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Nonsymmetric metric tensor and anticommutative geometry

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Abstract

In the framework of nonsymmetric gravitational theories we consider the equations of motion for matter fields. It is found that the antisymmetric part of the metric is the Pauli matrix in 4 dimensions, suggesting a possible deep relation between spin and geometry. Some arguments about the possibility of building a fermionic space-time instead the ordinary bosonic space-time are discussed.

1 Introduction

The possible extensions of General Relativity is a subject that has experimented a lot of different theoretical approximations since the formulation of the general theory in 1915. For example: Einstein, Schrodinger, Weyl, and many others [1]–[5], tried to unify electromagnetism and gravity using a formalism which defined both, a metric tensor and an affine connection that were non symmetric. This theory, despite its high degree of mathematical elegance did not work, and failed in the attempt to recover some classical results like the Lorentz Force. Years before the formulation of Einstein-Schrodinger theory, Cartan [6, 7], studied how to extend general relativity in order to incorporate torsion(the antisymmetric part of the affine connection). His efforts yields to the conception of space-time with curvature and torsion, unlike usual General Relativity where torsion is zero.

More recently, other physicists like John Moffat [8] have studied in detail the field equations of general theories based in nonsymmetric metric tensors. We will accept this theoretical framework as our starting point, with the aim of investigate the implications for particle physics. In particular, we want to study how the wave equations of the matter fields will be affected by the addition of a non symmetric contribution in the metric tensor. The mathematical discussion that follows provides the result that allow us to identify the Pauli matrix in 4 dimensions with the antisymmetric part of the metric.

2 The line element in General Relativity. Bosonic space-time and fermionic space-time

We begin with the standard definition for the line element in General Relativity:

$$ds^2 = g_{\alpha\beta}(x) dx^\alpha dx^\beta, \tag{1}$$

where the metric tensor is regarded as symmetric.Note that if we add a skew symmetric contribution to the metric tensor, $w_{\alpha\beta}{=}-w_{\beta\alpha}$, the line element remains unchanged due to a simple fact: usual space-time coordinates commute. In such sense, they are bosonic coordinates and represent bosonic degrees of freedom:

$$ds^2 = (g_{\alpha\beta} + w_{\alpha\beta}) dx^\alpha dx^\beta = g_{\alpha\beta} dx^\alpha dx^\beta, \tag{2}$$

$$[x^\alpha, x^\beta] = 0 \Longrightarrow dx^\alpha dx^\beta = dx^\beta dx^\alpha. \tag{3}$$

For this reason the term $w_{\alpha\beta} dx^\alpha dx^\beta$ vanishes identically.

We want to generalize (1) for a more general space-time configurations. Let us investigate how the line element in (1) can be extended in situations where the metric tensor is not generally symmetric, and where coordinates do not generally commute. Let us assume the hypothesis that this general expression of the line element is preserved:

$$\begin{aligned} ds^2 = G_{\alpha\beta} dx^\alpha dx^\beta &= (g_{\alpha\beta} + w_{\alpha\beta})(\frac{1}{2}\{dx^\alpha, dx^\beta\} + \frac{1}{2}[dx^\alpha, dx^\beta]) \\ &= \frac{1}{2} g_{\alpha\beta}\{dx^\alpha, dx^\beta\} + \frac{1}{2} w_{\alpha\beta}[dx^\alpha, dx^\beta], \end{aligned} \tag{4}$$

where we have made the following decomposition:

$$dx^\alpha dx^\beta = \frac{1}{2}\{dx^\alpha, dx^\beta\} + \frac{1}{2}[dx^\alpha, dx^\beta] \tag{5}$$

and where the nonsymmetric metric $G_{\alpha\beta}$ consists in the sum of two contributions:

