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Unified Theory of Obtaining a novel Class of Mixed Trilateral Generating Relations of Certain Special Functions by Group Theoretic Method

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Abstract

In this paper, we have presented a unified theory of obtaining a novel class of mixed trilateral generating relations involving certain special functions by group theoretic method. A good number of theorems in connection with mixed trilateral generating relations involving certain special functions are obtained in course of application of our result.

Key words: Mixed trilateral generating functions, Gegenbauer polynomials, Hypergeomtric polynomials, Biorthogonal polynomials, Laguerre polynomials.
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1 Introduction

Unified theory of bilateral or trilateral generating relations for various special functions are of greater importance in the study of special functions. In this direction several attempts have been made by some researchers [1-15].

In this present article, we have discussed a group theoretic method for deriving a unified presentation of a novel class of mixed trilateral generating relations for certain special functions subject to the condition of construction of one parameter continuous transformations group for the special function under consideration. Furthermore, we would like to mention that a good number of theorems on mixed trilateral generating relations for various special functions can be easily obtained in course of application of our result (Theorem 1). In fact the main result of our investigation is given in section 2(Theorem 1).

2 Group-Theoretic Discussion

We first consider a bilateral generating relation involving a particular special function $p_n^{(\alpha+n)}(x)$ of degree n and of parameter $(\alpha+n)$ as follows:

$$G(x,\ u,\ w) = \sum_{n=0}^{\infty} a_n\ p_n^{(\alpha+n)}(x)\ g_n(u)\ w^n, \tag{1}$$

where $g_n(u)$ is an arbitrary polynomial of degree n and a_n is independent of $x,\ u,\ w$. Replacing w by wvz and then multiplying both sides of (2.1) by y^α, we get

$$y^\alpha\ G(x,\ u,\ wvz) = \sum_{n=0}^{\infty} a_n\ (p_n^{(\alpha+n)}(x)\ y^\alpha z^n)\ g_n(u)\ (wv)^n. \tag{2}$$

We now suppose that for the above special function $p_n^{(\alpha+n)}(x)$, it is possible to define a linear partial differential operator R, which generates one parameter continuous transformations group as follows:

$$R = A_1(x,y,z)\ \frac{\partial}{\partial x} + A_2(x,y,z)\ \frac{\partial}{\partial y} + A_3(x,y,z)\ \frac{\partial}{\partial z} + A_0(x,y,z) \tag{3}$$

such that

$$R(p_n^{(\alpha+n)}(x)\ y^\alpha z^n) = \rho_n\ p_{n+1}^{(\alpha-1)(n+1)}(x)\ y^{\alpha-1} z^{n+1}, \tag{4}$$

where ρ_n is function of $n,\ \alpha$ and independent of x, y and

$$e^{wR} f(x,y,z) = \Omega(x,y,z,w) f\Big(g(x,y,z,w), h(x,y,z,w), k(x,y,z,w)\Big). \tag{5}$$

Operating both sides of (2.2) by e^{wR}, we get

$$e^{wR}\Big(y^\alpha\ G(x,\ u,\ wvz)\Big) = e^{wR}\Big(\sum_{n=0}^{\infty} a_n\ (p_n^{(\alpha+n)}(x)\ y^\alpha z^n)\ g_n(u)(wv)^n\Big). \tag{6}$$

Now the left number of (2.6), with the help of (2.5),becomes

$$\Omega(x,y,z,w)(h(x,y,z,w))^\alpha G\Big(g(x,y,z,w), u, vwk(x,y,z,w)\Big). \tag{7}$$

The right number of (2.6), with the help of(2.4), becomes

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n \frac{w^m}{m!}\ \rho_n\ \rho_{n+1}\ \cdot\ \cdot\ \cdot\ \rho_{n+m-1}\ p_{n+m}^{(\alpha+n)}(x) y^{\alpha-m} z^{n+m} g_n(u)(wv)^n. \tag{8}$$

