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Formalizing Antinomic Terms and Predicates

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1 Preliminaries

Circularity has been an important and vexing logical subject from the time of the Ancient Greeks on. A distinction has always been made between (i) a circular argument and (ii) a circular definition. An argument, of course, is a proof, a proof being an arrangement of statements in sequence or in a tree pattern following definite rules of deduction. A circular argument is one in which the conclusion of the argument is used as a premise in the proof, a fallacious reasoning to be discarded. A circular definition in turn is one in which the concept $\boldsymbol{A}$ is defined in terms of a concept $\boldsymbol{B}$, and then $\boldsymbol{B}$ defined in terms of $\boldsymbol{A}$. Most authorities have thought to this day that circular definitions are also to be discarded, although this has proved to be not always possible. Generally, though, the almost universal agreement has been to consider both kinds of circularity as logical defects. An extreme example of such position is that of Sextus Empiricus who claimed that all definitions, circular or not, are useless because they end up in an unavoidable regressus ad infinitum, hence his total scepticism concerning the viability of logic [1].

In this work we are not interested in circularity in general, only in the special case in which circular definitions involve concepts which are opposite to one another, and then, also in the way in which we understand opposite concepts whether we attempt to define them or not. Such close relationship between opposite concepts we can call "antinomic circularity," "antinomic" because when a concept $\boldsymbol{A}$ depends on an opposite concept $\boldsymbol{B}$ to be fully intelligible — that is, when the meaning of $\boldsymbol{B}$ is an indispensable part of the meaning of $\boldsymbol{A}$ — then $\boldsymbol{A}$ is inevitably self-contradictory, not in the sense in which a statement such as "I am lying" is self-contradictory because of its being true and false — concepts are neither true or false by themselves — but on account of the meaning of A containing contrary meanings without which it would be incomprehensible. The meaning of an antinomic concept is a function of itself and of the meaning of its opposite.

Antinomic concepts are then useful precisely because of their antinomicity. Thus, truth is unintelligible without falsity, and so are categories of thought inseparable from their opposites. Kurt Gödel put it this way: We must admit "the amazing fact that our logical intuitions (i.e. intuitions containing such notions as truth, concept, being, class, etc.) are self-contradictory.[2]"

But what is opposition? It is a primitive idea impossible to define properly just as so many mathematical ideas that occur in axioms and whose meaning — not always unique — gradually develops in the various interpretations such ideas have in the models of the axioms: we come to form an intuition of what a set is by applying the notion — hazy as it is before we use it. Even so, Cantor himself ventured an informal characterization of what a set is by saying that "a set is a multiplicity taken as a unit," which of course calls for the prior definitions of multiplicity and unit. Regardless, Cantor's characterization is useful for a purely intuitive first approach. Similarly, we can say that two concepts are opposite if they are semantically contrary to each other, that is, if the meaning of one is clearly placed against the meaning of the other. We do perceive this opposition of two ideas when it is present.

2 Antinomic Terms

When we define or merely understand a concept $\boldsymbol{A}$ in terms of another concept $\boldsymbol{B}$, and then do the same with $\boldsymbol{B}$ in terms of $\boldsymbol{A}$, we have what is commonly called a "vicious circle". Here we will be especially interested in the case in which $\boldsymbol{A}$ and $\boldsymbol{B}$ are opposite concepts, one example being when $\boldsymbol{B}$ is in some sense a negation of $\boldsymbol{A}$, negation constituting a particular case of opposition. In general, a "vicious circle" occurs when $\boldsymbol{A}$ and $\boldsymbol{B}$ are inseparable notions each in need of the other for any reasonable understanding. This is a much more common semantic event than what we tend to think, and an inevitable one. Given the pejorative slant with which the expression "vicious circle" is taken as denoting something to be avoided, we shall use alternatively the plain expression "circle" or, when opposition is involved, "antinomic circle" or, more specifically, "antinomic term" or "antinomic predicate" as the case may be. Nevertheless, we shall still use "vicious circle", but in a definitely positive sense.

