Rev. Real Academia de Ciencias. Zaragoza. **65:** 7–33, (2010).

Infinite dimensional linear groups: how we have studied them

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Premio a la Investigación de la Academia 2010. Sección de Exactas

Abstract

Let V be a vector space over a field F. *A linear group* is a subgroup G of the general linear group $GL(V, F)$ of V over F, the group of all F–automorphisms of V under composition. Since the early times of the theory of groups, finite dimensional linear groups (that is, $\dim_F V$ finite) are one of the mathematical objects most studied (see [51]). However, the case in which the dimension is infinite has only received systematic attention in the late years. In this survey paper we review the approach we have made to this topic in the last decade as well as the main results we have obtained. The more achieved parts of this research consider as fundamental tool *the central dimension of a linear group* G, that is the F–dimension of $V/C_V(G)$. We focus our study on the set of all proper infinite central dimensional subgroups of G. A wide description of certain generalized soluble infinite central dimensional linear groups in which that set satisfies one of the standard chain conditions is given as well the relationship among them is established. We also survey other related topics that we studied with occasion of this investigation, namely *linear groups admitting an infinite central deviation* and *antifinitary linear groups*.

Key Words: Soluble group. Radical group. Linear group. Central dimension of a linear group. Minimal and maximal conditions. Weak minimal and weak maximal conditions. Infinite central deviation. Augmentation dimension of a linear group. Antifinitary linear group.
2001 MSC: Primary: 20F22, Secondary: 20H20.

Layout of the paper

Let V be a vector space over a field F. *The general linear group* $GL(V, F)$ of V over V consists of all F–automorphisms of V made group under composition of maps. *A linear* *group* G (over F) is exactly a subgroup $G \leq GL(V, F)$. If $\dim_F V = n$ is finite, G is said to be *a finite dimensional linear group* (of degree n); otherwise G is said to be *infinite dimensional*. Finite dimensional linear groups are known to be matrix groups, and so they are one of the mathematical objects more studied due to the rich interplay between them and their applications inside or outside of Mathematics.

However this is not the case of infinite dimensional linear groups, who have not received a systematic attention until the last 25 years. This paper is a survey that reviews a substantial part of that is known until now. Section 1 briefly describes that is known up the end of the past century, finite dimensional or infinite dimensional as the finitary linear groups. In Section 2 we introduce and explain the role of the fundamental tool in our study: the central dimension of a linear group. Sections 3 and 4 quote with much care the main results obtained in the early years of this century in this direction of research. Finally, Section 5 is devoted to stablish the structure of the first type of infinite dimensional linear groups we considered, namely the antifinitary linear groups.

Our group-theoretical notation is standard and is taken out from [48].

1 Antecedents of the theory

We briefly summarize in this Section some of the previous facts studied before our study.

1.1 Finite dimensional linear groups

Suppose that $\dim_F V = n$. Then it is elementary to establish an isomorphism $GL(V, F) \cong GL(n, F)$ and therefore a linear group $G \leq GL(V, F)$ can be thought as a group of invertible $n \times n$ matrices over F. This elementary fact gave rise to a rich interplay between algebraical and geometrical ideas associated with such groups, as well as many applications inside and outside the theory of (finite dimensional) linear groups. The story of these groups goes back to the early years of the past century with the pioneer work of Burnside and Schur particularly. In fact one of the most famous previous results of finite dimensional linear groups (here used as a test for an abstract problem of the theory of groups) is the affirmative answer given by Schur to the general Burnside's problem for these type of groups.

Theorem 1.1 ([49]). *A finitely generated finite dimensional linear group whose elements have finite order is finite.*

Thus, periodic implies locally finite for finite dimensional linear groups, although this is far from being true for arbitrary groups (see [43]).

Finite dimensional groups arise in group theory in a number of contexts. One of the most common is as the automorphism groups of certain abelian groups (Maltsev). Since the middle of the past century, the use of properties of linear groups in the theory of abstract groups was increasing, and nowadays the theory of finite dimensional linear groups is one of the most developed topics in mathematics and played a very important role in algebra and other branches of mathematics (see [6, 50, 51] and their references). Anyway, to give a more concrete idea of the most achievement results of the theory, we mention some of the most impressive results. We start quoting an standard result of Jordan on finite subgroups of finite dimensional linear groups in characteristic 0.

Theorem 1.2 (C. Jordan: see [51]). *Let $G \leq GL(n, F)$ be finite. If char $F = 0$, there is an integer-valued function $\beta(n)$ such that G contains an abelian normal subgroup of finite index at most $\beta(n)$.*

Later on this result was extended by L. E. Dickson to prime characteristic (see [51, Theorems 9.3]), and by I. Schur to periodic groups in the following way.

Theorem 1.3 ([49]). *Suppose char $F = p$. Then a periodic linear group $G \leq GL(n, F)$ containing no elements of order p is abelian–by–finite.*

We mention that this is the abstract structure of these groups (see below). Actually *the arithmetic* structure of them is satisfactory as Platonov showed (see [51, Theorems 9.10 and 9.18]).

Theorem 1.4 ([46]). *The Sylow subgroups of a periodic finite dimensional linear group G are conjugate and are conjugately complemented in G.*

They are also well behaved with respect to chain conditions (Brauer-Feit, Maltsev).

Theorem 1.5 ([2, 37]). *Let $G \leq GL(n, F)$ be a finite dimensional linear group.*

(1) If G is finitely generated (for example, if G satisfies Max, the maximal condition on subgroups), then G is residually finite. Further, if G is simple, then G is finite.

(2) If G satisfies Min, the minimal condition on subgroups (G is periodic), then G is abelian–by–finite.

One of the most achieved structural result on finite dimensional linear groups (which resembles the spectral theory of one automorphism) is a result due to Maltsev that gathers results due to Lie and Kolchin. This result, and this extensions based on results by Zassenhaus [56] says us that some generalized soluble finite dimensional linear groups do not differ too greatly from triangular matrix groups.

Theorem 1.6 ([38]). *A soluble linear group G of degree n contains a triangularizable normal subgroup of finite index bounded by a function of n. In particular G is nilpotent–by–abelian–by–finite, that is G has a finite normal series*

$$\langle 1 \rangle \leq T \leq A \leq G$$

such that T is nilpotent, A/T is abelian and G/A is finite.

Actually locally soluble groups has the above structure.

Theorem 1.7 ([56]). *A locally soluble finite dimensional linear group is soluble and then it is nilpotent–by–abelian–by–finite.*

We recall that *a radical group* is a group having an ascending series whose factors are locally nilpotent groups. Radical groups is a wide extension of both soluble and locally nilpotent groups. As a corollary of Theorem 1.7 is possible to show that a lot of generalized soluble finite dimensional groups are soluble (see [48, Corollary to Theorem 3.23]) and generalizing a famous result due to Plotkin [47, Theorem 13].

Theorem 1.8 ([47]). *A radical finite dimensional linear group is soluble and then it is nilpotent–by–abelian–by–finite.*

Due to these last results, we considered the structure given in Theorems 1.7 and 1.8 something like the Theorem-type in our investigations.