Now equating (2.7) and (2.8) and then putting $y = z = 1$, we get

$$\begin{aligned}
&\Omega(x,1,1,w)(h(x,1,1,w))^\alpha G(g(x,1,1,w), u, vwk(x,1,1,w)) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n \frac{w^m}{m!}\ \rho_n\ \rho_{n+1}\ \cdot\ \cdot\ \cdot\ \rho_{n+m-1}\ p_n^{(\alpha+n-m)}(x) g_n(u)\ (wv)^n \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n} a_{n-m} \frac{w^n}{m!}\ \rho_{n-m}\ \rho_{n-m+1}\ \cdot\ \cdot\ \cdot\ \rho_{n-1}\ p_n^{(\alpha+n-m)}(x) g_{n-m}(u)\ v^{n-m} \\
&= \sum_{n=0}^{\infty} w^n\ \sigma_n(x,\ u,\ v),
\end{aligned}$$

where

$$\sigma_n(x,\ u,\ v) = \sum_{m=0}^{n} \frac{a_m}{(n-m)!} \prod_{i=m}^{n-1} \rho_i \ p_n^{(\alpha+m)}(x) g_m(u) v^m.$$

Thus we arrive at the following theorem.

Theorem 1: If

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n \ p_n^{(\alpha+n)}(x) g_n(u) \ w^n$$

then

$$\Omega(x,1,1,w)(h(x,1,1,w))^{\alpha} G(g(x,1,1,w),u,vwk(x,1,1,w)) = \sum_{n=0}^{\infty} w^n \sigma_n(x,u,v),$$

where

$$\sigma_n(x,u,v) = \sum_{m=0}^{n} \frac{a_m}{(n-m)!} \prod_{i=m}^{n-1} \rho_i \ p_n^{(\alpha+m)}(x) g_m(u) v^m.$$

3 Applications

We now proceed to give a good number of applications of our result stated in Theorem 1.

3.1 Application-1

At first we take

$$p_n^{(\alpha+n)}(x) = f_n^{(\beta+n)}(x) \quad with \quad \alpha = \beta.$$

We now consider the following partial differential operator R :

$$R = xy^{-1}z \ \frac{\partial}{\partial x} - z \ \frac{\partial}{\partial y} - 2z^2 y^{-1} \ \frac{\partial}{\partial z} - xy^{-1}z$$

such that

$$R(f_n^{(\beta+n)}(x) \ y^{\beta} z^n) = -(n+1) \ f_{n+1}^{(\beta+n)}(x) \ y^{\beta-1} z^{n+1} \tag{9}$$

and

$$e^{wR} f(x,y,z) = exp(-wxy^{-1}z) f\left(x(1+wy^{-1}z), \frac{y}{(1+wy^{-1}z)}, \frac{z}{(1+wy^{-1}z)^2} \right). \tag{10}$$

Comparing (2.4), (2.5) with (3.1), (3.2), we get

$$\rho_n = -(n+1), \Omega(x,y,z,w) = exp(-wxy^{-1}z), g(x,y,z,w) = x(1+wy^{-1}z),$$

$$h(x,y,z,w) = \frac{y}{(1+wy^{-1}z)}, \quad k(x,y,z,w) = \frac{z}{(1+wy^{-1}z)^2}.$$

So by the application of our Theorem 1, we get the following theorem on bilateral generating relations involving modified Laguerre polynomials.

Theorem 2: If

$$G(x,u,w)=\sum_{n=0}^{\infty}\ a_n\ f_n^{(\beta+n)}(x)g_n(u)\ w^n \tag{11}$$

then

$$exp(wx)(1-w)^{\beta}G\Big(x(1-w),u,vw(1-w)^{-2}\Big)=\sum_{n=0}^{\infty}w^n\ \sigma_n(x,u,v), \tag{12}$$

where

$$\sigma_n(x,\ u,v)=\sum_{m=0}^{n}\ a_m\ \begin{pmatrix} n\\ m\end{pmatrix}\ f_n^{(\beta+m)}(x)g_m(u)v^m.$$

