Rev. Real Academia de Ciencias. Zaragoza. **68:** 57–71, (2013).
ISSN: 0370-3207

Asymptotic behavior of certain multiplicative functions *

C. Calderón, M.J. de Velasco and M.J. Zárate

Departamento de Matemáticas. Facultad de Ciencia y Tecnología

Universidad del País Vasco, Bilbao, 48940

Abstract

Let $h_{\overline{\alpha}}(n, \overline{a}, \overline{b})$ be a sequence of complex numbers, coefficients of the Dirichlet series of $L^{\alpha}(a_1 s - b_1, \chi_1) L^{\beta}(a_2 s - b_2, \chi_2)$ where χ_1, χ_2 are characters modulo q_1, q_2 respectively. The main purpose of this paper is to prove approximative formulae for the logarithmic Riesz mean of $h_{\overline{\alpha}}(n, \overline{a}, \overline{b})$.

2010 Math. Subject Classification: 11N37, 30E10.
Key words: Dirichlet series, Riesz mean, Inversion formulas.

1 Introduction

Let $f(n)$ be the coefficient of some Dirichlet series $Z(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$ convergent in $Re\ s > \sigma_c$. For $\kappa > \max\{0, \sigma_c\}$, $r > 0$ and $x \geq 1$, the logarithmic Riesz mean of f has the following representation:

$$\frac{1}{\Gamma(r+1)} \sum_{n \leq x} f(n) \log^r (x/n) = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} Z(s) x^s s^{-r-1} ds.$$

Berndt [3] has derived asymptotic formulae for sums of type

$$\sum_{\lambda_n \leq x} f(n) \log^q (x/\lambda_n)$$

where q is a nonnegative integer, $\{\lambda_n\}$ is a sequence of positive numbers tending to ∞ and $f(n)$ is the coefficient of a generalized Dirichlet series

$$\sum_{n=1}^{\infty} f(n) \lambda_n^{-s}.$$

*Research supported by University of the Basque Country. EHU10/05

In [14], the authors studied a variant of the classical circle problem using the logarithmic Riesz mean

$$\frac{1}{\Gamma(1+\kappa)}\sum_{n\le x} r_2(n)\log^{\kappa}(x/n)$$

$r_2(n)$ is the number of representations of n as $n_1^2+n_2^2$, $n_1, n_2 \in \mathbb{Z}$. Vorhauer [13], obtained approximate formulae for the sum

$$\frac{1}{\Gamma(1+\kappa)}\sum_{n\le x} f(n)\log^{\kappa}(x/n),$$

for real $\kappa \ge 0$, subject to suitable conditions on the function f.

Let α, β, a_1, a_2 be positive integers and let b_1, b_2 be nonnegative integers. We denote $\overline{\alpha} = (\alpha, \beta)$, $\overline{a} = (a_1, a_2)$, $\overline{b} = (b_1, b_2)$. We consider the logarithmic Riesz mean

$$\frac{1}{\Gamma(r+1)}\sum_{n\le x} h_{\overline{\alpha}}(n;\overline{a},\overline{b})\log^{r}(x/n) \tag{1.1}$$

$h_{\overline{\alpha}}(n,\overline{a},\overline{b})$ being a sequence of complex numbers, not identically zero, coefficients of the Dirichlet series

$$L^{\alpha}(a_1 s - b_1, \chi_1)L^{\beta}(a_2 s - b_2, \chi_2) = \sum_{n=1}^{\infty}\frac{h_{\overline{\alpha}}(n;\overline{a},\overline{b})}{n^s}.$$

Thus

$$h_{\overline{\alpha}}(n;\overline{a},\overline{b}) = \sum_{n_1^{a_1}n_2^{a_2}=n} n_1^{b_1}n_2^{b_2}\tau_{\alpha}(n_1)\tau_{\beta}(n_2)\chi_1(n_1)\chi_2(n_2), \tag{1.2}$$

where $\tau_c(n)$ denotes the number of representations of n as a product of c factors. We denote

$$A(s;\overline{a},\overline{b}) = L^{\alpha}(a_1 s - b_1, \chi_1)L^{\beta}(a_2 s - b_2, \chi_2). \tag{1.3}$$

Here χ_1, χ_2 are primitive characters modulo q_1, q_2 respectively, where q_1 and q_2 are positive integers. Let us suppose for example that

$$\frac{1+b_1}{a_1} = \max\{\frac{1+b_j}{a_j}, j = 1, 2\},$$

therefore $a_1 \le a_2 + k$, with $k = a_2 b_1 - a_1 b_2$. One purpose of the present paper it to provide asymptotic estimates for the sums (1.1) using complex methods and properties of exponential integrals. Throughout the paper $s = \sigma + it$ and $w = u + iv$ with σ, t and u, v real numbers and C denotes a constant, not necessarily the same in each occurrence.

2 Statement of results

We prove the following theorem, which we shall apply of the functions $h_{\overline{\alpha}}(n;\overline{a},\overline{b})$.

Theorem. *Let $h_{\overline{\alpha}}(n;\overline{a},\overline{b})$ be the arithmetical function defined in (1.2) with*

$$\frac{b_2}{a_2} < \frac{b_1}{a_1} \le \frac{1+b_2}{a_2} \le \frac{1+b_1}{a_1}.$$

Let r be a positive number such that $2r+1-2(b_1\alpha+b_2\beta)-\alpha-\beta > 0$. Then the logarithmic Riesz mean (1.1) of $h_{\overline{\alpha}}$ holds the following asymptotic formula:

$$\frac{1}{\Gamma(r+1)}\sum_{n\leq x} h_{\overline{\alpha}}(n;\overline{a},\overline{b})\log^r\left(\frac{x}{n}\right) = S_r(x;\overline{a},\overline{b}) + O(x^{-\frac{2r+1-2(b_1\alpha+b_2\beta)-\alpha-\beta}{2(a_1\alpha+a_2\beta)}}) \qquad (2.1)$$

where $S_r(x;\overline{a},\overline{b})$ is the sum of the residues of the function $A(s;\overline{a},\overline{b})x^s s^{-r-1}$. Note that the power of x in the error term (2.1), is negative.