1.2 Finitary linear groups

Until the current century, the systematic study of infinite dimensional linear groups was much more limited probably due the absence of fruitful approaches. One approach there consisted in the application of finiteness conditions to the study of infinite groups. One such restriction that has enjoyed considerable attention in linear groups is the notion of a *finitary linear group*. In the late 1980's, R.E. Phillips, J.I. Hall and others studied infinite dimensional linear groups under finiteness conditions, namely finitary linear groups (see [44, 15, 40, 45, 16, 17])). Inspired probably in *finitary permutation groups*, a finitary linear group is composed of transformations that stabilize all but finitely many vectors of a basis of an arbitrary vector space on what they act. Specifically, a linear group $G \leq GL(V, F)$ is called *finitary* if, for each element $g \in G$, the subspace $C_V(g)$ has finite codimension in V, that is $\dim_F V/C_V(g)$ is finite; roughly speaking, a finitely generated finitary linear group behaves as a finite dimensional linear group, that is a finitary linear group looks like a locally (finite dimensional linear) group. The reader is referred to the above papers to see the type of results that were obtained. In particular, results near to the Theorem-type mentioned above have been showed, as for example the following result obtained by Meierfrankenfeld, Phillips and Puglisi [40].

Theorem 1.9. *A locally soluble finitary linear group is unipotent–by–abelian–by–locally finite.*

We recall that an element $g \in G$ is said to be *unipotent* if 1 is the unique eigenvalue of g; the group itself G is said to be unipotent if each one of its elements is unipotent. Finite dimensional unipotent linear groups are unitriangularizable and so they are nilpotent.

Theorem 3 is a good example of the effectiveness of finiteness conditions in the study of infinite dimensional linear groups. Actually, a finitary linear group can be viewed as the linear analogue of the abstract concept of an FC–group, that is a group with finite conjugacy classes; this association suggested that it was reasonable to start a systematic investigation of these "infinite dimensional linear groups" analogous to the fruitful study of finiteness conditions in infinite group theory in such a way that finitary linear groups were contained as a particular case.

2 The central dimension of a linear group

At the end of the past century, an idea to attack the study of infinite dimensional linear groups appearead. If $G \leq GL(V, F)$, then G acts trivially on the subspace

$$C_V(G) = \{v \in V \mid vg = v \text{ for all } g \in G\},$$

and hence G properly acts on the factor-space $V/C_V(G)$. Then *the central dimension of* G is defined to be the F–dimension of $V/C_V(G)$ and denoted by $\text{centdim}_F G$. A linear group G is said to have *finite central dimension* if $\text{centdim}_F G$ is finite; otherwise G is said to have *infinite central dimension.* Actually, a linear group G *is finitary if and only if* $\text{centdim}_F \langle g \rangle$ *is finite for every* $g \in G$.

At this point, it is worth mentioning that the above notion of central dimension of a linear group heavily relies in the way in which the linear group G is embedded in a *particular* general linear group. In fact, given an abstract group G, it is easy to construct embeddings of G in the same general linear group such that G has infinite or finite central dimension depending on the embedding (see [41])

Example 1. *Given a prime p, let $A = \langle a_1 \rangle \times \langle a_2 \rangle \times \ \dots$ and $B = \langle b_1 \rangle \times \langle b_2 \rangle \times \ \dots$ be two dos copies of the elementary abelian p–group. Then B acts on A as follows:*

$$\begin{cases} a_1 \cdot b_j = a_1 & j \geq 1 \\ a_{j+1} \cdot b_j = a_{j+1} a_1 & j+1 \geq 2 \\ a_k \cdot b_j = a_k & k \neq j+1 \end{cases}$$

We think of A as a vector space V over the prime field F_p of p elements and of B as a subgroup G of the general linear group $GL(V, F)$. We have that $C_V(G) = \langle a_1 \rangle$ and hence G has infinite central dimension.

Example 2. *As above, let B act on A as follows:*

$$\begin{cases} a_1 \cdot b_j = a_1 a_{j+1} & j \geq 1 \\ a_k \cdot b_j = a_k & k \geq 2, \ j \geq 1 \end{cases}$$

Again we think of A as a vector space V on F_p and of B as a subgroup G of $GL(V, F)$. Since $C_V(G) = \langle a_2 \rangle \times \langle a_3 \rangle \ \dots$, G has finite central dimension.

The above examples show that is necessary to indicate that a linear group is a particular subgroup of a general linear group rather than an abstract group that can be *linearized* in some way. Even the effect of the embedding can be very different of the idea we can intuitively have. For example, we have.

Example 3. *Let $A = \langle a_1 \rangle \times \langle a_2 \rangle \times \langle a_3 \rangle \times \dots$ be an elementary abelian p–group, p a prime and $G = \langle g \rangle$ be a cyclic group of order 2. We define an action of G on A as follows:*

$$\begin{cases} a_{2j} \cdot g = a_{2j-1} & j \geq 1 \\ a_{2j-1} \cdot g = a_{2j} & j \geq 1 \end{cases}$$

Think of A as a vector space V over the field F_p of p elements and of G as a subgroup of $GL(V, F_p)$. In this example $C_V(G) = \langle a_1 a_2 \rangle \times \langle a_3 a_4 \rangle \times \dots$ and so G has infinite central dimension.

Suppose that G has finite central dimension. If $C = C_G(C_V(G))$, then C is a normal subgroup of G and G/C is isomorphic to a subgroup of $GL(n, F)$, where $n = \text{centdim}_F G$. Since C stabilizes the series

$$\{0\} \leq C_V(G) \leq V,$$

we have that C is abelian. Moreover, if the characteristic of F is zero, then C is a torsion-free abelian group while if the characteristic of F is the prime p C is an elementary abelian p–group (see [14, Corollary to Theorem 3.8] and [13, section 43] for these assertions). Therefore, the structure of the given group G can be determined by the structure of its factor-group G/C, which is an ordinary finite dimensional linear group. In other words, finite dimensional central linear groups behave as finite dimensional linear groups (or, more precisely, their structure can be deduced from that the ordinary case), so the efforts must be focused in infinite dimensional central linear groups.

In the late years, a fundamental approach was develop to study these groups, an approach explained, discussed and investigated in several universities and research centers, giving rise to an intensive programme of research involving mathematicians of different countries. This approach could be summarized as follows. Let $\mathcal{L}_{icd}(G)$ be the set of all proper subgroups of G of infinite central dimension. In order to study infinite dimensional linear groups G that are close to finite dimensional, it is natural to proceed making $\mathcal{L}_{icd}(G)$

very small in some sense. For example, by imposing some restrictions to $\mathcal{L}_{icd}(G)$. Given the big success that the study of infinite groups with finiteness conditions had enjoyed, it seemed reasonable to study linear groups with finiteness conditions in a similar way as the above. In this survey-paper we start by reviewing the approach to the study of a infinite dimensional central linear groups we carried out in these years. This approach is based on the use of different finiteness conditions, namely the different chain conditions. We should mention that, in the same time, other approaches making use of the central dimension have also been carried out. For example, in [9, 4, 5, 3], an study of linear groups such that every $\mathcal{L}_{icd}(G)$–subgroup has a finite (types of) ranks can be found while the description of linear groups with boundedly finite G–orbits or related conditions can be seen in [11, 12, 10].