Let us characterize an "antinomic circle" as either (i) the conjunction of a term with one or more opposite terms, or (ii) the conjunction of a predicate with one or more opposite predicates. Antinomic circles must be distinguished from antinomic sentences such as "I am lying." The latter is true and false; antinomic circles are neither true nor false, they are complex linguistic entities to be fed to predicate formulas in the appropriate places. These formulas may then become true, false, true or false, or neither true nor false according to the semantic context in which they are interpreted. Now, the conjuncts of an antinomic circle are either inseparable opposite terms or inseparable opposite predicates; in each case we are dealing with intertwined concepts forming part of a complex linguistic entity. Opposition is a general case of antinomicity, and with the notion of antinomic circle we bring down antinomicity from the level of predicate formulas to the level of terms and predicates.

Most of the basic categories of thought are inseparable parts of antinomic circles. When we deepen our understanding of one such category, we come to realize that it is unthinkable without its opposite counterparts. Consider for example "one" and its opposite notion "many" in the sense of "more than one". Since our experience constantly recognizes pluralities, we perceive each "one" as "one of many" a chosen item. Similarly, each "many" is "many ones": "many" cannot be thought of in any way than as a gathering of "ones." The two notions depend on one another, they are semantically intertwined into an inseparable complex: "one-and-many", an antinomic term the meaning of each of whose components is included in the other's meaning, a conjunction without which neither concept can be comprehended. This semantic situation has been recognized before. The linguist Jost Trier remarked: "Every uttered word reminds us of the one of opposite sense. The value of a word is recognized only if it is bounded against the value of the neighboring and opposing words.[3]"

The inevitable antinomicity of our basic concepts is the force that makes up the semantic field. Charles Bally concurred: "Logical notions exist together in our mind with the opposite concepts, and these oppositions are ready to present themselves to the consciousness of the speaking subject. It can be said that the contrary of an abstract word is part of the sense of the word [4]." Interestingly, Martin Heidegger, an author not always clear, puts the matter in the most explicit way: "If interpretation must operate in that which is understood, how is it to bring any scientific results to maturity without moving in a circle? Yet according to the most elementary rules of logic this circle is a *circulus vitiosus... But if we sense this circle as a vicious one and look for ways of avoiding it, even if we just sense it as an inevitable imperfection, then the act of understanding has been misunderstood from the ground up...* What is decisive is not to get out of the circle but to come into it in the right way. In the circle is hidden a positive possibility of the most primordial kind of knowing [5]." It is not then a question of "tolerating" the vicious circles but that of turning them into efficacious instruments of our understanding.

Another antinomic term is matter-and-energy: energy being characterized by its effects on matter, and matter being concentrated energy lying in wait to be released. Whole-and-part represents another antinomic term whose components are impossible to separate entirely, and so does activity-and-passivity, as well as identity-and-difference, the latter being of the essence of becoming which is nothing other than the presence of change in what remains identifiably the same. Examples of antinomic predicates would be real-and-potential and universal-and-particular. But more about this in the following section.

3 Antinomic Predicates

Predicates will be understood as formalizing either properties (one-place predicates) or relations n-ary predicates n greater than 1). The expression “bittersweet”, a common word easily understood, represents an antinomic unary predicate that is the conjunction of two opposite properties. A situation, a memory, the taste of a food may be bitter and sweet at the same time. Experiencing them together transforms both sweetness and bitterness giving each a new enlarged meaning as each enters into the indivisible complex “bittersweet”.

In set theory, membership becomes an antinomic binary relation when we consider whether or not the set $\boldsymbol{T}$ of all sets that are not members of themselves is or is not a member of itself. This leads to Russell’s antinomy, of course, but at the same time it makes of the binary relation of membership an antinomic predicate. Let us make this more precise introducing now symbols to systematize the new situation.

Both terms and predicates can have more than one opposite. For example, the binary predicate “less than” has as opposites “greater than,” “equal to,” and “neither less than nor greater than,” that is, not comparable. In order to represent opposition both for terms and predicates let us use the negation symbol “$\neg$” extending its usual role as only applicable to formulas to being also applicable to terms and predicates, but with a different connotation. Let us look at the specifics. The negation of a statement $\boldsymbol{p}$ produces the unique compound statement “not-$\boldsymbol{p}$.” Now, if the statement $\boldsymbol{p}$ is in opposition with the statement $\boldsymbol{q}$, neither $\boldsymbol{p}$ nor $\boldsymbol{q}$ are necessarily each the negation of the other, negation of a statement is only a special case of opposition. The symbol “$\neg$” cannot possibly be used then to represent propositional opposition in general. For terms and predicates the situation is reversed.