$$G_{\alpha_\beta} = g_{\alpha\beta} + w_{\alpha\beta}, \tag{6}$$

with

$$g_{\alpha\beta} = \tfrac{1}{2}(G_{\alpha\beta} + G_{\beta\alpha}), \tag{7}$$

$$w_{\alpha\beta} = \tfrac{1}{2}(G_{\alpha\beta} - G_{\beta\alpha}). \tag{8}$$

The contravariant tensor $G^{\alpha\beta}$ is defined in terms of the equation

$$G^{\mu\nu} G_{\sigma\nu} = \delta^\mu_\sigma. \tag{9}$$

For usual commutative geometry and bosonic space-time :

$$[dx^\alpha, dx^\beta] \quad = 0, \tag{10}$$

$$\{dx^\alpha, dx^\beta\} \quad = 2dx^\alpha dx^\beta. \tag{11}$$

In this case, the second term in the right side of (4) vanishes, and we recover the usual expression (1), for the line element in General Relativity. But it is more interesting analys what would happen if we consider a fermionic configuration of space-time. This means that at each point exists a chart of coordinates that are Grassman numbers and verify the following relations:

$$\{d\theta^\alpha, d\theta^\beta\} \quad = 0, \tag{12}$$

$$[d\theta^\alpha, d\theta^\beta] \quad = 2d\theta^\alpha d\theta^\beta. \tag{13}$$

Under these conditions, the general line element (4) becomes:

$$ds^2 = G_{\alpha\beta}dx^\alpha dx^\beta = w_{\alpha\beta}d\theta^\alpha d\theta^\beta. \tag{14}$$

Automatically, it arises the question: What is the physical meaning of this construction? Does a fermionic space-time make sense after all? General Relativity is a theory formulated in a purely bosonic space-time where geometry is widely regarded as commutative. Meanwhile, Grassman variables represent fermions, and are present in the path formulation of fermionic fields in quantum field theory. Besides, exist in supersymmetry the superspace where bosonic coordinates are completed with Grassmann numbers, but in a framework where the metric is considered like a symmetric tensor. Given our hypothesis of a general metric with a decomposition in a symmetric and antisymmetric tensors, It seems that the last relations suggest that an unusual type of fermionic fields could be able to feel the antisymmetric part, while bosons and the other ordinary fermions only couple to the symmetric part. Despite the beauty and symmetry of this approach, we will show in the next section that if we accept the possibility of (12) and (13), the formalism leads to the existence of tachyons.

3 The metric tensor and the spin of the particles

In Minkowski space-time, a symmetric metric tensor given with signature $(+, -, -, -)$ we have the Casimir

$$P^\mu P_\mu = g_{\mu\nu}P^\mu P^\nu = m^2. \tag{15}$$

By application of the correspondence principle $P_\mu \to i\partial_\mu$ we obtain the free Klein Gordon wave equation

$$(g_{\mu\nu}\partial^\mu\partial^\nu + m^2)\phi(x) = 0. \tag{16}$$

Again, we make now the following observation: in the case free, if we add an antisymmetric field $w_{\mu\nu}$ to the metric tensor in (15) the Casimir is not affected by this addition, because $P^\mu P^\nu$ is a purely symmetric object describing bosonic matter. However, when we have interactions, the correspondence principle is modified by inserting covariant derivatives $P^\alpha \to iD^\alpha$ instead of usual derivatives, $P^\alpha \to i\partial^\alpha$. This substitution has the effect of changing the symmetry of $P^\alpha P^\beta$, because $D^\alpha D^\beta$ no longer commutes, and this will generate an additional term involving the antisymmetric part of the metric.