3 Minimal and maximal conditions on subgroups of infinite central dimension

In the theory of infinite groups with finiteness conditions the first problems considered concerned with the maximal and minimal conditions, which were first developed in ring and module theory. K. A. Hirsch and S. N. Chernikov started the research of soluble groups with the maximal condition while Cernikov himself began the investigation of groups with the minimal condition (the *Chernikov groups* are close to the Burnside's problem: see [19]). In the early years connected with these problems we also found the celebrated problem of O. Yu. Schmidt on groups all of whose proper subgroups are finite (*minimal infinite groups*), as well as other minimal groups. The investigations which resulted from these problems led the further development of the theory of groups with finiteness conditions.

It is said that a family $\mathcal{M}$ of subgroups of a given group G *satisfies the minimal condition* (or G *satisfies the minimal condition on* $\mathcal{M}$*–subgroups*) if given a descendant chain of $\mathcal{M}$–subgroups

$$H_1 \geq H_2 \geq H_3 \geq \cdots \geq H_n \geq H_{n+1} \geq \cdots,$$

then there exists some $m \geq 1$ such that $H_n = H_{n+1}$ for each $n \geq m$. $\mathcal{M}$ *satisfies the maximal condition* (or G *satisfies the maximal condition on* $\mathcal{M}$*–subgroups*) if given an ascendant chain of $\mathcal{M}$–subgroups

$$H_1 \leq H_2 \leq H_3 \leq \cdots \leq H_n \leq H_{n+1} \leq \cdots,$$

then there exists some integer $m \geq 1$ such that $H_n = H_{n+1}$ for every $n \geq m$. When $\mathcal{M}$ consists of all subgroups of G, we simply said that G *satisfies Min* or G *satisfies Max*, respectively.

Let $G \leq GL(V, F)$ be a linear group. We say that G *satisfies the minimal condition for subgroups of infinite central dimension*, or G satisfies Min–icd, if the set $\mathcal{L}_{icd}(G)$ satisfies the minimal condition. Similarly, we say that G *satisfies the maximal condition for subgroups of infinite central dimension*, or G satisfies Max–icd, if the set $\mathcal{L}_{icd}(G)$ satisfies the maximal condition.

In the beginning the first questions raised in this framework were analogous to the first founding problems of Infinite Group Theory, namely to classify linear groups $G \leq GL(V, F)$ under

- Schmidt's Problem: $\mathcal{L}_{icd}(G) = \emptyset$.
- Chernikov's Problem: $\mathcal{L}_{icd}(G)$ satisfies the minimal condition or G satisfies Min–icd.
- Hirsch-Baer's Problem: $\mathcal{L}_{icd}(G)$ satisfies the maximal condition or G satisfies Max–icd.

In the late years these questions were successfully studied, and we review here the main results. Actually linear groups satisfying Min–icd were studied by Dixon, Evans and Kurdachenko in [7] while Kurdachenko and Subbotin gave a complete description of the structure of linear groups satisfying Max–icd in [35].

We first note that both conditions above are trivially satisfied if $\mathcal{L}_{icd}(G) = \emptyset$, the problem from O.Yu. Schmidt's. In this case, we have an information that is very useful inside the theory.

Theorem 3.1 ([7])**.** *If $G \leq GL(V, F)$ is a (locally soluble)–by–finite subgroup of infinite central dimension whose proper subgroups have finite central dimension, then G is a Prüfer p–group, where $p \neq char\, F$.*

It is worth noting that there exists a connection among these groups, finitary linear groups and groups satisfying the chain conditions on all subgroups.

Proposition 3.2 ([7, 35])**.** *Suppose that $G \leq GL(V, F)$ is a linear group and $centdim_F G$ is infinite.*

(1) If G satisfies Min–icd, then G is finitary or G satisfies Min.

(2) If G satisfies Max–icd, then G is finitary or G is finitely generated.

Concerning those groups that satisfy the minimal condition, we see that locally soluble groups are soluble and locally finite groups are soluble–by–finite (*virtually soluble*).

Proposition 3.3 ([7])**.** *Let $G \leq GL(V, F)$ be a linear group satisfying Min–icd,* *(1) If G is locally soluble, then G is soluble.*

(2) If G is locally finite, then G is virtually soluble.

The main result of [7] classifies virtually soluble linear groups with Min–icd and gives a detailed structure of these groups. Of course this result fits in the terms of our theorem-type mentioned above.

Theorem 3.4 ([7]). *Let $G \leq GL(V, F)$ be a (locally soluble)–by–finite linear group. We suppose that G has infinite central dimension and satisfies Min–icd. Then either G is Chernikov or char $F = p > 0$ and G has a normal series $H \leq D \leq G$ such that:*

(1) G/D is finite and $D = H \rtimes Q$ is the semidirect product of H by a non-trivial divisible Chernikov p'–subgroup of infinite central dimension Q; and

(2) H is a nilpotent bounded p–subgroup of finite central dimension satisfying the minimal condition on Q–invariant subgroups.

In particular, G is a nilpotent–by–abelian–by–finite group that satisfies Min–n, the minimal condition on its normal subgroups.

The description of linear groups with Max–icd is more complicated than those given above. We mention here a weak version of the structure theorems, and refer to [35] for a detailed version.

The case when the linear group G is *infinitely generated* (that is, G cannot finitely generated: see Section 5) is studied first. In this case, we note that G is finitary by Proposition 3.2.

Theorem 3.5 ([35]). *Suppose that a linear group $G \leq GL(V, F)$ is soluble, has infinite central dimension and satisfies Max–icd. If G/G' is infinitely generated, then G has a normal series*

$$1 \leq H \leq N \leq L \leq G$$

such that

(1) L has finite index and infinite central dimension;

(2) L/H is abelian, N/H is finitely generated, $\text{centdim}_F N$ is finite and $L/N \cong C_{q^\infty}$, where $q \neq$ char F; and

(3) H is a torsion-free nilpotent group if char $F = 0$ and H is a nilpotent bounded p–subgroup if char $F = p > 0$.

In particular, G is nilpotent–by–abelian–by–finite.

The general description is achieved by the next result.

Theorem 3.6 ([35]). *Suppose that the linear group $G \leq GL(V, F)$ is soluble, has infinite central dimension and satisfies Max–icd. If G is infinitely generated, then G has a normal subgroup S of infinite central dimension such that G/S is a finitely generated abelian–by–finite group and S/S' is infinitely generated.*

Obviously, the structure of S is given by above Theorem 3.5 and, in particular, S is nilpotent–by–abelian–by–finite.