4 Types, Positive and Negative

We shall introduce term formulas in contrast with predicate formulas, terms assuming the role that predicates play in the usual predicate calculus, that is, forming atomic formulas when prefixed to the appropriate number of predicates, the latter playing the role that terms have in the usual predicate calculus. To this purpose, let us frame our logic within a system of types, the positive ones being the usual types of type theory, which will include all predicate formulas of any type definable in the formal language in case. Type 0 lodges the symbols for terms and predicates by themselves, no formulas. In turn, negative types harbor in type -1 (i) atomic term formulas of the form $\boldsymbol{t}(\boldsymbol{P})$ for example, $\boldsymbol{t}$ a term, $\boldsymbol{p}$ a predicate, or $\boldsymbol{t}(\boldsymbol{P}_1, \ldots \boldsymbol{P}_n)$, t a term of arity n, and each of the predicates $\boldsymbol{P}_1, \ldots \boldsymbol{P}_n$ having the same arity k; in addition, (ii) type -1 contains all the compound term formulas obtainable the usual way from the atomic ones using the propositional connectives and quantification over all predicates (we shall not go beyond type -1 here). Negative types of term formulas require of course a different semantics which will be described in this and the next two sections.

The notion of negative types was introduced by Hao Wang [6]. He used this notion only as a device to show the independence of the axiom of infinity in set theory, not to analyze terms. Wang states: "It may be argued that in logic and mathematics there is no reason for us to discuss individuals [7]." In his system of negative types "there are no more individuals, and all types are on an equal footing [8]." In contrast, we are here appealing to negative types precisely to formalize the structure of terms and predicates, and of their respective interpretations. Human thinking begins and ends with individuals; from them abstract properties and relations are abstracted. We claim that despite the power and achievements of abstraction it is a mistake to banish the analysis of individuals altogether from logic and mathematics. The present work is intended to reaffirm this position.

The system of types we are outlining differs from Wang's in that, for the latter's approach, in the semantics of all positive and negative types new objects are introduced for each type to form that type's domains of interpretation. Here, in the semantics of type -1, the "individuals" of a given domain of interpretation are classes of individuals or of k-tuples of individuals taken from a previously given domain of interpretation in type 1. Specifically, given that we wish to formalize terms in type -1 as performing a role similar to that of predicates in type 1 with each term being fed one or more predicates in accordance with the arity of the term, each predicate is to be interpreted by the class of individuals or k-tuples of individuals drawn from one specific domain in type 1, individuals that satisfy the predicate in the semantics of type 1. The principle behind this approach is that, concretely speaking, we can identify each real term as being the sum total of all the predicates the term displays, that is, all the properties that the term exhibits, plus all the relations the term is actually engaged with. In accordance to this principle, $\boldsymbol{t}(\boldsymbol{P})$ in type -1 implies that $\boldsymbol{t}(\boldsymbol{P})$ is the case in type 1, but it also implies that t is not an atomic, indivisible entity: no term is closed into itself. Contrary to logical atomism, "atomic" terms have structure, and the predicates that make $\boldsymbol{t}(\boldsymbol{P})$, $\boldsymbol{t}(\boldsymbol{P}_1, \ldots \boldsymbol{P}_n)$ etc., hold are the components of such structure. In addition, properties and relations in type -1 are not unbreakable either, they are made up in turn by components of even lower type, but this is beyond our scope.

5 Domains of Interpretation

In each type, the choice of a domain of interpretation is in principle arbitrary. In the applications however the choice responds of course to a specific purpose. Once chosen, the domain remains fixed to determine the particular interpretation intended. Let us represent by $\boldsymbol{D}_1$ a domain from type 1 and by $\boldsymbol{D}_{-1}$ a domain from type -1. The composition of any $\boldsymbol{D}_1$ is made up of individuals related to the given purpose of the interpretation. Select distinct members of a $\boldsymbol{D}_1$ are assigned one to one to each atomic term, each assigned individual constituting the meaning of the given term in $\boldsymbol{D}_1$. Each $\boldsymbol{D}_1$ is in turn and by itself the universe of discourse out of which there are no meaningful entities to talk about. Predicates are assigned each a subset of the set of all individuals of such $\boldsymbol{D}_1$. Then, meaning and satisfiability of formulas — and therefore truth and falsity — are recursively defined as a function of the temporarily fixed domain of interpretation $\boldsymbol{D}_1$, and will vary from domain to domain except for tautologies.

