the map ζ given by (1) is injective. Therefore:

$$\partial\bar\partial\text{-manifold} \implies \text{sGG manifold} \implies \text{sG manifold}.$$

The converses to these implications do not hold in general, and again one can find examples in the class of nilmanifolds (see Section 4).

In [36] two numerical characterizations of the sGG manifolds are obtained. The first one is given in terms of the Bott-Chern number $h_{\mathrm{BC}}^{0,1}(M, J)$ and the Hodge number $h_{\bar\partial}^{0,1}(M, J)$.

Theorem 2.3. *[36] On any compact complex manifold (M, J) we have $h_{\mathrm{BC}}^{0,1}(M, J) \leq h_{\bar\partial}^{0,1}(M, J)$. Moreover, (M, J) is an sGG manifold if and only if $h_{\mathrm{BC}}^{0,1}(M, J) = h_{\bar\partial}^{0,1}(M, J)$.*

The second numerical characterization of sGG manifolds involves the first Betti number $b_1(M)$ and the Hodge number $h_{\bar\partial}^{0,1}(M, J)$.

Theorem 2.4. *[36] On any compact complex manifold (M, J) we have $b_1(M) \leq 2\, h^{0,1}_{\bar\partial}(M, J)$. Moreover, (M, J) is an sGG manifold if and only if $b_1(M) = 2\, h^{0,1}_{\bar\partial}(M, J)$.*

It is well known that a compact complex surface is Kähler if and only if its first Betti number is even (a proof of this fact follows from Kodaira's classification of surfaces, [29] and [42]; see [9] and [25] for a direct proof). Hence, Theorem 2.4 makes the sGG manifolds reminiscent of the compact Kähler surfaces. It is clear that in complex dimension 2, the Kähler and the sGG conditions are equivalent. However, in dimension ≥ 3 the sGG property is much weaker than the Kähler one.

In the proof of Theorem 2.3 (see [36, Theorem 2.1]) it is shown that the map ζ given by (1) is always surjective, and that it is injective if and only if the manifold is sGG. In the following result we sum up the equivalent descriptions of the sGG property discussed above:

Proposition 2.5. *For a compact complex manifold (M, J), the following statements are equivalent:*

(i) *(M, J) is an sGG manifold;*

(ii) *every Gauduchon metric F on (M, J) is strongly Gauduchon;*

(iii) *the map ζ given by (1) is injective;*

(iv) *the map T given by (2) vanishes identically;*

(v) *the following special case of the $\partial\bar\partial$-lemma holds: for every d-closed $(n, n-1)$-form Ω on (M, J), if Ω is ∂-exact, then Ω is also $\bar\partial$-exact;*

(vi) $h^{0,1}_{\mathrm{BC}}(M, J) = h^{0,1}_{\bar\partial}(M, J)$;

(vii) $b_1(M) = 2\, h^{0,1}_{\bar\partial}(M, J)$.

3 Holomorphic deformations

In this section we address some problems about the behaviour of the properties considered in the previous section under holomorphic deformations of the complex structure.

Let Δ denote an open disc around the origin in $\mathbb{C}$. Following [34, Definition 1.12], a given property $\mathcal{P}$ of a compact complex manifold is said to be *open* under holomorphic deformations if for every holomorphic family of compact complex manifolds $(M, J_t)_{t\in\Delta}$ and for every $t_0 \in \Delta$ the following implication holds:

(M, J_{t_0}) has the property $\mathcal{P} \Longrightarrow (M, J_t)$ has the property $\mathcal{P}$ for all $t \in \Delta$ sufficiently close to t_0.

A given property $\mathcal{P}$ of a compact complex manifold is said to be *closed* under holomorphic deformations if for every holomorphic family of compact complex manifolds $(M, J_t)_{t\in\Delta}$ and for every $t_0 \in \Delta$ the following implication holds:

(M, J_t) has the property $\mathcal{P}$ for all $t \in \Delta\backslash\{t_0\} \Longrightarrow (M, J_{t_0})$ has the property $\mathcal{P}$.

Let us first consider the case when the property $\mathcal{P} = \mathcal{K}$ or $\mathcal{F}_k$ for some fixed k such that $0 \leq k \leq n$. Using the upper-semicontinuity of the Hodge numbers $h^{p,q}_{\bar\partial}(M, J_t)$ and that of the Bott-Chern numbers $h^{p,q}_{\mathrm{BC}}(M, J_t)$ as t varies in Δ (proved in [24, Theorem 4] and [41], respectively), it is easy to conclude that such properties are open under holomorphic deformations. In fact, for instance, for $\mathcal{P} = \mathcal{K}$, if $(M, J_t)_{t\in\Delta}$ is such that (M, J_{t_0}) has the property $\mathcal{K}$, then

$$b_2(M) = h^{1,1}_{\mathrm{BC}}(M, J_{t_0}) + 2\, h^{0,2}_{\bar\partial}(M, J_{t_0}) \geq h^{1,1}_{\mathrm{BC}}(M, J_t) + 2\, h^{0,2}_{\bar\partial}(M, J_t) \geq b_2(M),$$

for all t sufficiently close to t_0. Therefore, $\mathbf{k}_1(M, J_t) = 0$ and (M, J_t) also has the property $\mathcal{K}$.

However, the property $\mathcal{K}$ is not closed because, as proved in [26], there exists a holomorphic family of compact complex manifolds $\mathcal{X} = (M, J_t)_{t\in\Delta}$ such that $\mathbf{k}_1(M, J_0) \neq 0$ but $\mathbf{k}_1(M, J_t) = 0$ for all $t \in \Delta\backslash\{0\}$. The analytic family $\mathcal{X}$ is constructed on a 6-dimensional nilmanifold and the construction also suggests that one cannot expect a single property $\mathcal{F}_k$ to be closed. More concretely:

Theorem 3.1. *[26] The properties $\mathcal{K}$ and $\mathcal{F}_k$ for $k = 2$ are not closed.*

Concerning the $\partial\bar\partial$-lemma, a similar argument as above proves that it is an open property [7]. However, recently Angella and Kasuya have shown in [6] that it is not closed (see Remark 5.4 below). The upper-semicontinuity of the Hodge numbers also implies that the degeneration of the Frölicher spectral sequence at E_1 is an open property. Eastwood and Singer proved in [17] that this property is not closed by constructing a holomorphic family where all the fibres are twistor spaces. The analytic family $\mathcal{X}$ mentioned above provides another example, based on the complex geometry of nilmanifolds, showing the non-closedness of the property of degeneration of the Frölicher sequence at E_1 (see [10]).

The situation about openness and closedness of metric properties is as follows. Kodaira and Spencer [24] proved that the Kähler property is open, and Hironaka showed in [23] that in complex dimension ≥ 3 the Kähler property is not closed. Notice that, since a compact complex surface is Kähler if and only if its first Betti number is even, the Kähler property is closed in complex dimension 2.

Alessandrini and Bassanelli proved in [2] that the balanced property is not open. In contrast to the balanced case, Popovici has shown in [32] that the sG property is always open under holomorphic deformations, and conjectured in [34, Conjectures 1.21 and 1.23] that both the sG and the balanced properties are closed under holomorphic deformation. However, the family $\mathcal{X} = (M, J_t)_{t\in\Delta}$ satisfies that (M, J_t) has a balanced metric for all $t \in \Delta\backslash\{0\}$, but (M, J_0) does not admit any sG metric. In particular:

Theorem 3.2. *[10] The balanced and the sG properties are not closed.*

It follows from Theorem 2.4 and the upper-semicontinuity of the Hodge number $h^{0,1}_{\bar{\partial}}$ that the sGG property is open. Nevertheless, the analytic family $\mathcal{X}$ allows to show that the sGG property is not closed. Hence:

Theorem 3.3. *[36] The sGG property is open, but not closed.*

On the other hand, the existence of an sG metric in the central limit of a holomorphic deformation is guaranteed under the strong condition of the $\partial\bar{\partial}$-lemma. Concretely:

Proposition 3.4. *[34] Let $(M, J_t)_{t\in\Delta}$ be an analytic family of compact complex manifolds. If the $\partial\bar{\partial}$-lemma holds on (M, J_t) for every $t \in \Delta\backslash\{0\}$, then the central limit (M, J_0) has an sG metric.*

An interesting problem is if the conclusion in the previous proposition holds under weaker conditions than the $\partial\bar{\partial}$-lemma. The following result, which is a direct consequence of the properties of the family $\mathcal{X}$ mentioned above, shows that it is not true under the weaker property $\mathcal{K}$ or $\mathcal{F}_k$ for some particular k. Moreover:

Proposition 3.5. *There exists a holomorphic family of compact complex manifolds $\mathcal{X} = (M, J_t)_{t\in\Delta}$ of complex dimension 3, such that (M, J_t) satisfies the properties $\mathcal{F}_2$ and $\mathcal{K}$, admits balanced metric, is sGG and has degenerate Frölicher sequence for each $t \in \Delta\backslash\{0\}$, but (M, J_0) does not admit sG metrics.*

The result by Alessandrini and Bassanelli mentioned above about the non-openness of the balanced property is based on a holomorphic deformation of the Iwasawa manifold, which is a particular example of a complex manifold belonging to the class of 6-dimensional nilmanifolds endowed with an invariant complex structure. The analytic family $\mathcal{X}$ proving Theorems 3.1 and 3.2, the non-closedness of the sGG property in Theorem 3.3, and Proposition 3.5 is constructed by deforming appropriately an abelian complex structure J_0 on a 6-dimensional nilmanifold. In the next section we review the main results on the invariant complex geometry of 6-dimensional nilmanifolds.