For the sequel, we shall make use of the following notion. Let $G \leq GL(V, F)$. Then

$$FD(G) = \{x \in G \mid \langle x \rangle \text{ has finite central dimension}\}$$

is clearly a normal subgroup of G ([7]). This subgroup is called the *finitary radical* of G, and will be very useful in what follows. We remark that finitely generated subgroups of $FD(G)$ are finitary, that is $FD(G)$ *is locally finitary.*

We quote now the results for finitely generated groups. They split into two cases depending of the central dimension of the finitary radical of the linear group considered.

Theorem 3.7 ([35]). *Suppose that the linear group $G \leq GL(V, F)$ is finitely generated soluble, has infinite central dimension and satisfies Max–icd. If the central dimension of $FD(G)$ is finite, then G has a nilpotent normal subgroup $U \leq FD(G)$ of finite central dimension such that*

(1) G/U is polycyclic;

(2) U is a torsion-free group if char $F = 0$ and U is a bounded p–group if char $F = p > 0$; and

(3) U satisfies Max–$\langle g \rangle$ for every $g \in G \setminus FD(G)$.

In particular, G is nilpotent–by–polycyclic.

Theorem 3.8 ([35]). *Suppose that the linear group $G \leq GL(V, F)$ is finitely generated soluble, has infinite central dimension and satisfies Max–icd. If the central dimension of $FD(G)$ is infinite, then G has a normal subgroup $L \leq FD(G)$ of infinite central dimension such that*

(1) G/L is abelian–by–finite and L/L' is infinitely generated; and

(2) L satisfies Max–$\langle g \rangle$ for every $g \in G \setminus FD(G)$.

It follows that the complete structure of G is obtained applying Theorem 3.5 to L.

4 Weak minimal and weak maximal conditions on infinite central dimensional subgroups

The weak minimal and the weak maximal conditions were introduced simultaneously in 1968 by R. Baer [1] and D. I. Zaitsev [55] as the most natural generalization of the ordinary minimal and maximal conditions. We recall here the corresponding definitions.

Let G be a group and let $\mathcal{M}$ be a family of subgroups of G. It is said that $\mathcal{M}$ *satisfies the weak minimal condition* (or G *satisfies the weak minimal condition on* $\mathcal{M}$*–subgroups*) if given a descendant chain of $\mathcal{M}$–subgroups

$$H_1 \geq H_2 \geq H_3 \geq \cdots \geq H_n \geq H_{n+1} \geq \cdots ,$$

then there exists some integer $m \geq 1$ such that the index $|H_n : H_{n+1}|$ is finite for every integer $n \geq m$. Dually, $\mathcal{M}$ *satisfies the weak maximal condition* (or G *satisfies the weak maximal condition on* $\mathcal{M}$*–subgroups*) if given an ascendant chain of $\mathcal{M}$–subgroups

$$H_1 \leq H_2 \leq H_3 \leq \cdots \leq H_n \leq H_{n+1} \leq \cdots ,$$

then there exists some integer $m \geq 1$ such that the index $|H_{n+1} : H_n|$ is finite for every integer $n \geq m$. The former problems, If the family $\mathcal{M}$ is composed by all subgroups of G, it is said that G satisfies Wmin or G satisfies Wmax, respectively. After their already mentioned introduction as a topic within the Theory of Infinite Groups by Baer and Zaitsev, groups satisfying the weak minimal and the weak maximal conditions for several families of subgroups were successfully studied in many instances. For example, the weak chain conditions on normal subgroups ([21, 23, 24]), subnormal subgroups ([22]), non-normal subgroups ([25]) and non-subnormal subgroups ([31, 32]). The topic is now well developed and we refer to [36, Section 5.1] and [18] to obtain more information about it.

In the early years of the current century we learnt the research work done to classify linear groups infinite dimensional central linear groups G by imposing the ordinary chain conditions on the set $\mathcal{L}_{icd}(G)$. Having in mind the evolution of the ideas within the framework of the theory, it seemed that the natural step in this investigation was the imposing on the weak chain conditions on the mentioned set. This was the topic of the doctoral dissertation of Muñoz-Escolano in 2007 ([41]), which we developed since then.

The definitions are those one can expect. Let $G \leq GL(V, F)$ be a linear group. We say that G *satisfies the weak minimal condition for subgroups of infinite central dimension*, or G satisfies Wmin–icd, if the set $\mathcal{L}_{icd}(G)$ satisfies the weak minimal condition. Similarly, we say that G *satisfies the weak maximal condition for subgroups of infinite central dimension*, or G satisfies Wmax–icd, if the set $\mathcal{L}_{icd}(G)$ satisfies the weak maximal condition. In the memory [41], periodic linear groups satisfying the conditions Wmin–icd and Wmax–icd were characterized. Jointly with N. N. Semko from Kiev, co-adviser of the mentioned doctoral dissertation, Muñoz-Escolano and the author published the main results in [42], of which we give here a detailed reference.

We start by mentioning a result that extends Zassehhaus' theorem mentioned in Section 1.

Proposition 4.1 ([42]). *Let $G \leq GL(V, F)$ be a periodic linear group that either satisfies Wmin–icd or Wmax–icd. Then*

(1) If G is locally finite, then G is finitary or G is Chernikov.

(2) If G is locally soluble, then G is soluble.

In particular, periodic locally soluble non-Chernikov linear groups satisfying Wmin–icd or Wmax–icd are soluble finitary linear groups.

The main result of [42] is the following theorem that describes completely such groups.

Theorem 4.2 ([42]). *Let $G \leq GL(V, F)$ be a periodic locally soluble linear group of infinite central dimension satisfying either Wmin–icd or Wmax–icd. Then the following properties hold:*

(1) If char $F = 0$, then G is Chernikov; and

(2) If char $F = p > 0$, then either G is Chernikov or G has a series of normal subgroups $H \leq D \leq G$ satisfying the following conditions:

(2a) H is a nilpotent bounded p–subgroup whose central dimension is finite;

(2b) D has finite index and it is a semidirect product $D = H \leftthreetimes Q$, where Q is a Chernikov divisible p'–group whose central dimension is infinite; and

(2c) if K is a Prüfer q–subgroup of Q and centdim$_F K$ is finite, then H has a finite K–composition series.

With some extra work, the above results at once allow us to deduce the following consequences.

Corollary 4.3 ([42]). *Let $G \leq GL(V, F)$ be a periodic locally soluble linear group of infinite central dimension. Then the following conditions are equivalent.*

(1) G satisfies Wmin–icd;

(2) G satisfies Wmax–icd; and

(3) G satisfies Min–icd.

Moreover, if G is a periodic locally nilpotent group and one of these conditions holds then G is Chernikov.

For abstract groups, Baer [1] and Zaitsev [55] had already established in a rather easy way that the conditions Wmin and Wmax were equivalent for soluble groups (more general for (locally soluble)–by–finite groups) and even equivalent to *minimax* groups: see [36, 5.1.5]. We recall here that a group is called *minimax* if it has a subnormal series whose factors either satisfy Min or Max; see [48, 36].

Until now we do not know of a proof of Corollary 4.3 without using the classification's results mentioned above of the infinite central dimensional linear groups satisfying Min–icd, Wmin-icd or Wmx-icd.