As for any $\boldsymbol{D}_{-1}$, a parallel relativization of meaning as a function of the interpretation takes place. The composition of $\boldsymbol{D}_{-1}$ however differs from that of any $\boldsymbol{D}_1$. In accordance with what we stated in the previous section, the interpretation in $\boldsymbol{D}_1$ of $\boldsymbol{t}$ from $\boldsymbol{t}(\boldsymbol{P}_1, \ldots \boldsymbol{P}_n)$ for example, $\boldsymbol{t}$ and $\boldsymbol{P}_1, \ldots \boldsymbol{P}_n$ constant, consists of the class of all classes of k-tuples of individuals from $\boldsymbol{D}_1$ which in type 1 satisfy the constant components $\boldsymbol{P}_1, \ldots \boldsymbol{P}_n$ in $\boldsymbol{t}(\boldsymbol{P}_1, \ldots \boldsymbol{P}_n)$. Not going here beyond the types 1 and -1, we can use the same formal symbols $\boldsymbol{t}$ and the $\boldsymbol{P}_i$s to represent terms and predicates respectively in both types, but with different interpretations in each type as described. Within negative types no term or predicate can be represented by a point without parts: terms and predicates are not atomic, they have a structure that positive types hide and which negative types disclose. One must resist the subconscious inclination to fall back into thinking of terms and predicates in type -1 the way we do it in type 1. Whereas formal expressions are built and interpreted from the bottom up in positive types, they are constructed, analyzed, and interpreted from the top down in the negative ones.

6 The Syntax and Semantics of Term Formulas

The propositional operator of negation "$\neg$", which applies only to predicate formulas in type 1, will be extended in type -1 to apply to term formulas, opposite terms, and opposite predicates. Thus, the expressions "$\neg\boldsymbol{t}$" and "$\neg\boldsymbol{P}$" will indicate respectively a term and a predicate opposite to the term $\boldsymbol{t}$ and the predicate $\boldsymbol{P}$. If we need to represent that $\boldsymbol{t}$

and $\boldsymbol{P}$ have more than one opposite, we shall list them respectively thus $\neg_1\boldsymbol{t}, \neg_2\boldsymbol{t}, \ldots$, and $\neg_1\boldsymbol{P}, \neg_2\boldsymbol{P}, \ldots$,the terms in $\neg_1\boldsymbol{t}, \neg_2\boldsymbol{t}, \ldots, \neg_n\boldsymbol{t}$ and the predicates in $\neg_1\boldsymbol{P}, \neg_2\boldsymbol{P}, \ldots, \neg_n\boldsymbol{P}$ intended to represent respectively terms and predicates pairwise opposite to each other.

"$\neg\boldsymbol{t}(\boldsymbol{P})$" can be satisfied, not satisfied, or both simultaneously in a given interpretation. With $\boldsymbol{t}$ and $\boldsymbol{P}$ constant, $\neg\boldsymbol{t}(\boldsymbol{P})$ is satisfied in a domain $\boldsymbol{D}_{-1}$ iff the class of individuals that interprets $\boldsymbol{P}$ belongs to the class of classes of individuals that interprets $\neg\boldsymbol{t}$ in that domain. $\neg\boldsymbol{t}(\boldsymbol{P})$ is satisfied and not satisfied in $\boldsymbol{D}_{-1}$ iff the class of individuals that interprets P in such domain belongs and does not belong to the class of classes of individuals that interprets $\neg\boldsymbol{t}$ in $\boldsymbol{D}_{-1}$. We are assuming, of course, a semantics that allows for antinomies. If t is of arity n, the definition still obtains substituting "classes of n-tuples of classes of individuals" for "classes of classes of individuals."

"$\neg\boldsymbol{P}$" reads "the opposite of $\boldsymbol{P}$", and in a given domain $\boldsymbol{D}_{-1}\neg\boldsymbol{P}$ and $\boldsymbol{P}$ are respectively interpreted by classes of individuals, classes which may be disjoint, or have a nonempty intersection.

Let us extend now the connective "&" (and), used for the conjunction of predicate formulas, to the conjunction of term formulas, terms, and predicates; this makes of $\boldsymbol{t}$ & $\neg\boldsymbol{t}$ and $\boldsymbol{P} \,\&\, \neg\boldsymbol{P}$ formal expressions. As for the meaning, if $\boldsymbol{t} \,\&\, \neg\boldsymbol{t}$ represents the constant term one-and-many, we can interpret the term formula $(\boldsymbol{t} \,\&\, \neg\boldsymbol{t})(\boldsymbol{P})$ as stating that "the predicate $\boldsymbol{P}$ shares both the characteristics of being a unity and a multiplicity." Similarly, $\boldsymbol{P} \,\&\, \neg\boldsymbol{P}$ can be given the meaning of "bittersweet."