4 Complex geometry of nilmanifolds

In this section we focus on the complex geometry of nilmanifolds and their interesting properties in relation to the problems considered in Sections 2 and 3. Notice that the problems addressed in those sections are nontrivial only for $n \geq 3$; in fact, for compact complex manifolds of complex dimension 2 the Frölicher spectral sequence always degenerate at E_1, the balanced condition is the same as the Kähler condition, and the existence of a Kähler metric is equivalent to the first Betti number be even. Therefore, we will mainly focus on complex nilmanifolds of (real) dimension 6.

In what follows, M will denote a nilmanifold of (real) dimension $2n$ and J an invariant complex structure on M, i.e. $M = \Gamma\backslash G$ is a compact quotient of a simply-connected nilpotent Lie group G by a lattice Γ of maximal rank in G, and J stems naturally from a "complex" structure J on the Lie algebra $\mathfrak{g}$ of G.

A crucial result in the theory of nilmanifolds is Nomizu's theorem [31] which asserts that the de Rham cohomology of M is canonically isomorphic to the cohomology of its underlying Lie algebra $\mathfrak{g}$, i.e. $H^k_{\mathrm{dR}}(M) \cong H^k_{\mathrm{dR}}(\mathfrak{g})$. Using this result, Hasegawa [22] proved that the Chevalley-Eilenberg complex $(\bigwedge^*(\mathfrak{g}^*), d)$ of $\mathfrak{g}$ provides a minimal model of M and that it is formal if and only if the Lie algebra is abelian, that is to say, the nilmanifold M is a torus. Therefore, by [16] a complex nilmanifold never satisfies the $\partial\bar{\partial}$-lemma, unless it is a complex torus.

Concerning a Nomizu type result for the Dolbeault cohomology of (M, J), several advances have been obtained under additional conditions on the invariant complex structure J. First, we recall that Salamon gave in [40] a characterization of the invariant complex structures as those endomorphisms $J\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $J^2 = -\mathrm{Id}$ for which there exists a basis $\{\omega^j\}_{j=1}^n$ of the i-eigenspace $\mathfrak{g}^{1,0}$ of the extension of J to $\mathfrak{g}^*_{\mathbb{C}} = \mathfrak{g}^* \otimes_{\mathbb{R}} \mathbb{C}$ satisfying

$$d\omega^1 = 0, \qquad d\omega^j \in \mathcal{I}(\omega^1, \ldots, \omega^{j-1}), \qquad \text{for } j = 2, \ldots, n,$$

where $\mathcal{I}(\omega^1, \ldots, \omega^{j-1})$ is the ideal in $\bigwedge^* \mathfrak{g}^*_{\mathbb{C}}$ generated by $\{\omega^1, \ldots, \omega^{j-1}\}$.

A generic invariant complex structure J satisfies $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{2,0}(\mathfrak{g}^*) \oplus \bigwedge^{1,1}(\mathfrak{g}^*)$ with respect to the bigraduation induced by J on the exterior algebra $\bigwedge^* \mathfrak{g}^*_{\mathbb{C}}$. When J is *abelian* [3] the Lie algebra differential d satisfies $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{1,1}(\mathfrak{g}^*)$, a condition which is equivalent to the complex subalgebra $\mathfrak{g}_{1,0} = (\mathfrak{g}^{1,0})^*$ being abelian. On the other hand, the complex structures associated to complex Lie algebras satisfy $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{2,0}(\mathfrak{g}^*)$ and we will refer to them as *complex-parallelizable* structures. Both abelian and complex-parallelizable structures are particular classes of *nilpotent* complex structures, introduced and studied in [15], for which there is a basis $\{\omega^j\}_{j=1}^n$ for $\mathfrak{g}^{1,0}$ satisfying

$$d\omega^1 = 0, \qquad d\omega^j \in \bigwedge\nolimits^2 \langle \omega^1, \ldots, \omega^{j-1}, \omega^{\bar{1}}, \ldots, \omega^{\overline{j-1}} \rangle, \qquad \text{for } j = 2, \ldots, n,$$

where $\omega^{\bar{i}}$ stands for $\overline{\omega^i}$.

When J is complex-parallelizable, Sakane proved in [39] that the natural inclusion

$$\left(\bigwedge\nolimits^{p,q}(\mathfrak{g}^*), \bar{\partial}\right) \hookrightarrow (\Omega^{p,q}(M), \bar{\partial}) \tag{3}$$

induces an isomorphism

$$\iota\colon H^{p,q}_{\bar\partial}(\mathfrak{g}, J) \longrightarrow H^{p,q}_{\bar\partial}(M, J) \tag{4}$$

between the Lie-algebra Dolbeault cohomology of $(\mathfrak{g}, J)$ and the Dolbeault cohomology of (M, J). More general conditions under which the inclusion (3) induces an isomorphism (4) can be found in [11, 15, 38]; in particular, it is always true for abelian complex structures on nilmanifolds. We will discuss below the 6-dimensional case in detail.

Concerning the calculation of the Bott-Chern cohomology and the Frölicher spectral sequence of nilmanifolds with invariant complex structure, one has the following Nomizu type result. The first part is proved by Angella in [4, Theorem 2.8] and the second part follows from an inductive argument given in [14, Theorem 4.2].

Theorem 4.1. *If the natural inclusion* (3) *induces an isomorphism* (4) *between the Lie-algebra Dolbeault cohomology of* $(\mathfrak{g}, J)$ *and the Dolbeault cohomology of the complex nilmanifold* (M, J)*, then:*

(*i*) *the natural map*
$$\iota\colon H^{p,q}_{\mathrm{BC}}(\mathfrak{g}, J) \longrightarrow H^{p,q}_{\mathrm{BC}}(M, J)$$
between the Lie-algebra Bott-Chern cohomology of $(\mathfrak{g}, J)$ *and the Bott-Chern cohomology of* (M, J) *is also an isomorphism for any* $0 \le p, q \le n$*;*

(*ii*) *the natural map*
$$\iota\colon E^{p,q}_r(\mathfrak{g}, J) \longrightarrow E^{p,q}_r(M, J)$$
between the term in the Lie-algebra Frölicher sequence of $(\mathfrak{g}, J)$ *and the term in the Frölicher spectral sequence of* (M, J) *is also an isomorphism for any* $0 \le p, q \le n$.

The only 4-dimensional nilmanifolds having invariant complex structures are the torus $\mathbb{T}^4$ and the Kodaira-Thurston manifold [43]. The latter was the first known example of a compact symplectic manifold not admitting Kähler metric. In six dimensions, the nilmanifolds admitting invariant complex structures are classified through their underlying Lie algebras. The following result provides a classification of such nilmanifolds in terms of the different types of complex structures that they admit.