3.2 Application-2

Now let us take

$$p_n^{(\alpha+n)}(x)=L_{a,\ b,\ (m+n),\ n}(x)\ \ with\ \ \alpha=m.$$

Then we consider the following partial differential operator R, where

$$R=bxy^{-1}z\ \frac{\partial}{\partial x}+bz\ \frac{\partial}{\partial y}+2by^{-1}z^2\ \frac{\partial}{\partial z}-axy^{-1}z$$

such that

$$R(L_{a,\ b,\ (m+n),\ n}(x)\ y^m z^n)=(n+1)\ L_{a,\ b,\ (m+n),\ n+1}(x)\ y^{m-1}z^{n+1} \tag{13}$$

and

$$e^{wR}f(x,y,z)=exp\bigg(\frac{-awxy^{-1}z}{1-bwy^{-1}z}\bigg)f\bigg(x(1-bwy^{-1}z)^{-1},y(1-bwy^{-1}z)^{-1},z(1-bwy^{-1}z)^{-2}\bigg). \tag{14}$$

Comparing (2.4), (2.5) with (3.5), (3.6), we get

$$\rho_n=(n+1),\Omega(x,y,z,w)=exp\bigg(\frac{-awxy^{-1}z}{1-bwy^{-1}z}\bigg),g(x,y,z,w)=x(1-bwy^{-1}z)^{-1},$$

$$h(x,y,z,w)=y(1-bwy^{-1}z)^{-1},\quad k(x,y,z,w)=z(1-bwy^{-1}z)^{-2}.$$

So by the application of our Theorem 1, we get the following result on bilateral generating relations involving modified Laguerre polynomials.

Theorem 3: If

$$G(x,u,w)=\sum_{n=0}^{\infty}\ a_n\ L_{a,\ b,\ (m+n),\ n}(x)g_n(u)\ w^n \tag{15}$$

then

$$exp\bigg(\frac{-awx}{1-bw}\bigg)(1-bw)^{-m}G\Big(x(1-bw)^{-1},u,vw(1-bw)^{-2}\Big)=\sum_{n=0}^{\infty}w^n\ \sigma_n(x,u,v), \tag{16}$$

where

$$\sigma_n(x,u,v) = \sum_{p=0}^{n} a_p \begin{pmatrix} n \\ p \end{pmatrix} L_{a,\ b,\ (m+p),\ n}(x) g_p(u) v^p.$$

Corollary 1: On specializing the parameters as $a = b = 1$ and $m = 1 + \alpha$ in Theorem 3 we get the following result on modified Laguerre polynomials.

Theorem 4: If

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n\ L_n^{(\alpha+n)}(x) g_n(u)\ w^n \tag{17}$$

then

$$exp\left(\frac{-wx}{1-w}\right)(1-w)^{-(1+\alpha)} G\left(x(1-w)^{-1}, u, vw(1-w)^{-2}\right) = \sum_{n=0}^{\infty} w^n\ \sigma_n(x,u,v), \tag{18}$$

where

$$\sigma_n(x,u,v) = \sum_{p=0}^{n} a_p \begin{pmatrix} n \\ p \end{pmatrix} L_n^{(\alpha+p)}(x) g_p(u) v^p,$$

which is found derived in [17].

3.3 Application-3

Now we take

$$p_n^{(\alpha+n)}(x) = C_n^{(\lambda+n)}(x) \quad with \quad \alpha = \lambda.$$

From (16), we see that

$$R = (x^2-1)y^{-1}z\ \frac{\partial}{\partial x} + 2xz\ \frac{\partial}{\partial y} + 3xy^{-1}z^2\ \frac{\partial}{\partial z}$$

such that

$$R(C_n^{(\lambda+n)}(x)\ y^{\lambda} z^n) = (n+1)\ C_{n+1}^{(\lambda+n)}(x)\ y^{\lambda-1} z^{n+1} \tag{19}$$

and

$$\begin{aligned} e^{wR} f(x,y,z) &= \\ & f\left(\frac{xy-wz}{(wz^2-2wxyz+y^2)^{\frac{1}{2}}}, \frac{y^3}{(wz^2-2wxyz+y^2)}, \frac{y^3 z}{(wz^2-2wxyz+y^2)^{\frac{3}{2}}}\right). \end{aligned} \tag{20}$$