In mathematics and logic, and even in concrete contexts, the number of opposite terms and predicates relevant to the discourse in case is always limited. The usual opposite of "matter" is "energy," although "emptiness" could also be pertinent in a given physical discussion. And we already mentioned three opposites to the predicate of "less than." It is then the domain of discourse, the limited universe of meanings that effectively controls the number and use of the opposites to a term or to a predicate.

As for quantification, the term formula $\boldsymbol{t}(\boldsymbol{P})$ in type -1 is quantified over all predicates in expressions such as $(\forall\boldsymbol{P})\boldsymbol{t}(\boldsymbol{P})$ and $(\exists\boldsymbol{P})\boldsymbol{t}(\boldsymbol{P})$, $\boldsymbol{P}$ variable.

We mentioned that $\neg\boldsymbol{t}$ with $\boldsymbol{t}$ constant is interpreted by a class of classes of individuals, and of course so is $\boldsymbol{t}$, the interpretation of which is not necessarily the set-theoretic complement of the interpretation of $\neg\boldsymbol{t}$ in a given domain $\boldsymbol{D}_{-1}$. Thus, $(\boldsymbol{t} \,\&\, \neg\boldsymbol{t})(\boldsymbol{P})$ is satisfied iff there is a class of individuals in $\boldsymbol{D}_{-1}$ that satisfies $\boldsymbol{P}$ that belongs to the intersection of the two classes of classes that satisfy $\boldsymbol{t}$ and $\neg\boldsymbol{t}$ respectively; $(\boldsymbol{t} \,\&\, \neg\boldsymbol{t})(\boldsymbol{P})$ can then be true, false, true and false, or neither. Predicates are usually thought of as standing for independnet properties or relations; that is, formal objects whose meaning is floating by itself in some heaven of ideas removed from the concreteness of real terms.

But predicates are in actuality first seen as properties of a term, or in the relations the term exhibits. Originally, we do not think of whiteness by itself, we observe that some things are white. Similarly, we see in the term the relations that make up the term, its internal relations with other terms. To describe a term is to peel off in succession all the predicates that the term displays, i.e., to single out one after another the properties and the relations that the term offers. We perform the peeling of predicates from the term the same way we do with the folds of an onion. Then we turn each fold abstractedly into a formal item. The domain of interpretation $\boldsymbol{D}_{-1}$ must produce one such "onion" for each constant term $\boldsymbol{t}$, the "onion" being the class of all classes of individuals that satisfy $\boldsymbol{P}$ in $\boldsymbol{t}(\boldsymbol{P})$. If t is variable, $\boldsymbol{D}_{-1}$ must in turn produce an "onion space," with each constant assignment for $\boldsymbol{t}$ being given a single specific "onion." The "onion space" is then the class of all classes of classes of individuals such that each predicate or n-tuple of predicates are associated with one fold of one "onion."

Although $\boldsymbol{t}(\boldsymbol{P})$ may be false for all $\boldsymbol{P}$ in some abstract domain of interpretation, in the real world every existing term— every actual object— has properties, hence $\boldsymbol{t}(\boldsymbol{P})$is necessarily true for some $\boldsymbol{P}$. To feed an antinomic term such as $\boldsymbol{t} \,\&\, \neg\boldsymbol{t}$ to a nonantinomic predicate $\boldsymbol{P}$ does not necessarily produce in type 1 an antinomic formula: $\boldsymbol{P}(\boldsymbol{t} \,\&\, \neg\boldsymbol{t})$ can be simply true or simply false. Similarly, to feed an antinomic predicate such as $\boldsymbol{P} \,\&\, \neg\boldsymbol{P}$ to a nonantinomic term $\boldsymbol{t}$ in type -1 does not make $\boldsymbol{t}(\boldsymbol{P} \,\&\, \neg\boldsymbol{P})$ antinomic, the latter can be simply true or simply false. So may be the case also in type 1 for $(\boldsymbol{P} \,\&\, \neg\boldsymbol{P})(\boldsymbol{t})$ and $(\boldsymbol{P} \,\&\, \neg_1\boldsymbol{P} \,\&\, \ldots \,\&\, \neg_k\boldsymbol{P})(\boldsymbol{t})$: for example, let us take the antinomic predicate $\epsilon \,\&\, (\neg\epsilon)$ with ϵ being the binary relation of membership; let $\boldsymbol{T}$ be the set of all sets that are not members of themselves, then $(\epsilon \ \& \ (\neg\epsilon))(\boldsymbol{T}, \boldsymbol{T})$ is simply true, not antinomic.