A fairly consequence from the definitions and from Corollary 4.3 is the following property that has no analogue for abstract groups.

Corollary 4.4 ([42]). *Let $G \leq GL(V,F)$ be a periodic locally soluble linear group of infinite central dimension. If G satisfies Max–icd, then G satisfies Min–icd.*

For non-periodic groups the situation was more complicated and it was impossible to obtain results as the previous ones. We began this study in the doctoral dissertation by Muñoz-Escolano [41] dealing first with nilpotent groups. Later on, a systematic study of locally nilpotent linear groups satisfying Wmin–icd and Wmax–icd was the topic of the papers [27] and [29], and we collect here the main results.

The first result we quote shows that for nilpotent linear groups the weak chain conditions on subgroups of infinite central dimensional and the weak chain conditions on subgroups are equivalent.

Theorem 4.5 ([27]). *Let $G \leq GL(V,F)$ be a nilpotent linear group of infinite central dimension that either satisfies Wmin–icd or Wmax–icd. Then G is minimax.*

Other results of [27] were deduced in characteristic prime only since the tools needed to proceed in characteristic zero were rather different. We denote by $t(G)$ *the torsion subgroup* of the locally nilpotent group G, by $G^{\mathfrak{F}}$ *the finite residual* of G (the intersection of all subgroups of G of finite index), and by $G^{\mathfrak{N}}$ *the nilpotent residual* of G (the intersection of all normal subgroups H of G such that G/H is nilpotent). By means of these characteristic subgroups we have the following result.

Theorem 4.6 ([27]). *Let $G \leq GL(V,F)$ be a locally nilpotent linear group of infinite central dimension that either satisfies Wmin–icd or Wmax–icd. Suppose that char $F = p > 0$. Then*

(1) $G/t(G)$ is minimax;

(2) $G/G^{\mathfrak{F}}$ is nilpotent minimax; and

(3) $G/G^{\mathfrak{N}}$ is minimax.

In particular, if $t(G)$ has infinite central dimension, then G itself is minimax.

For linear groups that satisfies the weak minimal condition Wmin-icd something more can be said.

Theorem 4.7 ([29]). *Let $G \leq GL(V, F)$ be an infinite central dimensional linear group that satisfies Wmin–icd.*

(1) If G is locally nilpotent, then G either is minimax or finitary.

(2) If G is hypercentral and char $F = p > 0$, then G is minimax.

Similar results for the condition Wmax–icd are false. Indeed in [29, Section 4], an example of a hypercentral linear group satisfying Wmax–icd, which is neither minimax nor finitary was constructed.

Example 4.

Let F be a field of char $F = p > 0$. We consider a countably infinite dimensional F–vector space V. Let $\{u, v_n | n \in \mathbb{N}\}$ be a basis of V. Thus

$$V = uF \oplus v_1F \oplus v_2F \oplus \cdots \oplus v_nF \oplus \cdots = uF \oplus (\bigoplus_{n\in\mathbb{N}} v_nF)$$

Let $GL(\mathbb{N}, F)$ be the set of infinite matrices such that there are only finitely non-zero entries in each row and every column. Then we may define the product of two of such matrices by the same rule as finite dimensional matrices. Let $\gamma = \| c_{jk} \|_{j,k\in\mathbb{N}}$ be an infinite matrix where

$$\begin{cases} c_{11} = c_{12} = 1 & \\ c_{22} = 1 & \\ c_{j,j-1} = c_{jj} = 1 & \text{whenever } j \geq 3; \\ c_{jk} = 0 & \text{otherwise} \end{cases}$$

We define also the matrices $\alpha_n = \| a_{jk}^{(n)} \|_{j,k\in\mathbb{N}}, n \in \mathbb{N}$ such that

$$\begin{cases} a_{jj}^{(n)} = 1 & \text{for all } n \in \mathbb{N}, \\ a_{1n+1}^{(n)} = 1 & \\ a_{jk}^{(n)} = 0 & \text{for all remaining pairs } (j,k) \end{cases}$$

An straightforward computation gives that

- γ has infinite order;
- α_n has order p, for every $n \in \mathbb{N}$;
- $\alpha_n\alpha_m = \alpha_m\alpha_n$, for every $n, m \in \mathbb{N}$;
- $\gamma^{-1}\alpha_n\gamma = \alpha_n\alpha_{n-1}$, for every $n \in \mathbb{N}$

Let $G = \langle \gamma, \alpha_n \mid n \in \mathbb{N}\rangle$. Then $G = A \leftthreetimes \langle\gamma\rangle$ where $A = \mathrm{Dr}_{n\in\mathbb{N}}\langle\alpha_n\rangle$ is an infinite elementary abelian p–subgroup. Moreover, G is hypercentral.

From now on we shall freely make use of concepts related in the book [30]. If we consider A as an $\mathbb{F}_p\langle\gamma\rangle$–module, then it readily follows that A is a P–module over the ring $\mathbb{F}_p\langle\gamma\rangle$ where $P = (\gamma - 1)\mathbb{F}_p\langle\gamma\rangle$. Moreover it is not hard to see that A is a Prüfer P–module. By construction, $C_V(A) = \bigoplus_{n\in\mathbb{N}} v_nF$, so that $\dim_F(V/C_V(A)) = 1$ and A has finite central dimension. Furthermore, $C_V(\langle\gamma\rangle) = v_1F + (u - v_2)F$, so that $\mathrm{centdim}_F\langle\gamma\rangle$ is infinite. In conclusion, $\mathrm{centdim}_F G$ is infinite.

Finally, we claim that G satisfies Wmax–icd. Let

$$H_1 \leq H_2 \leq \cdots \leq H_n \leq \ldots$$

be an ascending chain of subgroups such that the indexes $|H_{j+1} : H_j|$ are infinite for all $j \in \mathbb{N}$. If $H_j \leq A$, then $\mathrm{centdim}_F H_j$ is finite. So, we can suppose that H_1 does not lie in A. We have $H_1 = (H_1 \cap A)\langle x\rangle$ where $x = \gamma^{m_1}\beta$ for some $\beta \in A$. For the positive integer m_1, we have a decomposition $m_1 = p^{k_1}q_1$ where $(p, q_1) = 1$. Put $s_1 = p^{k_1}$ and choose $\delta = \gamma^{s_1}$. By construction, the $\mathbb{F}_p\langle\gamma\rangle$–module A is artinian and $[A, \gamma] = A$ or in additive notation, $A(\gamma - 1) = A$. The finiteness of index $|\langle\gamma\rangle : \langle\delta\rangle|$ implies that A is an artinian $\mathbb{F}_p\langle\delta\rangle$–module (see [30, Theorem 5.2]). Since A is an elementary abelian p–group, for every $\alpha \in A$ we have