Theorem 4.2. *[40, 44] A nilmanifold* M *of (real) dimension* 6 *has an invariant complex structure if and only if its underlying Lie algebra is isomorphic to one in the following* *list:*

$$\begin{array}{llll}
\mathfrak{h}_1 = (0,0,0,0,0,0), & & \mathfrak{h}_{10} = (0,0,0,12,13,14), \\
\mathfrak{h}_2 = (0,0,0,0,12,34), & & \mathfrak{h}_{11} = (0,0,0,12,13,14+23), \\
\mathfrak{h}_3 = (0,0,0,0,0,12+34), & & \mathfrak{h}_{12} = (0,0,0,12,13,24), \\
\mathfrak{h}_4 = (0,0,0,0,12,14+23), & & \mathfrak{h}_{13} = (0,0,0,12,13+14,24), \\
\mathfrak{h}_5 = (0,0,0,0,13+42,14+23), & & \mathfrak{h}_{14} = (0,0,0,12,14,13+42), \\
\mathfrak{h}_6 = (0,0,0,0,12,13), & & \mathfrak{h}_{15} = (0,0,0,12,13+42,14+23), \\
\mathfrak{h}_7 = (0,0,0,12,13,23), & & \mathfrak{h}_{16} = (0,0,0,12,14,24), \\
\mathfrak{h}_8 = (0,0,0,0,0,12), & & \mathfrak{h}_{19}^- = (0,0,0,12,23,14-35), \\
\mathfrak{h}_9 = (0,0,0,0,12,14+25), & & \mathfrak{h}_{26}^+ = (0,0,12,13,23,14+25).
\end{array}$$

Moreover:

(*a*) *For* $\mathfrak{h}_{19}^-$ *and* $\mathfrak{h}_{26}^+$*, any complex structure is non-nilpotent;*

(*b*) *For* $\mathfrak{h}_k$*,* $1 \leq k \leq 16$*, any complex structure is nilpotent;*

(*c*) *For* $\mathfrak{h}_1$*,* $\mathfrak{h}_3$*,* $\mathfrak{h}_8$ *and* $\mathfrak{h}_9$*, any complex structure is abelian;*

(*d*) *For* $\mathfrak{h}_2$*,* $\mathfrak{h}_4$*,* $\mathfrak{h}_5$ *and* $\mathfrak{h}_{15}$*, there exist both abelian and non-abelian nilpotent complex structures;*

(*e*) *For* $\mathfrak{h}_6$*,* $\mathfrak{h}_7$*,* $\mathfrak{h}_{10}$*,* $\mathfrak{h}_{11}$*,* $\mathfrak{h}_{12}$*,* $\mathfrak{h}_{13}$*,* $\mathfrak{h}_{14}$ *and* $\mathfrak{h}_{16}$*, any complex structure is not abelian.*

Here, for instance, the notation $\mathfrak{h}_2 = (0,0,0,0,12,34)$ means that there exists a basis $\{e^i\}_{i=1}^6$ of the dual of the Lie algebra (or equivalently, a basis of invariant real 1-forms on the nilmanifold) such that $de^1 = de^2 = de^3 = de^4 = 0$, $de^5 = e^1 \wedge e^2$ and $de^6 = e^3 \wedge e^4$. Notice that $\mathfrak{h}_2$ is isomorphic to the product of two copies of the 3-dimensional real Heisenberg algebra $(0,0,12)$ (see Section 5 for more details).

By Theorem 4.2, if a 6-dimensional nilmanifold M admits invariant complex structures then all of them are either nilpotent or non-nilpotent. This special property does not hold in higher dimensions [15].

An interesting problem is to obtain a description of the moduli space of invariant complex structures on each nilmanifold. Andrada, Barberis and Dotti classified in [3] the abelian complex structures in dimension 6, whereas the classification of the non-nilpotent complex structures was given in [45] and the general classification was obtained recently in [10]. Let J and J' be two invariant complex structures on a nilmanifold M with underlying Lie algebra $\mathfrak{g}$. Recall that J and J' are said to be *equivalent* if there is an automorphism $F\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ of the Lie algebra such that $J' = F^{-1} \circ J \circ F$. Now, if $\mathfrak{g}_J^{1,0}$ and $\mathfrak{g}_{J'}^{1,0}$ denote the $(1,0)$-subspaces of $\mathfrak{g}_{\mathbb{C}}^*$ associated to J and J', respectively, then the complex structures J and J' are equivalent if and only if there exists a $\mathbb{C}$-linear isomorphism $F^*\colon \mathfrak{g}_J^{1,0} \longrightarrow \mathfrak{g}_{J'}^{1,0}$ such that $d \circ F^* = F^* \circ d$.

It is well known that, up to equivalence, there are only two complex-parallelizable structures, which are defined by the complex equations

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \rho\,\omega^{12},$$

where $\rho = 0$ or 1. The corresponding Lie algebras are $\mathfrak{h}_1$ when $\rho = 0$, and hence the complex nilmanifold is a complex torus $\mathbb{T}^3_{\mathbb{C}}$, and $\mathfrak{h}_5$ when $\rho = 1$, and the complex nilmanifold is the Iwasawa manifold given by a quotient of the *complex* Heisenberg group.

According to [10], the remaining complex structures in dimension 6 can be parametrized by the following three families of complex equations:

Family I: $d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \rho\,\omega^{12} + \omega^{1\bar{1}} + \lambda\,\omega^{1\bar{2}} + D\,\omega^{2\bar{2}},$

where $\rho \in \{0, 1\}$, $\lambda \in \mathbb{R}^{\geq 0}$ and $D \in \mathbb{C}$ with $\mathfrak{Im}\, D \geq 0$. The complex structure is abelian if and only if $\rho = 0$. The Lie algebras admitting complex structures in this family are $\mathfrak{h}_2, \ldots, \mathfrak{h}_6$ and $\mathfrak{h}_8$.

Family II: $d\omega^1 = 0, \quad d\omega^2 = \omega^{1\bar{1}}, \quad d\omega^3 = \rho\,\omega^{12} + B\,\omega^{1\bar{2}} + c\,\omega^{2\bar{1}},$

where $\rho \in \{0, 1\}$, $B \in \mathbb{C}$ and $c \in \mathbb{R}^{\geq 0}$, with $(\rho, B, c) \neq (0, 0, 0)$. The complex structure is abelian if and only if $\rho = 0$, and the Lie algebras admitting complex structures in this family are $\mathfrak{h}_7$ and $\mathfrak{h}_9, \ldots, \mathfrak{h}_{16}$.

Family III: $d\omega^1 = 0, \quad d\omega^2 = \omega^{13} + \omega^{1\bar{3}}, \quad d\omega^3 = \varepsilon\, i\,\omega^{1\bar{1}} \pm i(\omega^{1\bar{2}} - \omega^{2\bar{1}}),$

where $\varepsilon = 0$ or 1. The complex structures are non-nilpotent, and the Lie algebras are $\mathfrak{h}_{19}^-$ (for $\varepsilon = 0$) and $\mathfrak{h}_{26}^+$ (for $\varepsilon = 1$).

Tables 1, 2 and 3 below contain the general classification of invariant complex structures on 6-dimensional nilmanifolds in terms of its underlying Lie algebra and the values of the coefficients $(\rho, \lambda, D = x + iy)$ for Family I, (ρ, B, c) for Family II, and ε for Family III. Different values of the parameters in Table 1, resp. Tables 2 and 3, correspond to non-equivalent complex structures in Family I, resp. Families II and III (see [10] for more details).

Theorem 4.1 above asserts that if the natural isomorphism (4) holds then, in addition to the Dolbeault cohomology of $(M = \Gamma\backslash G, J)$, we also know other complex invariants as the Bott-Chern cohomology and the terms in the Frölicher spectral sequence. And moreover, such complex invariants can be obtained directly from the underlying Lie algebra $\mathfrak{g}$ together with the structure J.

In dimension 4 the natural isomorphism (4) holds for any invariant complex structure J. In dimension 6, Rollenske proved in [38, Section 4.2] that if $\mathfrak{g} \not\cong \mathfrak{h}_7$ then the natural inclusion (3) induces an isomorphism (4) between the Lie-algebra Dolbeault cohomology of $(\mathfrak{g}, J)$ and the Dolbeault cohomology of M.

Remark 4.3. Let $(M = \Gamma\backslash G, J)$ be a 6-dimensional nilmanifold endowed with an invariant complex structure J such that $\mathfrak{g} = \mathfrak{h}_7$. In [38, Theorem 4.4] it is proved that there is a dense subset of the space of all invariant complex structures for which the complex nilmanifold has the structure of a principal holomorphic bundle of elliptic curves over a Kodaira surface, but this does not hold for all complex structures. In fact, the invariant complex structure J may not be compatible with the lattice Γ, as [38, Example 1.14] shows, and hence, one cannot ensure the existence of the natural isomorphism (4) for any invariant complex structure on the nilmanifold.

Concerning the Bott-Chern cohomology of nilmanifolds, in [41] Schweitzer computed it for the Iwasawa manifold and in [4] Angella calculated the Bott-Chern cohomology groups of its small deformations. Notice that by [37, Theorem 2.6], if such deformations are sufficiently small then they are again invariant complex structures. Thus, the Bott-Chern cohomology determined in [5] and [26] for any pair $(\mathfrak{g}, J)$ covers that of any invariant complex structure and its sufficiently small deformations on any 6-dimensional nilmanifold with underlying Lie algebra not isomorphic to $\mathfrak{h}_7$, accordingly to Remark 4.3 and Theorem 4.1 (i).