7 A Fuzzy Logic of Opposition

Let us introduce a binary predicate of opposition that will apply (i) to pairs of statements, (ii) to pairs of terms, and (iii) to pairs of predicates. If $\boldsymbol{p}$ and $\boldsymbol{q}$ represent two statements, Opp($\boldsymbol{p}$, $\boldsymbol{q}$) will be interpreted as saying that "$\boldsymbol{p}$ and $\boldsymbol{q}$ are opposite statements." If Opp($\boldsymbol{p}$, $\boldsymbol{q}$) is the case, then it is always the case that Opp($\boldsymbol{q}$, $\boldsymbol{p}$). In turn, Opp($\boldsymbol{t}_1$, $\boldsymbol{t}_2$) states that the term $\boldsymbol{t}_1$ is opposite to the term $\boldsymbol{t}_2$. Similarly, Opp($\boldsymbol{P}$, $\boldsymbol{Q}$) states that the predicate $\boldsymbol{P}$ is opposite to the predicate $\boldsymbol{Q}$. If both cases Opp($\boldsymbol{t}_2$, $\boldsymbol{t}_1$) and Opp($\boldsymbol{Q}$, $\boldsymbol{P}$) follow.

But opposition is only in particular cases absolute. There is a grading of opposition that may apply to the three cases given above. For example, Peter may be the opposite of Mary in that Peter loves Mary but Mary dislikes Peter. If the love is stronger than the dislike, then Opp($\boldsymbol{P}$, $\boldsymbol{M}$) with $\boldsymbol{P}$ for Peter and $\boldsymbol{M}$ for Mary, has a greater degree than

Opp($\boldsymbol{M}$, $\boldsymbol{P}$). To represent this situation let us attach a real number from the interval [0,1] to the predicate of opposition in this manner: Opp($\boldsymbol{P}$, $\boldsymbol{M}$) and Opp($\boldsymbol{M}$, $\boldsymbol{P}$). If $a = 1$ or $b = 1$, the opposition is in each such case total; if $a = 0$ or $b = 0$ there is no opposition at all. For any real number between 0 and 1 the opposition is partial, which implies that, to some degree, the opposition is absent as well. In the case of Peter and Mary $a > b$. The following expressions $\mathrm{Opp}_c(\boldsymbol{p},\ \boldsymbol{q})$, $\mathrm{Opp}_d(\boldsymbol{t}_1,\ \boldsymbol{t}_2)$ and $\mathrm{Opp}_f(\boldsymbol{P},\ \boldsymbol{Q})$ will be read respectively "the statement $\boldsymbol{p}$ is the opposite of the statement $\boldsymbol{q}$ with a degree c" "the term $\boldsymbol{t}_1$ is the opposite of the term $\boldsymbol{t}_2$ with a degree d," and "the predicate $\boldsymbol{P}$ is the opposite of the predicate $\boldsymbol{Q}$ with a degree f." When Mary dislikes Peter only up to a point, there is no complete opposition, then $0 < b < 1$. If Peter truly loves Mary but not madly, then $0 < b < a < 1$.

To grade the predicate of opposition is tantamount to make the logic of opposition an infinite-valued one. The real number a represents the truth value of the predicate formulas $\mathrm{Opp}_a(\boldsymbol{p},\ \boldsymbol{q})$, $\mathrm{Opp}_a(\boldsymbol{t}_1,\ \boldsymbol{t}_2)$ and $\mathrm{Opp}_a(\boldsymbol{P},\ \boldsymbol{Q})$, with $a = 1$ corresponding to absolute truth, $a = 0$ to complete falsity, and $0 < a < 1$ to a partial degree of truth. From the considerable practical applications of fuzzy set theory and fuzzy logic one can learn how to go from a purely qualitative, subjective estimation of the graded truth values to a more quantitative, objective one. As for the truth value of $\neg\mathrm{Opp}_a(\boldsymbol{p},\ \boldsymbol{q})$, etc., the simplest assignment would be $1 - a$, but there are other ways of dealing with the operation of negation in fuzzy logic [9].