To show this in detail, let us write the Klein-Gordon equation in general curved space-time:

$$(G^{\alpha\beta}(x)D_\alpha D_\beta + m^2)\phi(x) = [(g^{\alpha\beta} + w^{\alpha\beta})(\frac{1}{2}\{D_\alpha, D_\beta\} + \frac{1}{2}[D_\alpha, D_\beta]) + m^2]\phi(x) = 0. \quad (17)$$

By a direct computation of the product in the last equation we obtain:

$$(\frac{1}{2}g^{\alpha\beta}\{D_\alpha, D_\beta\} + \frac{1}{2}w^{\alpha\beta}[D_\alpha, D_\beta] + m^2)\phi(x) = 0. \quad (18)$$

Straightforward manipulations show that the commutator of the covariant derivative can be written as:

$$[D_\alpha, D_\beta]\phi(x) = -(\Gamma^c_{\alpha\beta} - \Gamma^c_{\beta\alpha})\partial_c\phi(x). \quad (19)$$

General Relativity is torsion- free and this means that the Levi-Civita connection is symmetric. In these conditions the last commutator vanishes. Nevertheless, in our analysis this term gives an additional contribution that we shall bear in mind.

Similarly, it can be found an expression for the anti-commutator of the covariant derivatives, but involving the symmetric part of the affine connection:

$$\{D_\alpha, D_\beta\}\phi(x) = 2\partial_\alpha\partial_\beta\phi(x) - (\Gamma^c_{\alpha\beta} + \Gamma^c_{\beta\alpha})\partial_c\phi(x). \quad (20)$$

Eqs. (19) and (20) can be used to write the compact expression for the Klein-Gordon field in curved nonsymmetric space-time. Replacing these relations in (18) we find after straightforward calculations:

$$(g^{\alpha\beta}\partial_\alpha\partial_\beta + m^2)\phi(x) = G^{\alpha\beta}\Gamma^c_{\alpha\beta}\partial_c\phi(x), \quad (21)$$

where

$$\Gamma^c_{\alpha\beta} = \frac{1}{2}(\Gamma^c_{\alpha\beta} + \Gamma^c_{\beta\alpha}) + \frac{1}{2}(\Gamma^c_{\alpha\beta} - \Gamma^c_{\beta\alpha}). \quad (22)$$

The left side of (21) is identical to the corresponding Klein-Gordon equation in General Relativity. The difference lies in the right side: now the metric and the affine connection are not symmetric, but it is worth to note that the form of the equation remains the same.

What about Dirac fields? The explicit and detailed treatment of Dirac fields in a general curved space-time is a much more complicated task (see for instance [9]), but we only want to take a general picture in order to inquire some aspects of the field $w_{\alpha\beta}$. For this reason, we will not solve the covariant derivative over Dirac fields. We will limit to the task of requiring that the Dirac equation could be expressed as the square root of the Klein-Gordon field.

The Dirac equation in curved space-time can be written as

$$(i\gamma^{\alpha} D_{\alpha} - m)\psi(x) = 0, \tag{23}$$

where $D_{\alpha} = \partial_{\alpha} + \Gamma_{\alpha}$. As we have said before, we make the assumption that the Dirac field in general curved space-time can be expressed as the square root of the Klein-Gordon equation. This allows us to write

$$(-i\gamma^{\alpha} D_{\alpha} - m)(i\gamma^{\beta} D_{\beta} - m)\psi(x) = 0. \tag{24}$$

If we assume $D_{\alpha}\gamma^{\beta} = 0$, which seems a plausible generalization of the condition $\partial_{\alpha}\gamma^{\beta} = 0$ that is verified by the Dirac matrices in a flat space-time, we find

$$(\gamma^{\alpha}\gamma^{\beta} D_{\alpha} D_{\beta} + m^2)\psi(x) = 0. \tag{25}$$

This is nothing but the Klein-Gordon equation in general nonsymmetric curved space-time (17). Thus, we can make the identification

$$\gamma^{\alpha}\gamma^{\beta} = \frac{1}{2}\{\gamma^{\alpha}, \gamma^{\beta}\} + \frac{1}{2}[\gamma^{\alpha}, \gamma^{\beta}] = G^{\alpha\beta} = g^{\alpha\beta} + w^{\alpha\beta}. \tag{26}$$

That provides

$$g^{\alpha\beta} = \frac{1}{2}\{\gamma^{\alpha}, \gamma^{\beta}\}, \tag{27}$$

$$w^{\alpha\beta} = \frac{1}{2}[\gamma^{\alpha}, \gamma^{\beta}]. \tag{28}$$

Equation (27) is a well known result that remit us to the field of Clifford algebra. On the other hand, the commutator of the Dirac matrices transforms as a tensor, and is a clue concept to understand the behavior of the Dirac field under general Lorentz transformations. We suggest a new interpretation of this tensor in the framework of nonsymmetric space-time, where the metric tensor has an antisymmetric part.