In Tables 1, 2 and 3 we include the Bott-Chern numbers $h^{p,q}_{\mathrm{BC}}(\mathfrak{g}, J)$ for any J in the Families I, II and III above (see [26] for an explicit description of the generators of the Bott-Chern cohomology groups in terms of the complex equations in Families I, II and III). Therefore, the tables cover all the invariant complex geometry of 6-dimensional nilmanifolds, except for the complex torus and the Iwasawa manifold, which are well known and already given in [41] as we reminded above.

It is clear that in all cases $H^{3,0}_{\mathrm{BC}} = \langle[\omega^{123}]\rangle$ and $H^{3,3}_{\mathrm{BC}} = \langle[\omega^{123\bar{1}\bar{2}\bar{3}}]\rangle$, so $h^{3,0}_{\mathrm{BC}} = h^{3,3}_{\mathrm{BC}} = 1$. Notice that by the duality in the Bott-Chern cohomology it suffices to show the dimensions $h^{p,q}_{\mathrm{BC}}$ for $(p, q) = (1, 0), (2, 0), (1, 1), (2, 1), (2, 2), (3, 1)$ and $(3, 2)$. In fact, the dimension of any other Bott-Chern cohomology group is obtained by $h^{q,p}_{\mathrm{BC}} = h^{p,q}_{\mathrm{BC}}$.

<table>
<tr><td></td><td colspan="5">Family I</td><td colspan="7">Bott-Chern numbers</td></tr>
<tr><td>$\mathfrak{g}$</td><td>ρ</td><td>λ</td><td colspan="2">$D = x + iy$</td><td>$h^{1,0}_{\mathrm{BC}}$</td><td>$h^{2,0}_{\mathrm{BC}}$</td><td>$h^{1,1}_{\mathrm{BC}}$</td><td>$h^{2,1}_{\mathrm{BC}}$</td><td>$h^{3,1}_{\mathrm{BC}}$</td><td>$h^{2,2}_{\mathrm{BC}}$</td><td>$h^{3,2}_{\mathrm{BC}}$</td></tr>
<tr><td rowspan="5">$\mathfrak{h}_2$</td><td rowspan="2">0</td><td rowspan="2">0</td><td rowspan="2">$y = 1$</td><td>$x = 0$</td><td rowspan="2">2</td><td rowspan="2">1</td><td rowspan="2">4</td><td rowspan="2">6</td><td rowspan="2">3</td><td>7</td><td rowspan="2">3</td></tr>
<tr><td>$x \neq 0$</td><td>6</td></tr>
<tr><td rowspan="3">1</td><td rowspan="3">1</td><td rowspan="3">$y > 0$</td><td>$x = -1 \pm \sqrt{1 - y^2}$</td><td rowspan="3">2</td><td rowspan="3">1</td><td>5</td><td rowspan="3">6</td><td rowspan="3">2</td><td rowspan="2">6</td><td rowspan="3">3</td></tr>
<tr><td>$x \neq -1 \pm \sqrt{1 - y^2}$
$x \neq 1$</td><td rowspan="2">4</td></tr>
<tr><td>$x = 1$</td><td>7</td></tr>
<tr><td>$\mathfrak{h}_3$</td><td>0</td><td>0</td><td colspan="2">± 1</td><td>2</td><td>1</td><td>4</td><td>6</td><td>3</td><td>7</td><td>3</td></tr>
<tr><td rowspan="4">$\mathfrak{h}_4$</td><td>0</td><td>1</td><td colspan="2">$\frac{1}{4}$</td><td>2</td><td>1</td><td>4</td><td>6</td><td>3</td><td>6</td><td>3</td></tr>
<tr><td rowspan="3">1</td><td rowspan="3">1</td><td colspan="2">-2</td><td rowspan="3">2</td><td rowspan="3">1</td><td>5</td><td rowspan="3">6</td><td rowspan="3">2</td><td rowspan="2">6</td><td rowspan="3">3</td></tr>
<tr><td colspan="2">$D \in \mathbb{R} - \{-2, 0, 1\}$</td><td rowspan="2">4</td></tr>
<tr><td colspan="2">1</td><td>7</td></tr>
<tr><td rowspan="16">$\mathfrak{h}_5$</td><td rowspan="2">0</td><td rowspan="2">1</td><td colspan="2">0</td><td rowspan="2">2</td><td>2</td><td>6</td><td rowspan="2">6</td><td rowspan="2">3</td><td rowspan="2">6</td><td rowspan="2">3</td></tr>
<tr><td colspan="2">$D \in \left(0, \frac{1}{4}\right)$</td><td>1</td><td>4</td></tr>
<tr><td rowspan="14">1</td><td rowspan="5">0</td><td rowspan="3">$y = 0$</td><td>$x = 0$</td><td rowspan="14">2</td><td>2</td><td rowspan="13">4</td><td rowspan="14">6</td><td rowspan="14">2</td><td>7</td><td rowspan="14">3</td></tr>
<tr><td>$x = \frac{1}{2}$</td><td rowspan="4">1</td><td>8</td></tr>
<tr><td>$x \neq 0, \frac{1}{2},\ \ x > -\frac{1}{4}$</td><td rowspan="2">7</td></tr>
<tr><td>$0 < y^2 < \frac{3}{4}$</td><td>$x = \frac{1}{2}$</td></tr>
<tr><td>$y > 0$</td><td>$x \neq \frac{1}{2},\ \ x > y^2 - \frac{1}{4}$</td><td rowspan="10">6</td></tr>
<tr><td rowspan="2">$0 < \lambda^2 < \frac{1}{2}$</td><td rowspan="2">$x = 0$</td><td>$y = 0$</td><td>2</td></tr>
<tr><td>$0 < y < \frac{\lambda^2}{2}$</td><td>1</td></tr>
<tr><td rowspan="2">$\frac{1}{2} \leq \lambda^2 < 1$</td><td rowspan="2">$x = 0$</td><td>$y = 0$</td><td>2</td></tr>
<tr><td>$0 < y < \frac{1-\lambda^2}{2}$</td><td>1</td></tr>
<tr><td rowspan="2">$1 < \lambda^2 \leq 5$</td><td rowspan="2">$x = 0$</td><td>$y = 0$</td><td>2</td></tr>
<tr><td>$0 < y < \frac{\lambda^2-1}{2}$</td><td>1</td></tr>
<tr><td rowspan="3">$\lambda^2 > 5$</td><td rowspan="3">$x = 0$</td><td>$y = 0$</td><td>2</td></tr>
<tr><td>$0 < y < \frac{\lambda^2-1}{2},\ y \neq \sqrt{\lambda^2 - 1}$</td><td rowspan="2">1</td></tr>
<tr><td>$0 < y < \frac{\lambda^2-1}{2},\ y = \sqrt{\lambda^2 - 1}$</td><td>5</td></tr>
<tr><td>$\mathfrak{h}_6$</td><td>1</td><td>1</td><td colspan="2">0</td><td>2</td><td>2</td><td>5</td><td>6</td><td>2</td><td>6</td><td>3</td></tr>
<tr><td>$\mathfrak{h}_8$</td><td>0</td><td>0</td><td colspan="2">0</td><td>2</td><td>2</td><td>6</td><td>7</td><td>3</td><td>8</td><td>3</td></tr>
</table>

Table 1.— Classification of complex structures in Family I and dimensions of their Bott-Chern cohomology groups.