In order to place within fuzzy set theory the graded notion of logical opposition, especially that of antinomic terms and predicates, let us assume an already established set theory in which membership is a graded relation in a way that $\boldsymbol{x} \epsilon_a \boldsymbol{S}$ indicates that for every member of a universe $\boldsymbol{U}$ and every subset $\boldsymbol{S}$ of $\boldsymbol{U}$ the membership of $\boldsymbol{x}$ to $\boldsymbol{S}$ has a degree attached measured by a real number a from the interval [0, 1]. A fuzzy binary relation of opposition "Opp" on a universe $\boldsymbol{U}$ is a fuzzy set (a set with fuzzy membership) defined by a well-determined function on the Cartesian product $\boldsymbol{U} \times \boldsymbol{U}$ into the closed interval [0, 1] for each of the three following cases to consider: (i) $\boldsymbol{U}$ is the set of all statements in a given formal language, (ii) $\boldsymbol{U}$ is the set of all terms, simple or complex, in that language, (iii) $\boldsymbol{U}$ is the set of all predicates, simple or complex, in such language. Then $(\boldsymbol{p}, \boldsymbol{q}) \epsilon_a \mathrm{Opp}$, $(\boldsymbol{t}_1, \boldsymbol{t}_2) \epsilon_b \mathrm{Opp}$, and $(\boldsymbol{P}, \boldsymbol{Q}) \epsilon_c \mathrm{Opp}$, express each set-theoretically the logical fact that the two components of each pair are in opposition of one another. The numbers a, b, c measure the strength of the opposition between the components of each pair in one direction, i.e., the corresponding logical degree of truth.

Now, to state that $(\boldsymbol{t}_1, \boldsymbol{t}_2) \epsilon_b \mathrm{Opp}$ is partially true and b is its degree of truth implies that it is also partially false. In other words, unless $b = 0$ or $b = 1$, $\mathrm{Opp}_b(\boldsymbol{t}_1, \boldsymbol{t}_2)$ is necessarily an antinomic expression regardless of whether $\boldsymbol{t}_1$ or $\boldsymbol{t}_2$ are antinomic terms or not. What the degree of truth of $\neg\text{Opp}_b(\boldsymbol{t}_1, \boldsymbol{t}_2)$ may be would depend not only on the degree of falsity of $\text{Opp}(\boldsymbol{t}_1, \boldsymbol{t}_2)$ but also on the kind of fuzzy negation used. As we said, several options for negation are possible. The same can be said of $\text{Opp}_a(\boldsymbol{p}, \boldsymbol{q})$ and $\text{Opp}_c(\boldsymbol{P}, \boldsymbol{Q})$. Clearly, the degree of truth of the three kinds of opposition vary each with the given domains of interpretation and their corresponding functions $\boldsymbol{U} \times \boldsymbol{U} \longrightarrow [0, 1]$. And so it is also with the degree of falsity of each of the three expressions. If b is the degree of truth of $\text{Opp}(\boldsymbol{t}_1, \boldsymbol{t}_2)$ and b' its degree of falsity, we can represent its degree of antinomicity by the complex number $b + ib'$.

8 The Reality of Antinomic Terms and Predicates

We identify a term by the properties and relations it holds. If the term is real, so is the meaning of the predications that apply to it. Now, the characteristics of a term usually show themselves in succession; hence, to predicate about a term is to unfold it, and if the term is antinomic, it also implies to disclose one or a sequence of oppositions, hidden in it. In the example of the antinomic term one-and-many concealed underneath the obvious opposition of one and many, there is a second opposition, that of the antinomic pairing of two gerunds into the antinomic circle "gathering-and-choosing," which prior to one-and-many makes of one-of-many the outcome of a choice, and of many-ones the outcome of a gathering. There is here a vicious circle within a vicious circle. The conjugation of gathering and choosing is part of the conjugation of one and many. This disclosing can often be pursued even further, but let us now move on to a physical example of antinomic term.