So by comparing (3.11), (3.12) with (2.4), (2.5), we get

$$\rho_n = (n+1), \Omega(x,y,z,w) = 1, g(x,y,z,w) = \frac{xy-wz}{(wz^2-2wxyz+y^2)^{\frac{1}{2}}},$$

$$h(x,y,z,w) = \frac{y^3}{(wz^2-2wxyz+y^2)}, \quad k(x,y,z,w) = \frac{y^3 z}{(wz^2-2wxyz+y^2)^{\frac{3}{2}}}.$$

Then by the application of our Theorem 1, we get the following result on bilateral generating relations involving Gegenbauer polynomials.

Theorem 5: If

$$G(x,u,w)=\sum_{n=0}^{\infty}\ a_n\ C_n^{(\lambda+n)}(x)g_n(u)\ w^n \tag{21}$$

then

$$(w-2wx+1)^{-\lambda}G\left(\frac{x-w}{(w-2wx+1)^{\frac{1}{2}}},u,\frac{wv}{(w-2wx+1)^{\frac{3}{2}}}\right)=\sum_{n=0}^{\infty}w^n\ \sigma_n(x,u,v), \tag{22}$$

where

$$\sigma_n(x,u,v)=\sum_{m=0}^{n}\ a_m\ \left(\begin{array}{c} n \\ m \end{array}\right)\ C_n^{(\lambda+m)}(x)g_m(u)v^m.$$

3.4 Application-4

Now we take

$$p_n^{(\alpha+n)}(x)=\ {}_2F_1(-n,\beta;\alpha+n;x).$$

Then considering the following partial differential operator $R:$

$$R=(1-x)xy^{-1}z\ \frac{\partial}{\partial x}+z\ \frac{\partial}{\partial y}+2y^{-1}z^2\ \frac{\partial}{\partial z}-\beta xy^{-1}z$$

such that

$$R(\ {}_2F_1(-n,\beta;\alpha+n;x)\ y^{\alpha}z^n)=(2n+\alpha)\ \ {}_2F_1(-(n+1),\beta;\alpha+n;x)\ y^{\alpha-1}z^{n+1} \tag{23}$$

and

$$\begin{aligned} e^{wR}f(x,y,z)&=(1-wy^{-1}z)^{\beta}\{1-(1-x)wy^{-1}z\}^{-\beta} \\ &\times f\left(\frac{x}{1-(1-x)wy^{-1}z},\frac{y}{(1-wy^{-1}z)},\frac{z}{(1-wy^{-1}z)^2}\right). \end{aligned} \tag{24}$$

Comparing (3.15), (3.16) with (2.4), (2.5), we get

$$\rho_n=(2n+\alpha),\Omega(x,y,z,w)=\frac{(1-wy^{-1}z)^{\beta}}{\{1-(1-x)wy^{-1}z\}^{\beta}},g(x,y,z,w)=\frac{x}{1-(1-x)wy^{-1}z},$$

$$h(x,y,z,w)=\frac{y}{(1-wy^{-1}z)},\quad k(x,y,z,w)=\frac{z}{(1-wy^{-1}z)^2}.$$

So by the application of our Theorem 1, we at once get the following result on bilateral generating relations involving Hypergeometric polynomials.