If r is an integer and $\frac{1+b_2}{a_2} < \frac{1+b_1}{a_1}$, then we have

$$\sum_{n\leq x} h_{\overline{\alpha}}(n;\overline{a},\overline{b})\log^r(\frac{x}{n}) = r!\eta(\chi_1)x^{\frac{1+b_1}{a_1}}P_{\alpha-1}(\log x)+$$
$$+ r!\eta(\chi_2)x^{\frac{1+b_2}{a_2}}P_{\beta-1}(\log x) + r!P_r(\log x) + O(x^{-\frac{2r+1-2(b_1\alpha+b_2\beta)-\alpha-\beta}{2(a_1\alpha+a_2\beta)}}) \qquad (2.2)$$

where $\eta(\chi) = 1$ if χ is the principal character modulo q, and $\eta(\chi) = 0$ otherwise. The function $P_r(\log x)$ is $Res_{s=0}(A(s;\overline{a},\overline{b})x^s s^{-r-1})$.

If r is an integer and $\frac{1+b_1}{a_1} = \frac{1+b_2}{a_2}$, then

$$\sum_{n\leq x} h_{\overline{\alpha}}(n;\overline{a},\overline{b})\log^r(\frac{x}{n}) = r!\theta_{12}x^{\frac{1+b_2}{a_2}}P_{\lambda_{12}}(\log x)+$$
$$+r!P_r(\log x) + O(x^{-\frac{2r+1-2(b_1\alpha+b_2\beta)-\alpha-\beta}{2((a_2+k)\alpha+a_2\beta)}}). \qquad (2.3)$$

where

i) $\theta_{12} = 1$ and $\lambda_{12} = \alpha - 1$ if χ_1 is principal and χ_2 non-principal;
$\theta_{12} = 1$ and $\lambda_{12} = \beta - 1$ if χ_1 is non-principal and χ_2 principal;
$\theta_{12} = 1$ and $\lambda_{12} = \alpha + \beta - 1$ if χ_1 and χ_2 are principals; and
$\theta_{12} = 0$ otherwise.

ii) $P_n(\log x)$ is a polynomial in $\log x$ of nth-degree.

3 The necessary lemmas

We need to know the asymptotic behavior of the integral $\frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} x^s s^{-r-1}ds$. The case $r = 0$ can be found in [9] , Lemma-12.1. The case $r \geq 0$ real, is Lemma 1 [13] with $y = e^x$.

Lemma 1. *Let $r \geq 0$, $\kappa > 0$ be real constants. Then, uniformly in $y > 0, T \geq 1$, for $y \neq 1$*

$$\frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT}\frac{y^s}{s^{r+1}}ds - \frac{\lambda_r(y)}{\Gamma(r+1)} \ll y^\kappa T^{-r}\min\{1, T^{-1} \mid \log y \mid^{-1}\}$$

where $\lambda_r(y) = 0$ if $0 < y < 1$ and $\lambda_r(y) = \log^r y$, if $y > 1$. For $y = 1$,

$$\frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT}\frac{ds}{s} = \frac{1}{2} + O(\frac{\kappa}{T}), \qquad \left|\frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT}\frac{ds}{s^{r+1}}\right| \ll T^{-r}.$$

Note that when $y \neq 1$, one obtains the given estimation considering two different contours (as in [9] , Lemma-12.1): a U-shaped contour that opens to the right if $0 < y < 1$ and the another contour replacing the vertical segment of integration by a circular arc with center 0 and radius $R = \sqrt{c^2 + T^2}$. Analogously, if $y > 1$ the U-shaped contour opens to the left and for the circular arc we take the arc which lies to the left of the segment, also with center 0 and the same radius R.

Lemma 2. *Let*

$$A(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$$

be a Dirichlet series absolutely convergent for Re $s > \sigma_a$ *and* $|f(n)| < C\psi(n)$ *where* $C > 0$ *and for* $x \geq x_0$, $\psi(x)$ *is a nondecreasing function. Let further*

$$\sum_{n=1}^{\infty} |f(n)| n^{-\sigma} \ll (\sigma - \sigma_a)^{-\gamma} \tag{3.1}$$

as $\sigma \to \sigma_a^+$ *for some* $\gamma > 0$. *If* $w = u + iv$, $(u, v$ *real) is arbitrary,* $r \geq 0$ *be real number,* $x > 1$, $\kappa > 0$, $T \geq 1$ *and* $u + \kappa > \sigma_a$, *then*

$$\begin{aligned} \frac{1}{\Gamma(r+1)} \sum_{n \leq x} f(n) \log^r(x/n) n^{-w} &= \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} A(w+s) x^s s^{-r-1} ds \\ &+ O(x^{\kappa} T^{-r-1} (u + \kappa - \sigma_a)^{-\gamma}) + O(T^{-r-1} \psi(2x) x^{1-u} \log 2x) \\ &+ O(T^{-r} \psi(2x) x^{-u}). \end{aligned} \tag{3.2}$$

Proof. Following the proof of Lemma 3.12, case $r = 0$ of Titchmarsh [12], (Satz 3.1 p. 376-377 [11]), and using Lemma 1, the formula (3.2) is deduced. Here we require $\kappa > 0$ to allow the application of Lemma 1. Also $\kappa > \sigma_a$ in order to exchange summation and integration. ∎

For any character χ modulo q let $G(n, \chi)$ denote the Gauss sum

$$G(n, \chi) = \sum_{m=1}^{q} \chi(m) \exp(2\pi i m n / q).$$

If χ is a primitive character, then $G(n, \chi) = \overline{\chi}(n) G(1, \chi)$ (see Theorem V 4.12 Ayoub [2]) which holds for every n. Then

$$\sum_{n=1}^{\infty} G(n, \chi) n^{-s} = G(1, \chi) \sum_{n=1}^{\infty} \overline{\chi}(n) n^{-s}. \tag{3.3}$$

Lemma 3. *Let* χ *be a primitive character modulo* q *and let* $L(s, \chi)$ *be the associated Dirichlet L-function. For Re* $s > 1$, *let* $L^*(s, \overline{\chi}) = G(1, \chi) L(s, \overline{\chi})$ *given in (3.3). Then* $L(s, \chi)$ *and* $L^*(s, \overline{\chi})$ *satisfy the functional equation*

$$L(s, \chi) = (i/q)(2\pi/q)^{s-1} \Gamma(1-s) \{-e^{\pi i s/2} + \chi(-1) e^{-\pi i s/2}\} L^*(1-s, \overline{\chi}). \tag{3.4}$$