<table>
<tr><th></th><th colspan="3">Family II</th><th colspan="7">Bott-Chern numbers</th></tr>
<tr><th>$\mathfrak{g}$</th><th>ρ</th><th>B</th><th>c</th><th>$h^{1,0}_{\rm BC}$</th><th>$h^{2,0}_{\rm BC}$</th><th>$h^{1,1}_{\rm BC}$</th><th>$h^{2,1}_{\rm BC}$</th><th>$h^{3,1}_{\rm BC}$</th><th>$h^{2,2}_{\rm BC}$</th><th>$h^{3,2}_{\rm BC}$</th></tr>
<tr><td>$\mathfrak{h}_7$</td><td>1</td><td>1</td><td>0</td><td>1</td><td>2</td><td>5</td><td>6</td><td>2</td><td>5</td><td>3</td></tr>
<tr><td>$\mathfrak{h}_9$</td><td>0</td><td>1</td><td>1</td><td>1</td><td>1</td><td>4</td><td>5</td><td>3</td><td>6</td><td>3</td></tr>
<tr><td>$\mathfrak{h}_{10}$</td><td>1</td><td>0</td><td>1</td><td>1</td><td>1</td><td>4</td><td>5</td><td>2</td><td>5</td><td>3</td></tr>
<tr><td rowspan="2">$\mathfrak{h}_{11}$</td><td rowspan="2">1</td><td>$B \in \mathbb{R} - \{0, \frac{1}{2}, 1\}$</td><td>$|B-1|$</td><td rowspan="2">1</td><td rowspan="2">1</td><td rowspan="2">4</td><td rowspan="2">5</td><td rowspan="2">2</td><td>5</td><td rowspan="2">3</td></tr>
<tr><td>$\frac{1}{2}$</td><td>$\frac{1}{2}$</td><td>6</td></tr>
<tr><td rowspan="2">$\mathfrak{h}_{12}$</td><td rowspan="2">1</td><td>$\mathfrak{Re}B \neq \frac{1}{2},\ \mathfrak{Im}B \neq 0$</td><td rowspan="2">$|B-1|$</td><td rowspan="2">1</td><td rowspan="2">1</td><td rowspan="2">4</td><td rowspan="2">5</td><td rowspan="2">2</td><td>5</td><td rowspan="2">3</td></tr>
<tr><td>$\mathfrak{Re}B = \frac{1}{2},\ \mathfrak{Im}B \neq 0$</td><td>6</td></tr>
<tr><td rowspan="4">$\mathfrak{h}_{13}$</td><td rowspan="4">1</td><td rowspan="2">1</td><td>$0 < c < 2,\ c \neq 1$</td><td rowspan="4">1</td><td rowspan="4">1</td><td rowspan="2">5</td><td rowspan="4">5</td><td rowspan="4">2</td><td>5</td><td rowspan="4">3</td></tr>
<tr><td>1</td><td>6</td></tr>
<tr><td colspan="2">$B \neq 1,\ c \neq |B|, |B-1|,$
$(c, |B|) \neq (0, 1),$
$c^4 - 2(|B|^2+1)c^2 + (|B|^2-1)^2 < 0$</td><td rowspan="2">4</td><td>5</td></tr>
<tr><td colspan="2">$B \neq 1,\ c = |B| > \frac{1}{2},$
$|B| \neq |B-1|$</td><td>6</td></tr>
<tr><td rowspan="3">$\mathfrak{h}_{14}$</td><td rowspan="3">1</td><td>1</td><td>2</td><td rowspan="3">1</td><td rowspan="3">1</td><td>5</td><td rowspan="3">5</td><td rowspan="3">2</td><td>5</td><td rowspan="3">3</td></tr>
<tr><td>$|B| = \frac{1}{2}$</td><td>$\frac{1}{2}$</td><td rowspan="2">4</td><td>6</td></tr>
<tr><td colspan="2">$c \neq |B-1|,$
$(c, |B|) \neq (0, 1),\ (\frac{1}{2}, \frac{1}{2}),\ (2, 1),$
$c^4 - 2(|B|^2+1)c^2 + (|B|^2-1)^2 = 0$</td><td>5</td></tr>
<tr><td rowspan="8">$\mathfrak{h}_{15}$</td><td rowspan="3">0</td><td>0</td><td>1</td><td rowspan="3">1</td><td rowspan="2">1</td><td>5</td><td rowspan="3">5</td><td rowspan="3">3</td><td rowspan="3">5</td><td rowspan="3">3</td></tr>
<tr><td rowspan="2">1</td><td>$c \neq 0,\ 1$</td><td rowspan="2">4</td></tr>
<tr><td>0</td><td>2</td></tr>
<tr><td rowspan="5">1</td><td>0</td><td>0</td><td rowspan="5">1</td><td rowspan="2">2</td><td rowspan="2">4</td><td rowspan="5">5</td><td rowspan="5">2</td><td>7</td><td rowspan="5">3</td></tr>
<tr><td>$|B| \neq 0, 1$</td><td>0</td><td rowspan="2">5</td></tr>
<tr><td>1</td><td>$c > 2$</td><td rowspan="3">1</td><td>5</td></tr>
<tr><td>$|B| = c$</td><td>$0 < c < \frac{1}{2}$</td><td rowspan="2">4</td><td>6</td></tr>
<tr><td colspan="2">$c \neq 0,\ |B-1|,$
$B \neq 1,\ |B| \neq c,$
$c^4 - 2(|B|^2+1)c^2 + (|B|^2-1)^2 > 0$</td><td>5</td></tr>
<tr><td>$\mathfrak{h}_{16}$</td><td>1</td><td>$|B| = 1,\ B \neq 1$</td><td>0</td><td>1</td><td>2</td><td>4</td><td>5</td><td>2</td><td>5</td><td>3</td></tr>
</table>

Table 2.— Classification of complex structures in Family II and dimensions of their Bott-Chern cohomology groups.

The next result was proved in [10] and shows the general behaviour of the Frölicher spectral sequence in dimension 6. For that we applied Theorem 4.1 (ii) to ensure that $E_r^{p,q}(M,J) \cong E_r^{p,q}(\mathfrak{g},J)$ for any p,q and any $r \geq 1$, whenever $\mathfrak{g} \ncong \mathfrak{h}_7$.

Theorem 4.4. *[10] Let $M = \Gamma\backslash G$ be a 6-dimensional nilmanifold endowed with an invariant complex structure J such that the underlying Lie algebra $\mathfrak{g} \ncong \mathfrak{h}_7$. Then the Frölicher spectral sequence $\{E_r(M,J)\}_{r\geq 1}$ behaves as follows:*

- *(a) If $\mathfrak{g} \cong \mathfrak{h}_1$, $\mathfrak{h}_3$, $\mathfrak{h}_6$, $\mathfrak{h}_8$, $\mathfrak{h}_9$, $\mathfrak{h}_{10}$, $\mathfrak{h}_{11}$, $\mathfrak{h}_{12}$ or $\mathfrak{h}_{19}^-$, then $E_1(M,J) \cong E_\infty(M,J)$ for any J.*
- *(b) If $\mathfrak{g} \cong \mathfrak{h}_2$ or $\mathfrak{h}_4$, then $E_1(M,J) \cong E_\infty(M,J)$ if and only if J is non-abelian; moreover, any abelian complex structure satisfies $E_1(M,J) \ncong E_2(M,J) \cong E_\infty(M,J)$.*
- *(c) If $\mathfrak{g} \cong \mathfrak{h}_5$, then:*

	Family III	Bott-Chern numbers						
$\mathfrak{g}$	ε	$h^{1,0}_{\mathrm{BC}}$	$h^{2,0}_{\mathrm{BC}}$	$h^{1,1}_{\mathrm{BC}}$	$h^{2,1}_{\mathrm{BC}}$	$h^{3,1}_{\mathrm{BC}}$	$h^{2,2}_{\mathrm{BC}}$	$h^{3,2}_{\mathrm{BC}}$
$\mathfrak{h}_{19}^-$	0	1	1	2	3	2	4	2
$\mathfrak{h}_{26}^+$	1	1	1	2	3	2	3	2

Table 3.— Classification of complex structures in Family III and dimensions of their Bott-Chern cohomology groups.

(c.1) $E_1(M,J) \ncong E_2(M,J) \cong E_\infty(M,J)$ *for the complex-parallelizable structure* J*;*

(c.2) $E_1(M,J) \cong E_\infty(M,J)$ *if and only if* J *is not complex-parallelizable and* $\rho D \neq 0$ *in Table* 1*; moreover,* $E_1(M,J) \ncong E_2(M,J) \cong E_\infty(M,J)$ *when* $\rho D = 0$.

(d) If $\mathfrak{g} \cong \mathfrak{h}_{16}$ *or* $\mathfrak{h}_{26}^+$*, then* $E_1(M,J) \ncong E_2(M,J) \cong E_\infty(M,J)$ *for any* J.

(e) If $\mathfrak{g} \cong \mathfrak{h}_{13}$ *or* $\mathfrak{h}_{14}$*, then* $E_1(M,J) \cong E_2(M,J) \ncong E_3(M,J) \cong E_\infty(M,J)$ *for any* J.

(f) If $\mathfrak{g} \cong \mathfrak{h}_{15}$ *and* J *is a complex structure on* $\mathfrak{h}_{15}$ *given in Table* 2*, then:*

(f.1) $E_1(M,J) \ncong E_2(M,J) \cong E_\infty(M,J)$*, when* $c = 0$ *and* $|B - \rho| \neq 0$*;*

(f.2) $E_1(M,J) \cong E_2(M,J) \ncong E_3(M,J) \cong E_\infty(M,J)$*, when* $\rho = 1$ *and* $|B-1| \neq c \neq 0$*;*

(f.3) $E_1(M,J) \ncong E_2(M,J) \ncong E_3(M,J) \cong E_\infty(M,J)$*, when* $\rho = 0$ *and* $|B| \neq c \neq 0$.