The Bose-Einstein Condensate is a different state of matter that has been found to exist at extremely low temperatures. It is a very common state in nature, and is characteristic of certain quantum fluids in such a manner that they are observed to flow without friction, experimenting no viscosity, and therefore capable of passing through one another, occupying the same place in space, and then, although being distinct entities, becoming indistinguishable from one another: they are different individual entities without separate individuality, behaving as one single entity with the same quantum states. In short, they constitute physical antinomic terms. These condensates can rotate and generate vortices that have been compared to small black holes, and whose dynamics has now been studied with amazing mathematical detail, the shape, number, and position of the vortices being a function of the velocity of the rotation [10].

The condensates are just one more example of the many antinomic entities and properties that quantum mechanics is discovering with increasing frequency. Puzzled by the counterintuitive character of such discoveries when considered from our traditional way of thinking, many scientists claim that, in effect, there is no quantum reality, that what this branch of physics is producing is a succession of theoretical constructions useful for experimentation but which in themselves are not, cannot be a reflection of world reality as it truly is. Yet it has always been the belief of scientists that, subject to subsequent changes and adjustments, science does reveal to us a very good approximation to reality as it is. However, now that quantum mechanics defies our traditional ways of thinking as it is the case with the condensates as well as with the universal phenomenon of nonlocality, then the reality that physics constructs for us nowadays must be be denied despite all its verifications. This simply will not do. It is indeed our way of thinking that is in need of adjustment.

Compared to the positive attitude of physicists in pursuing the analysis and experimentation of phenomenons which they do not ultimately understand very well, mathematicians in contrast have for the most part displayed a negative attitude toward antinomic entities such as the Russell set, the set of all sets that are not members of themselves. In the eagerness to suppress such entities, these have not really been looked into with the same seriousness that is being devoted to the Bose-Einstein Condensate for example. Yet, it would be interesting to see what recurring circles of membership could be defined in the Russell set, what logical "vortices" could be discovered in its midst.

The Russell set is not the only mathematical antinomic term matching the interest of those of their physical antinomic counterparts. By suspending the axiom of choice the existence of new mathematical objects becomes possible, some of them antinomic, such as the mediate sets. A mediate set is one that (i) is not inductive, i.e., cannot be counted by a terminal sequence of positive integers in other words, infinite — and (ii) is also not reflexive, i.e., it is not equinumerous to a proper subset of itself — in other words, finite. Mediate sets are antinomic terms: at the same time finite and infinite. It can be proved that if the set of all functions on the cardinal number of the set ν into the cardinal number of the set μ is mediate, then μ or ν is mediate. However, the power set of the power set of a nonempty, noninductive, nonreflexive set is reflexive [11]. But if we drop the axiom of choice we must do the same with the continuum hypothesis, since the latter implies the former. This means that the cardinal number of the set of all subsets of a mediate set is not necessarily either an aleph or a beth number. Let us call it a "gimel number," which can be mediate or nonmediate, and, in the later case, not necessarily an aleph or a beth number. The cardinal number of the set of all subsets of the set of all subsets of a mediate set cannot be a mediate set because it is reflexive. Let us call it a "daleth number," still not necessarily an aleph or beth number: it could be a yet undefined kind of nonfinite reflexive cardinal. But we can narrow the options introducing a Mediate Continuum Hypothesis to require that every daleth equal a gimel number, i.e., that the cardinal number of the power set of the power set of a mediate set be the cardinal number of the power set of some mediate set. Gimel numbers can be antinomic terms, and daleth numbers are obtained from antinomic terms [12].

The objective of the previous remarks is to briefly introduce the reader to a look at domains of mathematical and physical antinomic terms which, far from ending in a rejection of such terms as pathological objects, views them as entities with interest in their own right. No doubt, opposition — and antinomicity in particular — introduce an unsettling element of ambiguity in our lives since it forces us to understand things and situations in more than one way. But ambiguity and fuzziness are of the essence of many real things and situations. True, there are regions where one can count on definiteness, on statements being either simply true or simply false, but there are also vast realms of reality where vagueness and ambiguity are part of the essence of things, not the result of faulty or incomplete information. We must realize this fact and adjust to it.

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[9] See for example the section on “Fuzzy Complements” in G.J. Klir and B. Yuan, “Fuzzy Sets and Fuzzy Logic”, Upper Saddle River, NJ: Prentice Hall, 1995.

[10] The literature on the Bose-Einstein Condensates is now very extensive and growing. For a detailed analysis of the intricate subject of vortices in particular see for example A. Aftalion, Vortices in Bose-Einstein Condensates, New York: Birkhäuser, 2006.

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