If we denote $H(s,\chi) = (2\pi)^{s-1}\Gamma(1-s)\{-e^{\pi i s/2} + \chi(-1)e^{-\pi i s/2}\}$, *then we can write (3.4) as*

$$L(s,\chi) = (i/q^s)H(s,\chi)L^*(1-s,\overline{\chi}). \tag{3.5}$$

Proof. See T.M. Apostol [1] Chapter 12, or K. Prachar [11] VII Satz 1.1 or K. Chandrasekharan [5] Chapter VI, Theorem 2. ∎

The asymptotic behaviour of $H(s,\chi)$ when σ is fixed and $|t| \to \infty$ can be determined by using Stirling′s formula in the form

$$\begin{aligned}\Gamma(s) =& \sqrt{2\pi}|t|^{\sigma-1/2}\times \\ &\times \exp\left\{-(1/2)\pi|t| + i\left(t\log|t| - t + \frac{\pi t}{2|t|}\left(\sigma - \frac{1}{2}\right)\right)\right\} \times \left(1 + O(|t|^{-1})\right)\end{aligned}$$

($|t| \geq 1$). So, one obtains

$$H(\sigma+it,\chi) = Ce^{-it\log t + it\log(2\pi) + it}t^{1/2-\sigma} + O(t^{-\sigma-1/2}) \tag{3.6}$$

where $t \geq T_0 > 0$, (see [10] A. Ivić and T. Meurman pag 347, see also E.T. Copson [6] or E.C.Titchmarsh pag 160 [12]). In (3.6) C and the implicit constants are depending on σ.

For $q > 1$, the principal character χ_0 is not primitive but it is real, so that $\overline{\chi}_0 = \chi_0$ and $L(s,\chi_0)$ holds a similar functional equation

$$L(s,\chi_0) = 2^s\pi^{s-1}\prod_{p|k}\frac{(1-p^{-s})}{(1-p^{s-1})}\Gamma(1-s)\sin\frac{\pi s}{2}L(1-s,\chi_0)$$

see [7] , Theorem 10, Chapter 12. If $q = q_1\cdots q_r$ is a decomposition of q into pairwise coprime integers and if χ is a character modulo q, then exists a unique decomposition of χ into characters χ_i modulo q_i, $\chi = \chi_1\cdots\chi_r$. The Gaussian sum $G(n,\chi)$ verifies $G(n,\chi) = \epsilon G(n_1,\chi_1)\cdots G(n_r,\chi_r)$with $n_i \equiv n(\bmod\, q_i)$ and $\epsilon = \chi_1(q/q_1)\cdots\chi_r(q/q_r)$(see [2] Theorem 4.1 and 4.2, Chapter V). If χ is a character modulo q, $f(\chi)$ its conductor and χ' the primitive character modulo $f(\chi)$ equivalent to χ, then

$$L(s,\chi) = \prod_{p|q}\left(1 - \frac{\chi'(p)}{p^s}\right)L(s,\chi')$$

(see [2] Theorem 4.7 Chapter V or Chapter VI, K. Chandrasekharan [4]). Hence every L-series $L(s,\chi)$ is equal to the L-series $L(s,\chi')$ of a primitive character , multiplied by a finite number of factors. Then it is sufficient to consider the functional equation (3.5).

Lemma 4. *Let* $0 < a \leq 1$. *Then for the Hurwitz zeta-function* $\zeta(s,a)$ *we have uniformly in* σ *as* $|t| \to \infty$

$$\zeta(\sigma+it,a) \ll |t|^{\tau(\sigma)}\log|t| \tag{3.7}$$

where

$$\tau(\sigma) = \begin{cases} \frac{1}{2} - \sigma, & \text{if } \sigma \le 0 \\ \frac{1}{2}(1-\sigma), & \text{if } 0 \le \sigma \le 1 \\ 0, & \text{if } \sigma \ge 1 \end{cases}$$

and $\log|t|$ *may be suppressed except when* $-\epsilon \le \sigma \le \epsilon$ *or* $1-\epsilon \le \sigma \le 1+\epsilon$.

Let χ *be a character modulo* q *and let* $L(s,\chi)$ *be the associated Dirichlet L-function. Then we have as* $|t| \to \infty$

$$L(s,\chi) \ll q|t|^{\tau(\sigma)} \log(q|t|) \tag{3.8}$$

where $\tau(\sigma)$ *is given above.*

Proof. (3.7) is proved in [15], p. 276, and (3.8) follows from (3.7) and from to the expression

$$L(s,\chi) = q^{-s} \sum_{n=1}^{q} \chi(n)\zeta(s, n/q) \qquad (s \ne 1). \quad \blacksquare$$

4 Proof of Theorem

We consider the function $A(s;\overline{a},\overline{b})$ defined as

$$\begin{aligned} A(s;\overline{a},\overline{b}) &= L^{\alpha}(a_1 s - b_1, \chi_1) L^{\beta}(a_2 s - b_2, \chi_2) \\ &= \sum_{n_1=1}^{\infty} \frac{n_1^{b_1} \tau_\alpha(n_1)\chi_1(n_1)}{n_1^{a_1 s}} \sum_{n_2=1}^{\infty} \frac{n_2^{b_2} \tau_\beta(n_2)\chi_2(n_2)}{n_2^{a_2 s}} \\ &= \sum_{n=1}^{\infty} h_{\overline{\alpha}}(n;\overline{a},\overline{b}) n^{-s}, \text{ in } Re\ s > \frac{1+b_1}{a_1}. \end{aligned}$$

Moreover, as $\frac{b_2}{a_2} \le \frac{b_1}{a_1}$ then $n_1^{b_1} n_2^{b_2} = (n_1^{a_1})^{\frac{b_1}{a_1}} (n_2^{a_2})^{\frac{b_2}{a_2}} \le (n_1^{a_1} n_2^{a_2})^{\frac{b_1}{a_1}}$ and

$$|h_{\overline{\alpha}}(n;\overline{a},\overline{b})| \le \sum_{n_1^{a_1} n_2^{a_2} = n} n_1^{b_1} n_2^{b_2} \tau_\alpha(n_1)\tau_\beta(n_2) \le n^{b_1/a_1} \sum_{n_1^{a_1} n_2^{a_2} = n} \tau_\alpha(n_1)\tau_\beta(n_2). \tag{4.1}$$

Therefore $|h_{\overline{\alpha}}(n;\overline{a},\overline{b})| \ll n^{\frac{b_1}{a_1}+\epsilon}$, $|h_{\overline{\alpha}}(n;\overline{a},\overline{0})| \ll n^{\epsilon}$ and we obtain the bound