According to Remark 4.3, we cannot ensure the existence of a canonical isomorphism between $E_r^{p,q}(\mathfrak{g},J)$ and $E_r^{p,q}(M,J)$ for any invariant complex structure J on a nilmanifold M with underlying Lie algebra $\mathfrak{g} \cong \mathfrak{h}_7$. However, it is worth noticing that up to equivalence there is only one complex structure J on $\mathfrak{h}_7$ whose sequence degenerates at the first step, that is, $E_1(\mathfrak{h}_7,J) \cong E_\infty(\mathfrak{h}_7,J)$.

Concerning the existence of balanced or sG metrics, if $(M = \Gamma\backslash G, J)$ is a nilmanifold endowed with an invariant complex structure, then it admits a balanced metric if and only if it has an invariant one. Thus, the existence of such a metric can be detected at the level of the underlying Lie algebra. This fact was proved in [18] and uses the so called *symmetrization process*. Moreover, in [10] it is proved that this process can also be applied to the existence of sG metrics on nilmanifolds, that is to say, $(M = \Gamma\backslash G, J)$ has an sG metric if and only if it has an invariant one. In dimension 6 we have:

Theorem 4.5. *[10, 44] Let* $M = \Gamma\backslash G$ *be a* 6*-dimensional nilmanifold admitting invariant complex structures* J*, and let* $\mathfrak{g}$ *be the underlying Lie algebra. Then:*

(*i*) *There exists J having balanced or sG metrics if and only if $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_1, \ldots, \mathfrak{h}_6$ or $\mathfrak{h}_{19}^-$.*

(*ii*) *If the complex structure J is abelian, then an invariant Hermitian metric is sG if and only if it is balanced.*

(*iii*) *For $\mathfrak{g} \cong \mathfrak{h}_2, \mathfrak{h}_4, \mathfrak{h}_5$ or $\mathfrak{h}_6$, if the complex structure J is non-abelian then any invariant Hermitian metric is sG.*

The following result implies that abelian complex structures on nilmanifolds with underlying Lie algebra isomorphic to $\mathfrak{h}_2$ or $\mathfrak{h}_4$ do not admit sG metrics.

Proposition 4.6. *[46] Let $M = \Gamma\backslash G$ be a 6-dimensional nilmanifold with an abelian complex structure J admitting an sG metric. Then the underlying Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_3$ or $\mathfrak{h}_5$.*

There exist compact complex manifolds having sG metrics but not admitting any balanced metric [34, Theorem 1.8]. The general situation for nilmanifolds in dimension 6 is as follows:

Proposition 4.7. *[10] Let $M = \Gamma\backslash G$ be a 6-dimensional nilmanifold with an invariant complex structure J such that $(M = \Gamma\backslash G, J)$ does not admit balanced metrics. If $(M = \Gamma\backslash G, J)$ has sG metric, then J is non-abelian nilpotent and $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_2$, $\mathfrak{h}_4$ or $\mathfrak{h}_5$. Moreover, according to the classification in Table 1, such a J is given by $\rho = 1$ and: $\lambda = 1, x + y^2 \geq \frac{1}{4}$ on $\mathfrak{h}_2$; $\lambda = 1, x \geq \frac{1}{4}$ on $\mathfrak{h}_4$; and $\lambda = 0, y \neq 0$ or $\lambda = y = 0, x \geq 0$ on $\mathfrak{h}_5$.*

Although by Theorem 4.5 (*i*) the underlying Lie algebras are the same both in the balanced and the sG case, Proposition 4.7 shows that the complex structures admitting such metrics differ.

We finish this section by considering those complex nilmanifolds $(M = \Gamma\backslash G, J)$ having the sGG property, that is, any Gauduchon metric is sG. By Theorem 2.4 the sGG condition is equivalent to $b_1(M) = 2\, h^{0,1}_{\bar\partial}(M, J)$. For instance, let us consider an invariant complex structure J in the Family I. If J is abelian then $\rho = 0$ and

$$H^{0,1}_{\bar\partial}(M, J) = \langle [\omega^{\bar 1}], [\omega^{\bar 2}], [\omega^{\bar 3}] \rangle,$$

whereas for a non-abelian J, i.e. $\rho = 1$, we have

$$H^{0,1}_{\bar\partial}(M, J) = \langle [\omega^{\bar 1}], [\omega^{\bar 2}] \rangle.$$

If M is not a torus then its first Betti number satisfies $b_1(M) \leq 5$, which implies that $(M = \Gamma\backslash G, J)$ in Family I cannot be sGG when J is abelian.

From a more detailed analysis of the non-abelian complex structures in Family I, together with the study of complex nilmanifolds in Families II and III, it follows:

Theorem 4.8. *[36] Let $M = \Gamma\backslash G$ be a 6-dimensional nilmanifold, not a torus, endowed with an invariant complex structure J. Then, (M, J) is sGG if and only if the Lie algebra underlying M is isomorphic to $\mathfrak{h}_2$, $\mathfrak{h}_4$, $\mathfrak{h}_5$ or $\mathfrak{h}_6$, and the complex structure J is not abelian.*

The complex geometry of 6-dimensional nilmanifolds allows to conclude that, apart from the obvious implications, there are no relations among the different properties of compact complex manifolds introduced in Section 2 (see [10] and [36] for more details):

- there exist sG manifolds that are not balanced;
- there are sG manifolds that are not sGG;
- the balanced property and the sGG property are unrelated;
- the degeneration of the Frölicher spectral sequence at E_1 and the sGG property are also unrelated.

In particular, the sGG class introduced and studied in [36] is a new class of compact complex manifolds.

5 On the complex geometry of the product of two Heisenberg manifolds

In this section we consider the product of two (3-dimensional real) Heisenberg (nil)manifolds, and we show that, despite the simplicity of this 6-dimensional nilmanifold, its complex geometry is surprisingly rich in relation to the deformation problems considered in Section 3.

Let us recall that the Heisenberg group H is the nilpotent Lie group constituted by all the matrices of the form

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}. \tag{5}$$

Since $\{\alpha^1 = dx, \alpha^2 = dy, \alpha^3 = xdy - dz\}$ is a basis of left-invariant 1-forms on H, the structure equations are given by $d\alpha^1 = d\alpha^2 = 0$, $d\alpha^3 = \alpha^{12}$, thus the Lie algebra of H is $\mathfrak{h} = (0, 0, 12)$. Let us consider the lattice Γ given by the matrices in (5) with (x, y, z)-entries lying in $\mathbb{Z}$. Hence, Γ is a lattice of maximal rank in H. From now on, we will denote by N the 3-dimensional nilmanifold $N = \Gamma\backslash H$ and we will refer to N as the Heisenberg nilmanifold.

Let us take another copy of N with basis of 1-forms $\{\beta^1, \beta^2, \beta^3\}$ satisfying $d\beta^1 = d\beta^2 = 0$ and $d\beta^3 = \beta^{12}$. Then, the Lie algebra underlying the 6-dimensional nilmanifold $N \times N$ is isomorphic to $\mathfrak{h} \oplus \mathfrak{h}$, i.e. to the Lie algebra $\mathfrak{h}_2 = (0, 0, 0, 0, 12, 34)$. On the product manifold $N \times N$ we consider the almost-complex structure J_0 defined by

$$J_0(\alpha^1) = -\alpha^2, \quad J_0(\beta^1) = -\beta^2, \quad J_0(\alpha^3) = -\beta^3. \tag{6}$$

It is easy to check that J_0 is integrable, i.e. its Nijehuis tensor vanishes identically, and abelian. Therefore, J_0 defines an invariant abelian complex structure on $N \times N$. The aim of this section is to show that the holomorphic deformations of this simple complex structure have very interesting properties in relation to the existence problem of sG metric, the invariants $\mathbf{f}_2, \mathbf{k}_1$ related to the $\partial\bar{\partial}$-lemma condition, the Frölicher spectral sequence and also with respect to the sGG condition.