$$\alpha(\delta - 1) = \alpha(\gamma^{s_1} - 1) = \alpha(\gamma - 1)^{s_1}$$

It follows that $A(\delta - 1) = A$. Then, by [30, Corollary 5.2],

$$A = C_1 \oplus C_2 \oplus \cdots \oplus C_t$$

where C_j is a Prüfer R–module where $R = (\delta - 1)\mathbb{F}_p\langle\delta\rangle$, $1 \leq j \leq t$. Without loss of generality, we can suppose that

$$H_1 \cap A = C_1 \oplus C_2 \oplus \cdots \oplus C_{r_1} \oplus D_1$$

where D_1 is finite and $\dim_{\mathbb{F}_p}\Omega_1(D_1) + r_1 \leq t$. We can suppose that

$$H_2 \cap A = C_1 \oplus C_2 \oplus \cdots \oplus C_{r_2} \oplus D_2$$

here D_2 is finite and $\dim_{\mathbb{F}_p}\Omega_1(D_2) + r_2 \leq t$. Since the index $|H_2 : H_1|$ is infinite and $|H_2A : H_1A|$ is finite, the index $|H_2 \cap A : H_1 \cap A|$ is infinite. It follows that $r_2 > r_1$. Proceeding in a similar way, we obtain that after finitely many steps the chain of subgroups $\{H_n \mid n \in \mathbb{N}\}$ breaks off so that G satisfies Wmax–icd. Since the periodic part of G is an infinite p–elementary abelian group, G is not minimax.

We realized that other extensions of these results to generalized soluble groups could be very difficult to deal with. However we were able to study soluble linear groups that either satisfy Wmin–icd or Wmax–icd in [28]. One more time, we found that the structure of these groups is rather like the structure of finite dimensional soluble groups, as the main result from [28] shows. We note that an element $x \in G \leq GL(V, F)$ is *unipotent* if there exists some $n \in \mathbb{N}$ such that $V(x-1)^n = 0$. A subgroup H of G is called *unipotent* if every element $x \in H$ is unipotent and *boundedly unipotent* if n is independent of x.

Theorem 4.8 ([28])**.** *Let $G \leq GL(V, F)$ be a soluble infinite central dimensional linear group that either satisfies Wmin–icd or Wmax–icd. Then either G is minimax or G satisfies the following conditions:*

(1) G has a boundedly unipotent normal subgroup L of finite central dimension such that G/L is minimax;

(2) L is a torsion-free nilpotent subgroup if char $F = 0$; and

(3) L is a bounded nilpotent p–subgroup if char $F = p > 0$.

For the weak minimal condition Wmin-icd we were able to say something more in a similar way to the locally nilpotent case.

Theorem 4.9 ([28])**.** *If $G \leq GL(V, F)$ is a soluble infinite central dimensional linear group that satisfies Wmin–icd, then G is either minimax or finitary.*

We mention that non-minimax soluble finitary linear groups satisfying Min–icd (hence Wmin–icd) have been constructed (see [7, Section 5]). Also, it is rather easy to construct non-finitary minimax soluble linear groups. On the other hand, analogous results to above for the weak maximal condition are not true. In fact, the example mentioned above ([29, Section 4]) is a metabelian linear group satisfying Wmax–icd which is neither minimax nor finitary.

Summing up, our next result collects the equivalence of the weak chain conditions on linear groups finishing this description.

Corollary 4.10. *Let $G \leq GL(V, F)$ an infinite central dimensional linear group. Suppose that G is* *(a) locally nilpotent non-finitary; or*

(b) soluble non-finitary; or

(c) hypercentral, being char $F = p > 0$.

Then the following assertions are equivalent

(1) G satisfies Wmin–icd,

(2) G satisfies Wmin,

(3) G satisfies Wmax,

(4) G is minimax

Moreover, if G is nilpotent, then the conditions are also equivalent to the weak maximal condition on subgroups of infinite central dimension Wmax–icd.

4.1 Appendix: Linear groups admitting an infinite central deviation

At the beginning of this investigation, we did not suspect that the two weak chain conditions on the set of infinite central dimensional subgroups $\mathcal{L}_{icd}(G)$ of a linear group G would become equivalent as they would in many cases (corollaries 4.3 and 4.10). Having in mind the different structural results obtained in the study of the ordinary chain conditions ([7, 35]), we started the research of the weak chain conditions considering first the weak minimal condition, leaving the study of the weak maximal condition as a second step. Very soon we realized that most of the proofs were similar and seemed to depend on the imposition to $\mathcal{L}_{icd}(G)$ of a more general finiteness condition, from which the mentioned chain conditions were a particular case. Our attempts tried to deal with chains having ascending or descending steps at the same time as it occurs in the weak chain conditions themselves. Later on groups with a deviation for non-subnormal subgroups were introduced and classified for small values (see [33, 33]).

At this point, to describe appropriately the found setting, a highlighting idea was to proceed imitating concepts and trends from the theory of noncommutative rings, concretely the way in which some types of rings generalizing artinian and noetherian rings (rings with *Krull dimension*) were constructed. A fundamental remark here is to note that for noncommutative rings the Krull dimension is measured on the lattice of right ideals instead of the maximal length of a chain of prime ideals.

Let $A = (A, \leq)$ be a partially ordered set. If $a, b \in A$, then we put

$$[a, b] = \{x \in A \mid a \leq x \leq b\}.$$

This subset $[a, b]$ is called the *closed interval* with ends a and b. By definition *the deviation of* A, $dev(A)$ for short, is given by the following rule: If A is trivial, then put $dev(A) = -\infty$. If A is nontrivial but satisfies the minimal condition, then $dev(A) = 0$. For a general ordinal α, we define $dev(A) = \alpha$ provided:

(i) $dev(A) \neq \beta < \alpha$;

(ii) in any descending chain $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots$ of elements of A all but finitely many closed intervals $[a_{n+1}, a_n]$ have deviation less than α.

Clearly $dev(A)$ cannot exist for an arbitrary partially ordered set. By definition, every partially ordered set satisfying the minimal condition has a deviation. On the other hand, if the partially ordered set satisfies the maximal condition, then it also has a deviation (see [39, 1.8]). Therefore, the imposition of the existence of a deviation is a generalization of both the condition minimal and the condition maximal. As we mentioned above, in the Theory of Rings, deviations were introduced to define interesting types of rings. In fact, for each ring R, one considers the partially ordered set by inclusion of all right ideals of R. If this set admits a deviation, it is said that R has Krull dimension. Hence the rings with Krull dimension are a generalization of both artinian and noetherian rings.

We applied the same idea to linear groups. Let $G \leq GL(V, F)$. Then $\mathcal{L}_{icd}(G)$ is a partially ordered set by inclusion. The deviation of the poset $\mathcal{L}_{icd}(G)$ is called *the infinite dimensional deviation* of G and is denoted by $dev_{icd}(G)$. If G satisfies Min–icd, then $dev_{icd}(G) = 0$. As we mentioned before, the groups satisfying Max-icd also have an infinite dimensional deviation. On the other hand, if G satisfies Wmin–icd then $dev_{icd}(G) = 1$, and if G satisfies Wmax–icd then there exists an infinite dimensional deviation $dev_{icd}(G)$. Therefore the existence of an infinite dimensional deviation generalizes the verification of one of the ordinary chain or weak chain conditions.