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{|h_{\overline{\alpha}}(n;\overline{a},\overline{b})|}{n^{\sigma}} &\ll \sum_{n_1=1}^{\infty} \left(\frac{1}{n_1^{a_1\sigma - b_1}} \right)^{\alpha} \sum_{n_2=1}^{\infty} \left(\frac{1}{n_2^{a_2\sigma - b_2}} \right)^{\beta} \\ &\ll \frac{1}{(\sigma - \sigma_a)^{\gamma}}, \quad \sigma_a = \frac{1+b_1}{a_1}, \quad \gamma = \alpha + \beta. \end{aligned} \tag{4.2}$$

We denote

$$M(x; h_\alpha, r) = \frac{1}{\Gamma(r+1)} \sum_{n \le x} h_{\overline{\alpha}}(n;\overline{a},\overline{b}) \log^r(x/n).$$

Let T be large, in fact larger than the imaginary part of any of the singularities of $A(s;\overline{a},\overline{b})$. From (4.1) and (4.2) we can use Lemma 2 with $w=0$, $\psi(x)=x^{\frac{b_1}{a_1}+\epsilon}$, $\kappa=\frac{1+b_1}{a_1}+\frac{1}{\log x}$ to deduce that

$$M(x;h_\alpha,r)=\frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT}A(s;\overline{a},\overline{b})x^s s^{-r-1}ds+R(x;\overline{a},\overline{b}) \tag{4.3}$$

where for any fixed $\epsilon>0$,

$$R(x;\overline{a},\overline{b})=O(x^{\frac{1+b_1}{a_1}}T^{-r-1}(\log x)^{\alpha+\beta})+O(T^{-r-1}x^{1+\frac{b_1}{a_1}+\epsilon})+O(T^{-r}x^{\frac{b_1}{a_1}+\epsilon}).$$

Since $\frac{1+b_1}{a_1}\leq 1+\frac{b_1}{a_1}$, the first O-term can be omitted, and one obtains

$$R(x;\overline{a},\overline{b})=O(T^{-r-1}x^{1+\frac{b_1}{a_1}+\epsilon})+O(T^{-r}x^{\frac{b_1}{a_1}+\epsilon}).$$

Therefore if we take $T=x^\delta$ $(\delta>1)$, we have that $R(x;\overline{a},\overline{b})\ll T^{-r}x^{\frac{b_1}{a_1}+\epsilon}$.

In order to calculate the integral of (4.3), we insert the line segments L_2, L_3, L_4 to form a closed contour to the left of $L_1=[\kappa-iT,\kappa+iT]$. That is the rectangle R_T with vertices $-h\pm iT,\kappa\pm iT$, (for some $h>0$). By the residue theorem of Cauchy and (4.3) we have

$$\begin{aligned}M(x;h_\alpha,r)=&\,S_r(x;\overline{a},\overline{b})+R(x;\overline{a},\overline{b})-\\&-\frac{1}{2\pi i}\int_{L_2+L_3+L_4}A(s;\overline{a},\overline{b})x^s s^{-r-1}ds\end{aligned} \tag{4.4}$$

where the function $S_r(x;\overline{a},\overline{b})$ is a sum of the residues of $A(s;\overline{a},\overline{b})x^s s^{-r-1}$ for the singularities inside the rectangle R_T. L_2 is the line segment from $\kappa+iT$ to $-h+iT$, L_3 is the line segment from $-h+iT$ to $-h-iT$, and L_4 is the line segment from $-h-iT$ to $\kappa-iT$.

Estimation to the integral over L_2+L_4

$$\left|\frac{1}{2\pi i}\int_{L_2}A(s;\overline{a},\overline{b})x^s s^{-r-1}ds\right|\leq\frac{1}{2\pi}\int_{-h}^{\kappa}|A(\sigma+iT;\overline{a},\overline{b})|x^\sigma T^{-r-1}d\sigma.$$

From Lemma 4 we deduce

$$L^\alpha(a_1\sigma-b_1+ia_1T,\chi_1)\ll\begin{cases}{q_1}^\alpha|a_1T|^{\alpha(\frac{1}{2}-a_1\sigma+b_1)}\log^\alpha(q_1|a_1T|), & \text{if } \sigma\leq b_1/a_1.\\ {q_1}^\alpha|a_1T|^{(\frac{\alpha}{2}(1-a_1\sigma+b_1))}\log^\alpha(q_1|a_1T|), & \text{if } \frac{b_1}{a_1}\leq\sigma\leq\frac{1+b_1}{a_1}.\\ {q_1}^\alpha\log^\alpha(q_1|a_1T|), & \text{if } \sigma>\frac{1+b_1}{a_1}.\end{cases}$$

Similarly we have

$$L^\beta(a_2\sigma-b_2+ia_2T,\chi_2)\ll\begin{cases}{q_2}^\beta|a_2T|^{\beta(\frac{1}{2}-a_2\sigma+b_2)}\log^\beta(q_2|a_2T|), & \text{if } \sigma\leq b_2/a_2.\\ {q_2}^\beta|a_2T|^{(\frac{\beta}{2}(1-a_2\sigma+b_2))}\log^\beta(q_2|a_2T|), & \text{if } \frac{b_2}{a_2}\leq\sigma\leq\frac{1+b_2}{a_2}.\\ {q_2}^\beta\log^\beta(q_2|a_2T|), & \text{if } \sigma>\frac{1+b_2}{a_2}.\end{cases}$$

In order to estimate the integral over L_2+L_4 we need obtain bounds of $L^\alpha(a_1\sigma-b_1+ia_1T,\chi_1)L^\beta(a_2\sigma-b_2+ia_2T,\chi_2)$ over the differents intervals:

$$-h<0<\frac{b_2}{a_2}<\frac{b_1}{a_1}\le\frac{1+b_2}{a_2}\le\frac{1+b_1}{a_1}.$$

Let $F(T)=q_1{}^{\alpha}q_2{}^{\beta}\log^{\alpha}(q_1|a_1T|)\log^{\beta}(q_2|a_2T|)$, then we deduce the estimates:

(i) When $-h<\sigma\le\frac{b_2}{a_2}$ then

$$A(s;\overline{a},\overline{b})\ll F(T)|a_1T|^{\alpha[\frac{1}{2}+b_1]}|a_2T|^{\beta[\frac{1}{2}+b_2]}\left(|a_1T|^{-\alpha a_1}|a_2T|^{-\beta a_2}\right)^{\sigma}$$

and we deduce the estimates

$$\begin{aligned}I_1=&\frac{1}{2\pi}\left(\int_{-h}^{b_2/a_2}\right)|A(\sigma+iT;\overline{a},\overline{b})|\frac{x^{\sigma}}{T^{r+1}}d\sigma\\ \ll&F(T)|a_1T|^{\alpha(\frac{1}{2}+b_1)}|a_2T|^{\beta(\frac{1}{2}+b_2)}T^{-r-1}\int_{-h}^{b_2/a_2}\left(\frac{x}{|a_1T|^{\alpha a_1}|a_2T|^{a_2\beta}}\right)^{\sigma}d\sigma\\ \ll&x^{-h}T^{\frac{1}{2}(\alpha+\beta)+(b_1\alpha+b_2\beta)+(a_1\alpha+a_2\beta)h-r-1}\log^{\alpha+\beta}T\end{aligned}$$

(ii) When $\frac{b_2}{a_2}<\sigma\le\frac{b_1}{a_1}$ the argument is similar and since

$$A(s;\overline{a},\overline{b})\ll F(T)|a_1T|^{\alpha[\frac{1}{2}+b_1]}|a_2T|^{\beta[\frac{1}{2}(1+b_2)]}\left(|a_1T|^{-\alpha a_1}|a_2T|^{-\beta a_2/2}\right)^{\sigma}$$

then

$$\begin{aligned}I_2=&\frac{1}{2\pi}\int_{b_2/a_2}^{b_1/a_1}|A(\sigma+iT;\overline{a},\overline{b})|\frac{x^{\sigma}}{T^{r+1}}d\sigma\\ \ll&F(T)|a_1T|^{\alpha(\frac{1}{2}+b_1)}|a_2T|^{\beta(\frac{1+b_2}{2})}T^{-r-1}\int_{b_2/a_2}^{b_1/a_1}\left(\frac{x}{|a_1T|^{\alpha a_1}|a_2T|^{\frac{a_2\beta}{2}}}\right)^{\sigma}d\sigma\end{aligned}$$

hence we have

$$I_2\ll x^{\frac{b_2}{a_2}}T^{\frac{1}{2}(\alpha+\beta)+(b_1\alpha+b_2\beta)-(a_1\alpha+a_2\beta)\frac{b_2}{a_2}-r-1}\log^{\alpha+\beta}T$$

(iii) When $\frac{b_1}{a_1}<\sigma\le\frac{1+b_2}{a_2}(<\frac{1+b_1}{a_1})$

$$\begin{aligned}I_3=&\frac{1}{2\pi}\int_{b_1/a_1}^{(1+b_2)/a_2}|A(\sigma+iT;\overline{a},\overline{b})|\frac{x^{\sigma}}{T^{r+1}}d\sigma\\ \ll&F(T)|a_1T|^{\alpha(\frac{1+b_1}{2})}|a_2T|^{\beta(\frac{1+b_2}{2})}T^{-r-1}\int_{b_1/a_1}^{(1+b_2)/a_2}\left(\frac{x}{|a_1T|^{\frac{\alpha a_1}{2}}|a_2T|^{\frac{a_2\beta}{2}}}\right)^{\sigma}d\sigma\end{aligned}$$

analogously we obtain

$$I_3\ll x^{\frac{b_1}{a_1}}T^{\frac{1}{2}(\alpha+\beta)+\frac{1}{2}(b_1\alpha+b_2\beta)-(a_1\alpha+a_2\beta)\frac{b_1}{2a_1}-r-1}\log^{\alpha+\beta}T.$$

(iv) When $\frac{1+b_2}{a_2} < \sigma \leq \frac{1+b_1}{a_1}$ then the function $A(s;\overline{a},\overline{b})$ is

$$A(s;\overline{a},\overline{b}) \ll F(T)|a_1T|^{\alpha[\frac{1+b_1}{2}]}\left(|a_1T|^{-\alpha a_1/2}\right)^{\sigma}$$

and we deduce the integral I_4 in a similar manner to above cases.

(v) Finally, if $\sigma > \frac{1+b_1}{a_1}$ we obtain $A(s;\overline{a},\overline{b}) \ll F(T)$ and it follows

$$I_5 \ll x^{\frac{1+b_1}{a_1}}T^{-r-1}\log^{\alpha+\beta}T\log^{-1}x \ll R(x;\overline{a},\overline{b}). \tag{4.5}$$

Thus, from the above estimates we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{L_2+L_4} & A(s;\overline{a},\overline{b})x^s s^{-r-1}ds \\ & \ll x^{-h}T^{\frac{\alpha+\beta}{2}+(\alpha b_1+\beta b_2)+h(\alpha a_1+\beta a_2)-r-1}\log^{\alpha+\beta}T \\ & +x^{\frac{b_2}{a_2}}T^{\frac{\alpha+\beta}{2}+(\alpha b_1+\beta b_2)-(\alpha a_1+\beta a_2)\frac{b_2}{a_2}-r-1}\log^{\alpha+\beta}T \\ & +x^{\frac{b_1}{a_1}}T^{\frac{\alpha+\beta}{2}+\alpha b_1+\frac{\beta b_2}{2}-(\alpha a_1+\frac{\beta a_2}{2})\frac{b1}{a_1}-r-1}\log^{\alpha+\beta}T\end{aligned} \tag{4.6}$$

From (4.4), (4.5) and (4.6) we obtain

$$\begin{aligned}M(x;h_\alpha,r) =& S_r(x;\overline{a},\overline{b}) + R(x;\overline{a},\overline{b}) \\ & +O(x^{-h}T^{\frac{\alpha+\beta}{2}+(\alpha b_1+\beta b_2)+h(\alpha a_1+\beta a_2)-r-1}\log^{\alpha+\beta}T) \\ & +O(x^{\frac{b_2}{a_2}}T^{\frac{\alpha+\beta}{2}+(\alpha b_1+\beta b_2)-(\alpha a_1+\beta a_2)\frac{b_2}{a_2}-r-1}\log^{\alpha+\beta}T) \\ & +O(x^{\frac{b_1}{a_1}}T^{\frac{\alpha+\beta}{2}+\alpha b_1+\frac{\beta b_2}{2}-(\alpha a_1+\frac{\beta a_2}{2})\frac{b1}{a_1}-r-1}\log^{\alpha+\beta}T) \\ & -\frac{1}{2\pi i}\int_{L_3} A(s;\overline{a},\overline{b})x^s s^{-r-1}ds.\end{aligned} \tag{4.7}$$