In order to fit the complex structure (6) in the general frame of Table 1, we will express J_0 in terms of the basis of 1-forms $\{e^1, e^2, e^3, e^4, e^5, e^6\}$ given by

$$e^1 = \alpha^1, \quad e^2 = \alpha^2, \quad e^3 = \beta^1, \quad e^4 = \beta^2, \quad e^5 = \alpha^3, \quad e^6 = \beta^3, \tag{7}$$

which satisfies the equations

$$de^1 = de^2 = de^3 = de^4 = 0, \quad de^5 = e^{12}, \quad de^6 = e^{34}. \tag{8}$$

Using (6) and (7) we have that the complex forms

$$\omega_0^1 = e^1 - iJ_0e^1 = e^1 + i\, e^2, \quad \omega_0^2 = e^3 - iJ_0e^3 = e^3 + i\, e^4, \quad \omega_0^3 = 2e^6 - 2iJ_0e^6 = 2e^6 - 2i\, e^5, \tag{9}$$

constitute a basis of forms of bidegree $(1,0)$ with respect to the abelian complex structure J_0. Now, it follows from (8) that the complex structure equations for $(N \times N, J_0)$ in the basis $\{\omega_0^1, \omega_0^2, \omega_0^3\}$ are

$$d\omega_0^1 = d\omega_0^2 = 0, \quad d\omega_0^3 = \omega_0^{1\bar{1}} + i\, \omega_0^{2\bar{2}}. \tag{10}$$

Notice that these equations correspond to take $\rho = \lambda = 0$ and $D = i$ for $\mathfrak{h}_2$ in Table 1.

The compact complex manifold $(N \times N, J_0)$ can be described as follows. The Lie group $H \times H$ endowed with the complex structure J_0 can be realized by the complex matrices of the form

$$(H \times H, J_0) = \left\{ \begin{pmatrix} 1 & -z_1 & -iz_2 & z_3 \\ 0 & 1 & 0 & \bar{z}_1 \\ 0 & 0 & 1 & \bar{z}_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid z_1, z_2, z_3 \in \mathbb{C} \right\}.$$

In terms of the complex coordinates (z_1, z_2, z_3), the left translation by an element (a_1, a_2, a_3) of $(H \times H, J_0)$ is given by

$$L^*_{(a_1,a_2,a_3)}(z_1, z_2, z_3) = (z_1 + a_1,\ z_2 + a_2,\ z_3 - a_1\bar{z}_1 - ia_2\bar{z}_2 + a_3).$$

The basis $\{\omega_0^1, \omega_0^2, \omega_0^3\}$ of left-invariant complex (1,0)-forms on $(H \times H, J_0)$ is expressed in these complex coordinates as

$$\omega_0^1 = dz_1, \quad \omega_0^2 = dz_2, \quad \omega_0^3 = dz_3 + z_1 d\bar{z}_1 + iz_2 d\bar{z}_2.$$

Now, the complex nilmanifold $(N\times N, J_0)$ can be realized as the quotient of $(H\times H, J_0)$ by the lattice defined by taking (z_1, z_2, z_3) as Gaussian integers.

As we reminded in Section 4, the Dolbeault cohomology of the compact complex manifold $(N\times N, J_0)$ can be computed explicitly from the pair $(\mathfrak{h}_2, J_0)$, i.e. $H^{p,q}_{\bar\partial}(N\times N, J_0)\cong H^{p,q}_{\bar\partial}(\mathfrak{h}_2, J_0)$ for any $0\leq p,q\leq 3$. In order to perform an appropriate holomorphic deformation of J_0 we first compute the particular Dolbeault cohomology group

$$H^{0,1}_{\bar\partial}(N\times N, J_0)\cong H^{0,1}_{\bar\partial}(\mathfrak{h}_2, J_0)=\langle[\omega_0^{\bar 1}], [\omega_0^{\bar 2}], [\omega_0^{\bar 3}]\rangle.$$

We consider the small deformation J_t given by

$$t\frac{\partial}{\partial z_2}\otimes\omega_0^{\bar 1}+it\frac{\partial}{\partial z_1}\otimes\omega_0^{\bar 2}\in H^{0,1}(X_0,\, T^{1,0}X_0),$$

where X_0 denotes the complex manifold $(N\times N, J_0)$. This deformation is defined for any $t\in\Delta=\{t\in\mathbb{C}\mid |t|<1\}$ and the analytic family of compact complex manifolds $(N\times N, J_t)$ has a complex basis $\{\omega_t^1, \omega_t^2, \omega_t^3\}$ of type (1,0) with respect to J_t given by

$$J_t:\quad \omega_t^1=\omega_0^1+it\,\omega_0^{\bar 2},\qquad \omega_t^2=\omega_0^2+t\,\omega_0^{\bar 1},\qquad \omega_t^3=\omega_0^3. \tag{11}$$

We can express the complex structures J_t in terms of the real basis of 1-forms $\{e^1,\ldots,e^6\}$ given by (7) as follows. Let us denote by t_1 the real part of t and by t_2 its imaginary part, i.e. $t=t_1+i\,t_2$. From (9) and (11) we get

$$\omega_t^1=e^1-t_2e^3+t_1e^4+i(e^2+t_1e^3+t_2e^4),\quad \omega_t^2=e^3+t_1e^1+t_2e^2+i(e^4+t_2e^1-t_1e^2),\quad \omega_t^3=2e^6-2i\,e^5.$$

Since the complex form ω_t^k, $1\leq k\leq 3$, is declared to be of bidegree (1,0) with respect to the complex structure J_t, necessarily

$$e^2+t_1e^3+t_2e^4=-J_t(e^1-t_2e^3+t_1e^4),\quad e^4+t_2e^1-t_1e^2=-J_t(e^3+t_1e^1+t_2e^2),\quad e^5=J_t(e^6),$$

and we have that the complex structure J_t in the basis $\{e^1,\ldots,e^6\}$ is given by

$$J_t=\frac{-1}{1+|t|^4}\begin{pmatrix} 2|t|^2 & |t|^4-1 & 2(t_2-t_1|t|^2) & -2(t_1+t_2|t|^2) & 0 & 0\\ 1-|t|^4 & 2|t|^2 & -2(t_1+t_2|t|^2) & -2(t_2-t_1|t|^2) & 0 & 0\\ 2(t_1-t_2|t|^2) & 2(t_2+t_1|t|^2) & -2|t|^2 & |t|^4-1 & 0 & 0\\ 2(t_2+t_1|t|^2) & -2(t_1-t_2|t|^2) & 1-|t|^4 & -2|t|^2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & -1 & 0\end{pmatrix},$$

for each $t\in\Delta$.

In the following result we find complex structure equations for every J_t, except for the central limit $t=0$, that fit in the classification given in Table 1.

Proposition 5.1. *Let $(N \times N, J_t)$ be the product of two copies of the 3-dimensional Heisenberg nilmanifold N endowed with the complex structure J_t given by (11). For each $t \in \Delta\backslash\{0\}$, there is a (global) basis $\{\eta_t^1, \eta_t^2, \eta_t^3\}$ of complex forms of bidegree $(1,0)$ with respect to J_t satisfying*

$$d\eta_t^1 = d\eta_t^2 = 0, \quad d\eta_t^3 = \eta_t^{12} + \eta_t^{1\bar{1}} + \eta_t^{1\bar{2}} + i\,\frac{1+|t|^4}{4|t|^2}\eta_t^{2\bar{2}}. \tag{12}$$

Proof. By a direct calculation using (10), we get that the (1,0)-basis $\{\omega_t^1, \omega_t^2, \omega_t^3\}$ given in (11) satisfies $d\omega_t^1 = d\omega_t^2 = 0$ and

$$d\omega_t^3 = \frac{2i\bar{t}}{1+|t|^4}\omega_t^{12} + \frac{1-i|t|^2}{1+|t|^4}\omega_t^{1\bar{1}} + \frac{i-|t|^2}{1+|t|^4}\omega_t^{2\bar{2}}.$$

For each $t \in \Delta\backslash\{0\}$, let us consider the new (1,0)-basis $\{\tau_t^1 = \omega_t^1, \tau_t^2 = \frac{2i\bar{t}}{1-i|t|^2}\omega_t^2, \tau_t^3 = \frac{1+|t|^4}{1-i|t|^2}\omega_t^3\}$. Hence, the complex structure equations for J_t, $t \neq 0$, in this basis are expressed as

$$d\tau_t^1 = d\tau_t^2 = 0, \quad d\tau_t^3 = \tau_t^{12} + \tau_t^{1\bar{1}} + D'\,\tau_t^{2\bar{2}}, \quad 0 < |t| < 1,$$

where $D' = -\frac{1}{2} + i\frac{1-|t|^4}{4|t|^2}$.