With these results in mind let us return to the previous section where we studied the notion of a fermionic space-time of Grassmann coordinates, that couple to the antisymmetric part of the metric tensor in the definition of the general line element(4). Let us begin writing the left side of the Casimir invariant (15), in a flat space-time doted with a nonsymmetric metric $G^{\alpha\beta} = \gamma^{\alpha}\gamma^{\beta}$. Then we have

$$G^{\alpha\beta} p_{\alpha} p_{\beta} = \gamma^{\alpha}\gamma^{\beta} p_{\alpha} p_{\beta}. \tag{29}$$

Making contact now with (12) and (13), for fermionic degrees of freedom

$$\{p_\alpha, p_\beta\} = p_\alpha p_\beta + p_\beta p_\alpha = 0. \tag{30}$$

This allow us to make the substitution: $p_\alpha p_\beta = -p_\beta p_\alpha$ in equation (29) and we have

$$-\gamma^\alpha \gamma^\beta p_\beta p_\alpha. \tag{31}$$

Note that $\gamma^\beta p_\beta$ is nothing but the Dirac equation in momentum space: $\gamma^\beta p_\beta \psi = m\psi$. Therefore, equation (29) provides after a direct computation a global term of $-m^2$, and this means that we are dealing with tachyons.

If we repeat the same reasoning for bosonic commutative variables, the result gives the correct sign for the Casimir invariant.

This result is intriguing. It likely means that we are not allowed to describe usual fermions with anticommutative variables in the external space-time, but only in their own internal vectorial space. Maybe this result is telling us something about tachyons. Tachyons would be Grassman fields that behave in the space-time being able to feel the antisymmetric part of the metric tensor, undetectable for us and the other ordinary matter. Indeed, bosons and the usual fermions that are represented by Grassmann variables in their internal spinor space, are all associated with standard bosonic coordinates when they move in the space-time.

In any case, this last result only questions the assumptions of the equations (12) and (13), but says nothing about the validity of nonsymmetrical gravitational theories.

4 Discussion

In this paper we have explored an alternative approach which combines some insights of nonsymmetrical gravitational theories with concepts of noncommutative geometry. In this approach, we have discussed that the inclusion of an antisymmetric part in the metric tensor has some interesting consequences when is considered the possibility of extend the conception of bosonic space-time with coordinates that do not commute. We have postulated a generalization for the invariant line element in General relativity, which puts bosonic and fermionic coordinates on an equal footing. Bosonic degrees of freedom couple to the symmetric part of the metric, while unusual fermionic degrees of freedom would do the same but with the other part.

Nevertheless, these assumptions lead to the wrong sign for the square of the mass in the Casimir invariant, which means that tachyons arise inevitably when we describe these unusual type of fermions. In other words: Grassman coordinates in a nonsymmetric space-time represent degrees of freedom that behave like tachyons.

On the other hand, it has been studied in some detail the wave equations for matter fields in the framework of nonsymmetric gravitational theories, suggesting a possible new interpretation for the commutator of the Dirac matrices, which emerges naturally as the antisymmetric part of the metric tensor.

Until now, it has not been found any experimental evidence of the nonsymmetric nature of the metric tensor. But we point out that if these nonsymmetric theories of gravity are finally found to be a correct description of nature, then such identification of the antisymmetric part of the metric tensor with the commutator of the gamma matrices could be naturally established as a theoretical consequence.

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