We succeed in studying the condition $dev_{icd}(G) = 1$, as we were able to show satisfactory results as the following ones.

Theorem 4.11. *Let G be a soluble subgroup of $GL(V, F)$ with $dev_{id}(G) = 1$. If char $F = 0$, then G either is finitary or minimax.*

Theorem 4.12. *Let G be a soluble subgroup of $GL(V, F)$ with $dev_{id}(G) = 1$. Suppose that char $F = p > 0$ and G further satisfies one of the following conditions:*

(a) G has infinite central dimension and has virtually finite central dimension; or

(b) G is neither finitary nor has virtually finite central dimension.

Then G has a normal subgroup H such that H is a nilpotent bounded p–group, G/H is abelian–by–finite minimax and H satisfies Min–$\langle g \rangle$ for every element $g \in G \setminus FD(G)$. In particular G is nilpotent–by–abelian–by–finite.

Later on, we were able to show a general result in describing the structure of a soluble linear group admitting an infinite dimensional deviation in the following way.

Theorem 4.13. *Let G be a soluble subgroup of $GL(V, F)$ and suppose that $dev_{icd}(G)$ exists. If G has infinite central dimension, then G either has virtually finite central dimensional or G has a finite series of normal subgroups*

$$H \leq T \leq L \leq G$$

such that

(1) G/L is a finitely generated abelian–by–finite group;

(2) L/T is a torsion-free nilpotent minimax group;

(3) T/H is a divisible Chernikov group;

(4) L/H is nilpotent minimax; and

(5) H is a nilpotent subgroup of finite central dimension.

The next ones were results obtained making use of rather different technics, which arose from that we call *local methods*. These results complement those quoted above.

Theorem 4.14. *Let G be a soluble subgroup of $GL(V, F)$ and suppose that $dev_{icd}(G)$ exists. If G has infinite central dimension and has virtually finite central dimensional, then G has a normal subgroup H such that H is nilpotent $\langle g\rangle$–minimax for every $g \in G\backslash FD(G)$, and G/H is abelian–by–finite minimax.*

Theorem 4.15. *Let G be a soluble subgroup of $GL(V, F)$ and suppose that $dev_{icd}(G)$ exists. If G is neither finitary nor has virtually finite central dimensional, then G has a normal subgroup H such that H is nilpotent of finite central dimension and $\langle g\rangle$–minimax for every $g \in G \setminus FD(G)$, and G/H is minimax.*

Note that G is nilpotent–by–abelian–finite and nilpotent–by–minimax, respectively. Moreover, in both results, the nilpotent normal subgroup H is a stability group of a finite series and becomes to be torsion-free if char $F = 0$ and a bounded p–group if char $F = p > 0$.

Despite of these goals we were unable to construct specific examples of infinite central dimensional linear groups having an infinite central deviation that did not satisfy one of the weak chain conditions Wmax–icd or Wmin-icd. That is the reason why we decided to leave this direction of research and to fucus in the investigation of the two conditions at the same time, as we described in the first part of this Section, founding that most of the proofs were similar (specially in the periodic case). However, up now, the following problem remains open.

QUESTION *Is there a linear group G admitting an infinite dimensional deviation that does not satisfy the weak maximal condition Wmax–icd nor the weak minimal condition Wmin–icd?*

5 Augmentation dimension: antifinitary linear groups

Though this work is strongly focused on the research done with occasion of the doctoral dissertation of Muñoz-Escolano [41], whose basic tool is the central dimension, our first approach to infinite dimensional linear groups was rather different since at first we wanted to find the dual concept of finitary group (see below). Having in mind the idea of proceeding through the finiteness conditions, it seemed to us that the concept of finitary group was close to the concept of an FC–group and then it could be imitated better making use of the definition of *the augmentation dimension* of a linear group as it appeared in other papers. We recall the definitions. Let $G \leq GL(V, F)$ be a linear group that acts on a vector space V over a field F. We consider *the group ring of G over F*, that is the ring

$$FG = \{\sum_{g\in G} t_g g \mid g \in G,\ t_g \in F,\ t_g = 0 \text{ for all but finitely many } g \in G\}$$

whose multiplication is induced by that of G. We look at the corresponding augmentation ideal $\omega FG = \langle g - 1 \mid g \in G\rangle$ and the subspace of G generated by it, $V(\omega FG)$. Then it is clear that G acts trivially on the factor-space $V/V(\omega FG)$, and hence G properly acts on the subspace $V(\omega FG)$. As in [8], the *augmentation dimension* of G is defined to be the F–dimension of $V(\omega FG)$ and is denoted by $augdim_F G$.

The study of finite augmentation dimensional linear groups can be reduced to the study of finite dimensional linear groups, just as it happens for finite central dimensional linear groups. Suppose that G has finite augmentation dimension, that is $V(\omega FG)$ is finite dimensional. Put $C = C_G(V(\omega FG))$. Clearly, C is a normal subgroup of G and G/C is isomorphic to a subgroup of $GL(n, F)$, where $n = \text{augdim}_F G$. Each element of C acts trivially in every factor of the series

$$\{0\} \leq V(\omega FG) \leq V,$$

so that C is abelian, torsion-free if the characteristic of F is zero and an elementary abelian p–subgroup if the characteristic of F is the prime p. Therefore the structure of G can be determined by the structure of G/C, which is an ordinary finite dimensional linear group. As in the past sections, we must pay attention to the set $\mathcal{L}_{iad}(G)$ of all proper subgroups of G having infinite augmentation dimension. This concept was introduced in [8]. Later on, linear groups with some rank restrictions on the same subgroups were studied ([5, 3]).

Though the concept of augmentation dimension resembles to be opposite of the concept of central dimension, there is an apparently strong relationship between them. In fact $\dim_F V/C_V(g)) = \dim_F V(g-1)$ for $g \in GL(V,F)$. More generally, if $G \leq GL(V,F)$ is finitely generated, then it is easy to see that the finiteness of one of the dimensions implies the finiteness of the other. However, in the general case, this does not hold as the following specific examples show.

Example 5. *Let V and G as in Example 1. Since $C_V(G) = \langle a_1 \rangle = V(\omega(F_pG))$, we have that G has infinite central dimension and finite augmentation dimension.*

Example 6. *Let V and G as in Example 2. In this case $C_V(G) = \langle a_2 \rangle \times \langle a_3 \rangle \times = V(\omega F_p G)$, and so G has finite central dimension and infinite augmentation dimension.*

Despite of these examples, in [20], more conditions over G such that the finiteness of $\text{centdim}_F G$ implies the finiteness of $\text{augdim}_F G$ are established. Given a prime p, it is said that an abstract group G has *finite factor p-rank r* (respectively, *finite factor 0-rank r*) if whenever U and V are normal subgroups of G such that V/U is an abelian p-group (respectively, torsion-free abelian) and H is an intermediate subgroup $U \leq H \leq V$ such that H/U is finitely generated, then the minimal number of elements required to generated H/U is less or equal to r, and r is the least such integer with this property.