Now, we consider another basis $\{\eta_t^1, \eta_t^2, \eta_t^3\}$ of bidegree $(1,0)$ with respect to J_t given by

$$\tau_t^1 = \eta_t^1 - D'\bar{\sigma}\,\eta_t^2, \quad \tau_t^2 = \sigma\,\eta_t^1 + \eta_t^2, \quad \tau_t^3 = (1 + D'|\sigma|^2)\eta_t^3,$$

where $\sigma = 2(1+|t|^2 i)/(1+|t|^4)$. By a long but direct calculation we arrive at

$$d\eta_t^1 = d\eta_t^2 = 0, \quad d\eta_t^3 = \eta_t^{12} + \eta_t^{1\bar{1}} + \eta_t^{1\bar{2}} + D''\,\eta_t^{2\bar{2}},$$

where $D'' = -i\,\frac{1+|t|^4}{4|t|^2}$. Finally, by [10, Proposition 2.4] there exists a (1,0)-basis with respect to which we can take the complex conjugate of D'' as the coefficient of $\eta_t^{2\bar{2}}$, that is, we arrive at the equations (12), and the proof is complete. □

The analytic family $\mathcal{X}$ mentioned in Section 3 was the first example of an analytic family of compact complex manifolds $(X_t)_{t\in\Delta}$ such that the complex invariants $\mathbf{f}_2(X_t) = \mathbf{k}_1(X_t) = 0$ for any $t \neq 0$, but $\mathbf{f}_2(X_0) \neq 0$ and $\mathbf{k}_1(X_0) \neq 0$ (Theorem 3.1), and its construction was based on an appropriate deformation of the abelian complex structure on the nilmanifold with underlying Lie algebra $\mathfrak{h}_4$ [10, 26]. Moreover, the Frölicher spectral sequence of any X_t degenerates at E_1 except for $t = 0$ [10]. The family $\mathcal{X}$ also allows to show that the sGG property is not closed under holomorphic deformations (Theorem 3.3) and, furthermore, the fibres X_t have balanced metric for any $t \in \Delta\backslash\{0\}$, but the central limit X_0 does not admit any strongly Gauduchon metric (Theorem 3.2).

The real nilmanifold underlying the compact complex manifolds X_t is not a product since the Lie algebra $\mathfrak{h}_4$ is irreducible. The following result is a bit of a surprise because it provides an example of a product manifold with a holomorphic family of complex structures satisfying similar properties to those of the analytic family $\mathcal{X}$.

Theorem 5.2. *Let $(N\times N, J_t)_{t\in\Delta}$ be the analytic family given by the product of two copies of the 3-dimensional Heisenberg nilmanifold N endowed with the complex structures J_t given by* (11)*. Then:*

(i) The invariants $\mathbf{f}_2(N\times N, J_t)$ and $\mathbf{k}_1(N\times N, J_t)$ vanish for each $t\in\Delta\backslash\{0\}$, however $\mathbf{f}_2(N\times N, J_0)=3$ and $\mathbf{k}_1(N\times N, J_0)=2$.

(ii) The Frölicher spectral sequence of $(N\times N, J_t)$ degenerates at the first step for each $t\in\Delta\backslash\{0\}$, however the central limit satisfies $E_1(N\times N, J_0)\ncong E_2(N\times N, J_0)\cong E_\infty(N\times N, J_0)$.

(iii) The complex manifold $(N\times N, J_t)$ is sGG for each $t\in\Delta\backslash\{0\}$, but $(N\times N, J_0)$ does not admit strongly Gauduchon metrics.

Proof. We will apply the general result on nilmanifolds given in Section 4 to the complex structures J_t on $N\times N$ given by (11). For the proof of (i) we first notice that the second Betti number of $N\times N$ is equal to 8. From Table 1 we have that $h^{1,1}_{\mathrm{BC}}(N\times N, J_t)=4$ for any $t\in\Delta$, but the Hodge number $\dim E_1^{0,2}(N\times N, J_t)=h^{0,2}_{\bar\partial}(N\times N, J_t)$ depends on the complex structure J_t. In fact, for $t\neq 0$ it follows from (12) that the Dolbeault class $[\eta_t^{\bar 1\bar 2}]$ vanishes because $\eta_t^{\bar 1\bar 2}=\bar\partial\eta_t^{\bar 3}$, hence $H^{0,2}_{\bar\partial}(N\times N, J_t)=\langle[\eta_t^{\bar 1\bar 3}],[\eta_t^{\bar 2\bar 3}]\rangle$ and $h^{0,2}_{\bar\partial}(N\times N, J_t)=2$, for any $t\neq 0$. For the central limit, by (10) we get $H^{0,2}_{\bar\partial}(N\times N, J_0)=\langle[\omega_0^{\bar 1\bar 2}],[\omega_0^{\bar 1\bar 3}],[\omega_0^{\bar 2\bar 3}]\rangle$, so $h^{0,2}_{\bar\partial}(N\times N, J_0)=3$. In conclusion, the complex invariant

$$\mathbf{k}_1(N\times N, J_t)=h^{1,1}_{\mathrm{BC}}(N\times N, J_t)+2\dim E_1^{0,2}(N\times N, J_t)-b_2(N\times N)$$

vanishes on $\Delta\backslash\{0\}$, but it equals 2 for the central limit.

For the invariant $\mathbf{f}_2$ we recall that

$$\mathbf{f}_2(N\times N, J_t)=2\,h^{2,0}_{\mathrm{BC}}(N\times N, J_t)+h^{1,1}_{\mathrm{BC}}(N\times N, J_t)+2\,h^{3,1}_{\mathrm{BC}}(N\times N, J_t)+h^{2,2}_{\mathrm{BC}}(N\times N, J_t)-2b_2(N\times N).$$

By Table 1 we have that $h^{2,0}_{\mathrm{BC}}(N\times N, J_t)$ is constant and equal to 1, whereas $h^{3,1}_{\mathrm{BC}}(N\times N, J_t)=2$ and $h^{2,2}_{\mathrm{BC}}(N\times N, J_t)=6$ for any $t\neq 0$, but $h^{3,1}_{\mathrm{BC}}(N\times N, J_0)=3$ and $h^{2,2}_{\mathrm{BC}}(N\times N, J_0)=7$. Hence, $\mathbf{f}_2(N\times N, J_t)=0$ for any $t\neq 0$, but $\mathbf{f}_2(N\times N, J_0)=3$.

Property (ii) follows directly from the general study given in Theorem 4.4 (b). In fact, the nilpotent complex structure J_t is abelian if and only if $t=0$, hence the Frölicher spectral sequence of $(N\times N, J_t)$ degenerates at the first step if and only if $t\neq 0$. Moreover, the Frölicher sequence of the central limit collapses at the second step.

For the proof of (iii), since J_t is non-abelian for $t\neq 0$, by applying Theorem 4.8 to the $\mathfrak{h}_2$ case we have that $(N\times N, J_t)$ is sGG for any $t\neq 0$. Nevertheless the central limit is not sG because J_0 is abelian and we can apply Proposition 4.6. $\square$

Remark 5.3. Notice that complex manifold $(N \times N, J_t)_{t\in\Delta}$ does not admit balanced metric for any value of t. In fact, for any invariant non-abelian complex structure J on $N \times N$, there exists a (1,0)-basis $\{\omega^1, \omega^2, \omega^3\}$ satisfying

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}} + D\,\omega^{2\bar{2}},$$

where $D = x + iy$ with $y > 0$, and by Proposition 4.7 we know that $(N \times N, J)$ admits a balanced metric if and only if $x + y^2 < \frac{1}{4}$. In our case, for J_t $(t \neq 0)$ it follows from Proposition 5.1 that $x = 0$ and $y = \frac{1+|t|^4}{4|t|^2}$, thus

$$x + y^2 - \frac{1}{4} = \left(\frac{1 - |t|^4}{4|t|^2}\right)^2 > 0$$

and there is no balanced metric on $(N \times N, J_t)$.

Remark 5.4. Angella and Kasuya found in [6] a holomorphic deformation that shows that the $\partial\bar{\partial}$-lemma property is not closed. More recently, in [19] it is proved the existence of an analytic family of compact complex manifolds $(X_t)_{t\in\Delta}$ such that X_t satisfies the $\partial\bar{\partial}$-lemma and admits balanced metric for any $t \in \Delta\backslash\{0\}$, but the central limit X_0 neither satisfies the $\partial\bar{\partial}$-lemma nor admits balanced metrics. Both constructions are based on the complex geometry of the real solvmanifold underlying the Nakamura manifold [30], which is not diffeomorphic to a product manifold.

We finish with the following related question:

Question 5.5. *Does there exist a holomorphic deformation of* $(N \times N, J_0)$ *admitting balanced metrics?*

To answer this question, a more detailed study of the Kuranishi space of deformations of J_0 would be required.

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