Estimation of integral over L_3. Over the line segment $[-h \pm iT]$ we can write

$$\begin{aligned}\frac{1}{2\pi i}\int_{-h-iT}^{-h+iT} & A(s;\overline{a},\overline{b})x^s s^{-r-1}ds = \\ & = \frac{1}{2\pi i}\int_{-h-iT}^{-h+iT} L^{\alpha}(a_1s-b_1,\chi_1)L^{\beta}(a_2s-b_2,\chi_2)x^s s^{-r-1}ds \\ & = \frac{1}{2\pi}\int_{-T}^{T}\frac{L^{\alpha}(a_1(-h+it)-b_1,\chi_1)L^{\beta}(a_2(-h+it)-b_2,\chi_2)x^{-h+it}}{(-h+it)^{r+1}}dt.\end{aligned}$$

Then an application of the functional equation given in Lemma 3 shows that

$$\frac{1}{2\pi i}\int_{-h-iT}^{-h+iT} A(s;\overline{a},\overline{b})x^s s^{-r-1}ds = \frac{i^{\alpha+\beta}}{2\pi}{q_1}^{\alpha(a_1h+b_1)}{q_2}^{\beta(a_2h+b_2)}x^{-h}\times$$
$$\times\sum_{N_1=1}^{\infty} G^*(N_1,\chi_1,\alpha)N_1^{-(1+b_1+a_1h)}\sum_{N_2=1}^{\infty} G^*(N_2,\chi_2,\beta)N_2^{-(1+b_2+a_2h)}$$
$$\times\int_{-T}^{T} H^{\alpha}(-a_1h-b_1+ia_1t,\chi_1)H^{\beta}(-a_2h-b_2+ia_2t,\chi_2)$$
$$\times\left(\frac{N_1^{a_1}N_2^{a_2}x}{q_1^{\alpha a_1}q_2^{\beta a_2}}\right)^{it}\frac{dt}{(-h+it)^{r+1}}. \tag{4.8}$$

From (3.6) we obtain

$$H^{\alpha}(-a_1h-b_1+ia_1t,\chi_1)H^{\beta}(-a_2h-b_2+ia_2t,\chi_2)=$$
$$=Ce^{-i(a_1\alpha+a_2\beta)t\log t+i(a_1\alpha\log(2\pi e/a_1)+a_2\beta\log(2\pi e/a_2))t}t^{(1/2+a_1h+b_1)\alpha}\times$$
$$\times t^{(1/2+a_2h+b_2)\beta}\{1+O(|t|^{-1})\}. \tag{4.9}$$

Furthermore

$$(-h+it)^{-r-1}=(it)^{-r-1}+O(|t|^{-r-2}),\quad (|t|>h) \tag{4.10}$$

Let us suppose $T_0>0$ enough large. Now, we will divide the integral $\int_{-T}^{T}$ as the sum

$$\int_{-T}^{T}=\int_{-T}^{-T_0}+\int_{-T_0}^{T_0}+\int_{T_0}^{T}.$$

Obviously, $\int_{-T_0}^{T_0}$ is a constant, and to estimate the other two integrals is sufficient to estimate one of them. Using (4.9) and (4.10) we have

$$\int_{T_0}^{T}\frac{H^{\alpha}(-a_1h-b_1+ia_1t,\chi_1)H^{\beta}(-a_2h-b_2+ia_2t,\chi_2)}{(-h+it)^{r+1}}\left(\frac{N_1^{a_1}N_2^{a_2}x}{q_1^{\alpha a_1}q_2^{\beta a_2}}\right)^{it}dt=$$
$$=C'\int_{T_0}^{T}\varphi(t)e(f(t)+Kt)dt+O\left(\int_{T_0}^{T}(\varphi(t)/t)dt\right) \tag{4.11}$$

C' being a constant, $e(\alpha)=e^{2\pi i\alpha}$,

$$\varphi(t)=t^{(1/2)(\alpha+\beta)+h(a_1\alpha+a_2\beta)+b_1\alpha+b_2\beta-r-1}$$
$$f(t)=-\frac{1}{2\pi}(a_1\alpha+a_2\beta)t\log t$$
$$K=\frac{1}{2\pi}\left(a_1\alpha\log(\frac{2\pi e}{a_1})+a_2\beta\log(\frac{2\pi e}{a_2})+\log\left(\frac{N_1^{a_1}N_2^{a_2}x}{q_1^{\alpha a_1}q_2^{\beta a_2}}\right)\right).$$

We take $h=\frac{2r+1-2(b_1\alpha+b_2\beta)-\alpha-\beta}{2(a_1\alpha+a_2\beta)}>0$, and obtain $\varphi(t)=t^{-1/2}$. To estimate the exponential integral of (4.11) we broke the integration interval

$$[T_0,T]=[T_0,t_0/2]\cup[t_0/2,3t_0/2]\cup[3t_0/2,T]$$

that is

$$\int_{T_0}^{T} \varphi(t)e(f(t)+Kt)dt =$$
$$= \left(\int_{T_0}^{t_0/2} + \int_{t_0/2}^{3t_0/2} + \int_{3t_0/2}^{T}\right)\varphi(t)e(f(t)+Kt)dt = J_1 + J_2 + J_3$$

being $f'(t_0) + K = 0$, it is

$$t_0 = (2\pi)\left(\frac{N_1^{a_1}N_2^{a_2}x}{(a_1q_1)^{\alpha a_1}(a_2q_2)^{\beta a_2}}\right)^{1/(a_1\alpha+a_2\beta)}.$$

Therefore

$$t_0 \ll x^{\frac{1}{a_1\alpha+a_2\beta}}.$$

As

$$f''(t) = -\frac{a_1\alpha + a_2\beta}{2\pi t} < 0$$

we can use the lemmas 5.1.2 and 5.1.3 of [8], so

$$J_1 \ll 1, \quad J_2 \ll 1 \quad J_3 \ll x^{-\frac{1}{2(a_1\alpha+a_2\beta)}}.$$