Theorem 5.1 ([20])**.** *Let G be a locally soluble subgroup of $GL(V,F)$ such that char $F = p \geq 0$. Suppose that G has finite factor p–rank. If $\text{centdim}_F G$ is finite, then so is $\text{augdim}_F G$.*

Theorem 5.2 ([20])**.** *Let $G \leq GL(V,F)$ be periodic and let char $F = p > 0$. Suppose that G has an ascending series of normal subgroups such that every factor either is a p–group or a p'–group, and there is an integer r such that every finitely generated p–subgroup of G can be generated by r elements. If $\text{centdim}_F G$ is finite, then so is $\text{augdim}_F G$.*

As mentioned, our approach to the study of infinite dimensional linear groups wanted to be based on the influence of the family of the subgroups having finite augmentation dimension on the structure of the whole linear group. For example, if every proper subgroup of a locally soluble linear group G has finite augmentation dimension, in [8] it was established that G is a Prüfer p–group for some prime p. On the contrary, if G has no subgroups of finite augmentation dimension, we have not information about the structure of G. As an exception, we should mention the cofinitary linear groups. A subgroup G of $GL(V,F)$ is called *cofinitary* if, for each element $g \in G$, the subspace $V(g-1) = [V,g]$ generated by all elements of the form $v(g-1)$, $v \in V$, has finite codimension in V. These groups were considered by B. A. F. Wehrfritz [52, 53, 54]. This is the dual concept of a finitary linear group since a subgroup G of $GL(V,F)$ is *finitary* if, for each element $g \in G$, the subspace $V(g-1)$ has finite dimension in V.

Every finitely generated subgroup of a finitary linear group has finite augmentation dimension, that is, as we mentioned above, finitary linear groups are locally finite dimensional in some sense.It is worth to mention that very often the structure of groups satisfying locally a property $\mathcal{P}$ is very different of that of the groups having the property $\mathcal{P}$. Therefore it is interesting to consider linear groups, in which the family of subgroups of finite augmentation dimension is bigger. Recall that a subgroup H of an arbitrary group G is called *infinitely generated* if H cannot have a finite set of generators. In almost all groups (with the exception of noetherian groups), the family of infinitely generated subgroups is bigger than that of finitely generated subgroups. We say that a group $G \leq GL(V, F)$ is *an anfinitary linear group* if each proper infinitely generated subgroup of G has finite augmentation dimension. Note that these groups are the antipodes of finitary linear groups. These groups were our first goal and were carefully studied in [26] (A paper accepted for publication in 2006 that appeared in 2008 due to inner problems of the journal).

The description of the main results of [26] is the aim of the current Section. We first noticed that the study had to be focused on non-finitary antifinitary linear groups because of the next result.

Theorem 5.3 ([8]). *Let $G \leq GL(V, F)$ be a (locally soluble)–by–finite linear group of infinite augmentation dimension such that every proper subgroup has finite augmentation dimension. Then G is a Prüfer p–group, where $p \neq char\ F$.*

In particular, this happens if G is (locally soluble)–by–finite finitary and an antifinitary linear group of infinite central dimension. For, $\mathcal{L}_{iac}(G) = \emptyset$ and it suffices to apply Theorem 5.3. Another interesting fact about locally finite antifinitary linear groups is the following result.

Proposition 5.4 ([26]). *Let $G \leq GL(V, F)$ be a locally finite antifinitary linear group. Then G either is finitary or Chernikov.*

After these preliminary results, we planned the study of antifinitary linear groups splitting it in two natural cases depending on whether or not the group considered was finitely generated. For infinitely generated groups, we were able to establish the following description.

Proposition 5.5 ([26]). *Let $G \leq GL(V, F)$ be an infinitely generated locally generalized radical antifinitary linear group. Then G either is finitary or locally finite.*

Here we recall that a group G is said to be *generalized radical* if G has an ascending series whose factors are locally nilpotent or locally finite. Proposition 5.4 and Proposition 5.5 at once ensure us that an infinitely generated antifinitary linear group G that is locally generalized radical is Chernikov. Even more, we have the following detailed description.

Theorem 5.6 ([26]). *Let G be an infinitely generated locally generalized radical antifinitary subgroup of $GL(V,F)$ such that $G \neq FD(G)$.*

(1) If $G/FD(G)$ is infinitely generated, then G is a Prüfer p–group for some prime p.

(2) If $G/FD(G)$ is finitely generated, then $G = K\langle g\rangle$ satisfies the following conditions:

(2a) K is a divisible abelian Chernikov normal q–subgroup of G, for some prime q

(2b) g is a p–element, where p is a prime such that $p = |G/FD(G)|$;

(2c) K is G–divisibly irreducible, i.e. K has no proper G–invariant subgroups;

(2d) if $q = p$, then K has finite special rank equal to $p^{m-1}(p-1)$ where $p^m = |\langle g\rangle/C_{\langle g\rangle}(K)|$; and

(2e) if $q \neq p$, then K has finite special rank $o(q,p^m)$ where as above $p^m = |\langle g\rangle/C_{\langle g\rangle}(K)|$ and $o(q,p^m)$ is the order of q modulo p^m.

In particular G is abelian–by–finite.

On the other hand, we were able to characterize finitely generated antifinitary linear groups as follows.

Theorem 5.7 ([26]). *Let G be a finitely generated radical antifinitary subgroup of $GL(V,F)$. If G has infinite augmentation dimension, then the following conditions holds:*

(1) $augdim_F FD(G)$ is finite;

(2) G has a normal subgroup U such that G/U is polycyclic;

(3) U is boundedly unipotent and, in particular, U is nilpotent;

(4) U is torsion-free if char $F = 0$ and is a bounded p–subgroup if char $F = p > 0$; and

(5) if $\langle 1\rangle = Z_0 \leq Z_1 \leq \cdots \leq Z_m = U$ is the upper central series of U, then the factors Z_1/Z_0, ..., Z_m/Z_{m-1} are finitely generated $\mathbb{Z}\langle g\rangle$–modules for each element $g \in G \setminus FD(G)$. In particular U satisfies the maximal condition on $\langle g\rangle$–invariant subgroups (Max–$\langle g\rangle$) for every $g \in G \setminus FD(G)$.

In particular G is nilpotent–by–polycyclic.

Acknowledgements

This research was supported by Proyecto MTM2010-19938-C03-03 of Dirección General de Investigación of MICINN (Spain). I also am very grateful to the University of Zaragoza (Spain) for the support given through specific grants and all the help I have received in some difficult years for me. I wish to express my deep gratitude to my colleagues and good friends Leonid A. Kurdachenko from Dnepropetrovsk (Ukraine) and Nikolaj N. Semko from Kiev (Ukraine), who shared with me this research and its related ideas for more than one decade. Special thanks are due to José M. Muñoz-Escolano, (*the scholar fellow*), a former graduate student of mine. For me, it was a great pleasure to be the advisor of his doctoral dissertation and to carry out this common investigation, whose more outstandings results are contained in this survey.

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