Theorem 6: If

$$G(x,u,w)=\sum_{n=0}^{\infty}\ a_n\ \ {}_2F_1(-n,\beta;\alpha+n;x)g_n(u)\ w^n \tag{25}$$

then

$$(1-w)^{\beta-\alpha}\{1-(1-x)w\}^{-\beta}G\left(\frac{x}{(1-w(1-x))},u,\frac{wv}{(1-w)^2}\right)=\sum_{n=0}^{\infty}w^n\ \sigma_n(x,u,v), \quad (26)$$

where

$$\sigma_n(x,u,v)=\sum_{k=0}^{n}\ a_k\ \begin{pmatrix} n+\alpha+k-1 \\ 2k+\alpha-1 \end{pmatrix}\ {}_2F_1(-n,\beta;\alpha+k;x)g_k(u)v^k.$$

3.5 Application-5

Finally we take

$$p_n^{(\alpha+n)}(x)=Y_n^{\alpha+n}(x;k),$$

k is a non-zero positive integer. Then from [17], we see that

$$R=xy^{-1}z\ \frac{\partial}{\partial x}+z\ \frac{\partial}{\partial y}+(k+1)y^{-1}z^2\ \frac{\partial}{\partial z}+(1-x)y^{-1}z$$

such that

$$R(Y_n^{\alpha+n}(x;k)\ y^{\alpha}z^n)=k(n+1)\ Y_{n+1}^{\alpha+n}(x;k)y^{\alpha-1}z^{n+1} \quad (27)$$

and

$$\begin{aligned} e^{wR}f(x,\ y,\ z) &= (1-kwy^{-1}z)^{-\frac{1}{k}}\ exp[\ x\{1-(1-kwy^{-1}z)^{-\frac{1}{k}}\}\ ] \\ &\times\ f\left(x(1-kwy^{-1}z)^{-\frac{1}{k}},y(1-kwy^{-1}z)^{-\frac{1}{k}},z(1-kwy^{-1}z)^{-\frac{k+1}{k}}\right). \end{aligned} \quad (28)$$

Comparing (3.19), (3.20) with (2.4), (2.5), we get

$$\rho_n=k(n+1),\Omega(x,y,z,w)=(1-kwy^{-1}z)^{-\frac{1}{k}}\ exp[x\{1-(1-kwy^{-1}z)^{-\frac{1}{k}}\}],$$

$$g(x,y,z,w)=x(1-kwy^{-1}z)^{-\frac{1}{k}},\ \ h(x,y,z,w)=y(1-kwy^{-1}z)^{-\frac{1}{k}},$$

$$k(x,y,z,w)=z(1-kwy^{-1}z)^{-\frac{k+1}{k}}.$$

So by the application of our Theorem 1, we get the following result on bilateral generating relations involving Konhauser biorthogonal polynomials suggested by the Laguerre polynomials.

Theorem 7: If

$$G(x,u,w)=\sum_{n=0}^{\infty}\ a_n\ Y_n^{\alpha+n}(x;k)g_n(u)\ w^n \quad (29)$$

then

$$(1-kw)^{-\frac{(1+\alpha)}{k}}exp[x\{1-(1-kw)^{-\frac{1}{k}}\}]G\left(\frac{x}{(1-kw)^{\frac{1}{k}}},u,\frac{wv}{(1-kw)^{\frac{k+1}{k}}}\right)=\sum_{n=0}^{\infty}w^n\sigma_n(x,u,v), \quad (30)$$

where

$$\sigma_n(x,u,v) = \sum_{m=0}^{n} a_m \begin{pmatrix} n \\ m \end{pmatrix} k^{n-m} \, Y_n^{\alpha+m}(x;k) g_m(u) v^m,$$

which is found derived in [17].

Corollary 2: On putting $k = 1$ in Theorem 7 we get the Theorem 4.

4 Conclusion

From the above discussion, it is clear that one may apply Theorem 1 in the case of other polynomials and functions existing in the field of special functions subject to the condition of construction of one parameter continuous transformations group for the said special function(s). Furthermore, the importance of the above theorems (2-7) lies in the fact that whenever one knows a bilateral generating function of the form (3.3,3.7 etc.) then the corresponding mixed trilateral generating function can at once be written down from (3.4,3.8 etc.). So one can get a large number of mixed trilateral generating functions by attributing different suitable values to a_n in (3.3,3.7 etc.).

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