Then the integral (4.11) is bounded

$$\int_{T_0}^{T} \frac{H^{\alpha}(-a_1h - b_1 + ia_1t, \chi_1)H^{\beta}(-a_2h - b_2 + ia_2t, \chi_2)}{(-h+it)^{r+1}}\left(\frac{N_1^{a_1}N_2^{a_2}x}{q_1^{\alpha a_1}q_2^{\beta a_2}}\right)^{it} dt \ll 1.$$

Therefore we obtain the following estimate

$$\frac{1}{2\pi i}\int_{L_3} A(s;\overline{a},\overline{b})x^s s^{-r-1}ds \ll x^{-h}. \tag{4.12}$$

If we choose $T = x^{1+2(h+\frac{b_1}{a_1})}$, we obtain that the order of each O-term of (4.7) is smaller than x^{-h}. Therefore

$$M(x;h_\alpha,r) = S_r(x;\overline{a},\overline{b}) + O(x^{\frac{\alpha+\beta}{2(a_1\alpha+a_2\beta)} - \frac{r-b_1\alpha-b_2\beta+1/2}{a_1\alpha+a_2\beta}}).$$

Hence (2.1) holds. If r is an integer, then $\Gamma(r+1) = r!$ and we obtain (2.2) and (2.3). The theorem is proved. ■

Corollary 1. *Let* $\frac{1+b_2}{a_2} < \frac{1+b_1}{a_1}$, *then*

$$\sum_{n_1^{a_1}n_2^{a_2}\le x} n_1^{b_1}n_2^{b_2}\chi_1(n_1)\chi_2(n_2)\log^r(x/n_1^{a_1}n_2^{a_2}) = r!K_1\eta(\chi_1)x^{\frac{1+b_1}{a_1}} +$$
$$+ r!K_2\eta(\chi_2)x^{\frac{1+b_2}{a_2}} + r!P_r(\log x) + O(x^{-\frac{r-1/2-b_1-b_2}{a_1+a_2}}) \tag{4.13}$$

$$\sum_{n_1^{a_1}n_2^{a_2}\le x} n_1^{b_1}n_2^{b_2}\log^r(x/n_1^{a_1}n_2^{a_2}) = r!K_1x^{\frac{1+b_1}{a_1}}+$$
$$+r!K_2x^{\frac{1+b_2}{a_2}}+r!P_r(\log x)+O(x^{-\frac{r-1/2-b_1-b_2}{a_1+a_2}}). \tag{4.14}$$

Proof. Taking $\alpha=\beta=1$, and using (2.2) we obtain the asymptotic formula (4.13). Note that in this case the corresponding polynomial $P_{\alpha-1}(\log x)$, $P_{\beta-1}(\log x)$, are constants, K_1, K_2.

If $\chi_1=\chi_2\equiv 1$, $q_1=q_2=1$, then $\eta(\chi_1)=\eta(\chi_2)=1$, we obtain the asymptotic formula (4.14).

If $b_1=b_2=0$ then

$$h_{\overline{1}}(n;\overline{a},\overline{0}) = d(n;a_1,a_2) = \#\{(n_1,n_2)\in\mathbb{N}\times\mathbb{N}; n_1^{a_1}n_2^{a_2}=n\}$$

and we can deduce the corresponding asymptotic formula. ∎

Corollary 2. *Let $\alpha,\beta\ge 2$, $\chi_1=\chi_2\equiv 1$, $q_1=q_2=1$ and $\frac{1+b_2}{a_2}<\frac{1+b_1}{a_1}$, then*

$$\sum_{n_1^{a_1}n_2^{a_2}\le x} n_1^{b_1}n_2^{b_2}\tau_\alpha(n_1)\tau_\beta(n_2)\log^r(x/n) = r!x^{\frac{1+b_1}{a_1}}P_{\alpha-1}(\log x)+$$
$$+r!x^{\frac{1+b_2}{a_2}}P_{\beta-1}(\log x)+r!P_r(\log x)+O(x^{-\frac{2r+1-2(b_1\alpha+b_2\beta)-\alpha-\beta}{2(a_1\alpha+a_2\beta)}}). \tag{4.15}$$

Proof. Is a consequence of (2.2) ∎

Corollary 3. *Let $\frac{1+b_2}{a_2}=\frac{1+b_1}{a_1}$, $\alpha=\beta=2$, and $k=a_2b_1-a_1b_2$, then we have*

$$\sum_{n_1^{a_1}n_2^{a_2}\le x} n_1^{b_1}n_2^{b_2}d(n_1)d(n_2)\chi_1(n_1)\chi_2(n_2)\log^r(\frac{x}{n}) = r!\theta_{12}x^{\frac{1+b_2}{a_2}}P_{\lambda_{12}}(\log x)+$$
$$+r!P_r(\log x)+O(x^{-\frac{r+1/2-2b_1-2b_2-2}{2((a_2+k)+a_2)}}). \tag{4.16}$$

Proof. If $\alpha=\beta=2$, then $\tau_\alpha(n)=\tau_\beta(n)=d(n)$ and using (2.3) we deduce the asymptotic formula (4.16). ∎

5 Application to Farey sequence

Let F_N be a Farey sequence of order N, that is

$$F_N=\{\frac{b}{a}\mid b\in\mathbb{Z}, a\in\mathbb{N}, (b,a)=1, 0\le\frac{b}{a}\le 1, 1\le a\le N\}$$

where every fraction is arranged in non-decreasing order. From theorem of Farey-Cauchy we know that : If $\frac{b_1}{a_1}$ is the immediately successor of $\frac{b_2}{a_2}$ in the Farey sequence F_N, then $k=a_2b_1-a_1b_2=1$. We consider

$$\frac{b_2}{a_2}<\frac{b_1}{a_1}\le\frac{1+b_2}{a_2}\le\frac{1+b_1}{a_1},\quad \text{with } 1\le a_1\le a_2+1,\ k=a_2b_1-a_1b_2=1.$$

We take the case: $\alpha = \beta = 2$, that is, $\tau_\alpha(n) = \tau_\beta(n) = d(n)$ and $\frac{b_1}{a_1}, \frac{b_2}{a_2} \in F_7$. where

$$F_7 = \left\{ \frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1} \right\}.$$

(A). The consecutive fractions $\frac{b_2}{a_2} < \frac{b_1}{a_1}$, with the condition $a_1 = a_2 + 1$ immediately imply that

$$\frac{1+b_2}{a_2} = \frac{1+b_1}{a_1}$$

for example the fractions (a) $(\frac{4}{5}, \frac{5}{6})$, and (b) $(\frac{5}{6}, \frac{6}{7})$,

a) In the case $(\frac{4}{5}, \frac{5}{6})$, we have $A(s;\overline{a},\overline{b}) = L^2(6s-5,\chi_1)L^2(5s-4,\chi_2)$

$$h_{(2,2)}(n;(6,5),(5,4)) = \sum_{n_1^6 n_2^5 = n} n_1^5 n_2^4 d(n_1)d(n_2)\chi_1(n_1)\chi_2(n_2).$$

The asymptotic formula is a particular case of (4.16) when $\overline{a} = (6,5)$ and $\overline{b} = (5,4)$.

b) In the case $(\frac{5}{6}, \frac{6}{7})$, we have $A(s;\overline{a},\overline{b}) = L^2(7s-6,\chi_1)L^2(6s-5,\chi_2)$ then

$$h_{(2,2)}(n;(7,6),(6,5)) = \sum_{n_1^7 n_2^6 = n} n_1^6 n_2^5 d(n_1)d(n_2)\chi_1(n_1)\chi_2(n_2).$$

As in the above case, we obtain the asymptotic formula from (4.16).

(B). For the other fractions, we have the conditions

$$\frac{b_2}{a_2} < \frac{b_1}{a_1} < \frac{1+b_2}{a_2} < \frac{1+b_1}{a_1}, \quad 1 < a_1 < a_2 + 1$$

and we obtain the asymptotic formula sustituying $\overline{a}, \overline{b}$ in the corresponding cases. We consider, for example, the following consecutive fractions of F_7:

i) Let $\frac{b_2}{a_2} = \frac{2}{7}, \frac{b_1}{a_1} = \frac{1}{3}$; then $A(s;\overline{a},\overline{b}) = L^2(3s-1,\chi_1)L^2(7s-2,\chi_2)$. The coefficient of its Dirichlet series is

$$h_{(2,2)}(n;(3,7),(1,2)) = \sum_{n_1^3 n_2^7 = n} n_1 n_2^2 d(n_1)d(n_2)\chi_1(n_1)\chi_2(n_2)$$

ii) If $(\frac{b_2}{a_2}, \frac{b_1}{a_1}) = (\frac{3}{7}, \frac{1}{2})$, then $A(s;\overline{a},\overline{b}) = L^2(2s-1,\chi_1)L^2(7s-3,\chi_2)$, and

$$h_{(2,2)}(n;(2,7),(1,3)) = \sum_{n_1^2 n_2^7 = n} n_1 n_2^3 d(n_1)d(n_2)\chi_1(n_1)\chi_2(n_2).$$

iii) For $(\frac{4}{7}, \frac{3}{5})$; we have $A(s;\overline{a},\overline{b}) = L^2(5s-3,\chi_1)L^2(7s-4,\chi_2)$, and

$$h_{(2,2)}(n;(5,7),(3,4)) = \sum_{n_1^5 n_2^7 = n} n_1^3 n_2^4 d(n_1)d(n_2)\chi_1(n_1)\chi_2(n_2).$$

iv) If $(\frac{3}{5}, \frac{2}{3})$; then $A(s; \overline{a}, \overline{b}) = L^2(3s-2, \chi_1)L^2(5s-3, \chi_2)$, and the function is

$$h_{(2,2)}(n; (3,5), (2,3)) = \sum_{n_1^3 n_2^5 = n} n_1^2 n_2^3 d(n_1)d(n_2)\chi_1(n_1)\chi_2(n_2).$$

v) For $(\frac{5}{7}, \frac{3}{4})$ we have $A(s; \overline{a}, \overline{b}) = L^2(4s-3, \chi_1)L^2(7s-5, \chi_2)$, then

$$h_{(2,2)}(n; (4,7), (3,5)) = \sum_{n_1^4 n_2^7 = n} n_1^3 n_2^5 d(n_1)d(n_2)\chi_1(n_1)\chi_2(n_2).$$

Using (2.2) or (4.15) with $\alpha = \beta = 2$, we obtain the asymptotic formula for the above functions.

References

[1] Apostol,T.M, *Dirichlet L-functions and character powers sums.* J. of Number Theory 2, 223-234 (1970).

[2] Ayoub,R. *An introduction to the analytic theory of numbers.* Mathematical surveys, Number 10, American Mathematical Society, Providence, R. I. 1964.

[3] Berndt, Bruce C, *Identities involving the coefficients of a class of Dirichlet series I,II, III* Trans. Amer. Math. Soc. 137, (1969) 345-359, ibid. (1969), 361-374 and 146, (1969), 323-348.

[4] Chandrasekharan, K. *Introduction to analytic number theory.* Springer-Verlag, Berlin Heidelberg New York, 1968.

[5] Chandrasekharan, K. *Arithmetical functions.* Springer-Verlag, Berlin Heidelberg New York, 1970.

[6] Copson, E.T, *An introduction to the theory of functions of a complex variable.* Clarendon Press, Oxford, (1970).

[7] Grosswald, E. *Topics from the theory of numbers.* Birhkauser, Boston Basel Stuttgart. 1984.

[8] Huxley, M.N. *Area, lattice points and exponential sums.* London Mathematical Society Monographs. New Series, 13. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1996.

[9] Ivić, A, *The Riemann Zeta-Function. Theory and applications.* J.Wiley and Sons. New York 1985. Dover Publications, Inc., Mineola, NY, 2003.

[10] Ivić, A. and Meurman, T, *Sums of coefficients of Hecke series.* Acta Arithmetica LXVIII.4 (1994), 341-367.

[11] Prachar, K, *Primzahlverteilung.* Springer-Verlag. Berlin-Heidelberg-New York 1957.

[12] Titchmarsh, E.C. *The theory of the Riemann zeta function (Revised by D.R. Heath-Brown).* Clarendon Press Oxford 1986.

[13] Vorhauer, U.M.A., *Three two-dimensional Weyl steps in the circle problem. II. The logarithmic Riesz mean for a class of arithmetic functions.* Acta-Arith. 91 (1) (1999), 57-73.

[14] Vorhauer, U.M.A., Wirsing, E *Three two-dimensional Weyl steps in the circle problem. III. Exponential integrals and application.* [Proceedings Paper] Number theory in progress, Vol. 2 (Zakopane-Koscielisko, 1997). de Gruyter, Berlin. (1999). 1131-1146.

[15] Whitaker, E.T. and Watson, G.N. *A course of modern analysis.* Cambridge 1963.