Rev. Real Academia de Ciencias. Zaragoza. **64:** 107–120, (2009).

Generalized Hermite Fractal Interpolation

A. K. B. Chand[1] and M. A. Navascués [2]

[1] Dept. of Mathematics, Indian Institute of Technology Madras

Chennai 600036, India.

[2] Depto. de Matemática Aplicada, CPS, Universidad de Zaragoza

C/ María de Luna 3, Zaragoza 50018, Spain.

Abstract

The classical methods of real data interpolation can be generalized with fractal interpolation. This paper extends the classical Hermite interpolation by means of a class of fractal interpolants. This problem prescribes at each support abscissa not only the value of a function but also its derivatives up to any finite order. Some error bounds of Hermite fractal interpolation function with respect to the data generating function are derived.

Keywords: Fractals, Iterated Function Systems, Fractal Interpolation Functions, Hermite Interpolation

MSC 2000 Classifications: 28A80, 41A15, 65D05, 65D07, 65D10

1 Introduction

Interpolation plays major roles in classical Numerical Analysis. The classical methods of numerical quadrature, numerical integration of ordinary differential equations, numerical differentiation can be studied by means of interpolation functions. We focus attention on a special class of continuous functions, called as fractal interpolation functions (FIFs) constructed by means of iterated function system (IFS). FIFs constitute an advance in the techniques of approximation, since all the classical interpolants can be generalized by means of fractal interpolants [9, 10].

Barnsley [1,2] first developed the theory and applications of FIFs for the approximation of naturally occurring phenomena showing some sort of self-similarity under magnification. Barnsley and Harrington [3] introduced an algebraic method for the construction of a restricted class of C^r-FIF f that interpolates the prescribed data by providing values of $f^{(k)}$, $k = 1, 2, \ldots, r$, at the initial end point of the given interval. However, in their method of construction, specifying boundary conditions similar to those for classical splines was found to be quite difficult to handle. Spline FIFs with general boundary conditions are proved recently (see for instance [5, 11]).

Hermite's interpolation formula [8] provided an expression for a polynomial which passes through given points with given slopes. The interpolation problem is called Hermite if it compromises only interpolation of consecutive derivatives, commencing with the value of the function itself at all the given mesh point and it is a particular case of the Hermite-Birkhoff interpolation problem [4]. Spitzbart [14] deduced the explicit Hermite polynomial formula so as to include the values of derivatives up to specified and arbitrary orders at the mesh points. The generalized Hermite interpolation plays an important role in radial basis functions [7] and computer graphics [13]. Navascués and Sebastián [11] generalized Hermite functions by fractal interpolation where the problem is prescribed with values of a function and its p derivatives at mesh points. Our approach is more general in the sense that the generalized Hermite FIF matches with the values of derivatives up to specified, arbitrary orders which depend on the points of the partition.

In Section 2, we discuss the basics of FIF. In Section 3, the generalized Hermite FIF is constructed by using classical Hermite interpolation and upper bounds for the error of the generalized Hermite FIF with the original function generating the data are deduced. The effect of the scaling factor is shown in Section 4 with a suitable example.

2 Fractal Interpolation Functions

Suppose $\{(t_n, x_n) \in [t_0, t_N] \times \mathbb{R} : n = 0, 1, 2, \ldots, N\}$ is a given set of interpolation data, where $-\infty < t_0 < t_1 < \cdots < t_N < \infty$. Let us consider $F = I \times \mathbb{R}$. Let $I = [t_0, t_N]$, $I_n = [t_{n-1}, t_n]$, and $L_n : I \longrightarrow I_n$; $n = 1, 2, \ldots, N$ be contractive homeomorphisms such that

$$L_n(t_0) = t_{n-1}, \quad L_n(t_N) = t_n. \tag{1}$$

Let $F_n : F \to \mathbb{R}$ be continuous functions such that

$$\left.\begin{aligned} &F_n(t_0, x_0) = x_{n-1},\ F_n(t_N, x_N) = x_n, \\ &|F_n(t, x) - F_n(t, x^*)| \le |\alpha_n||x - x^*|, \end{aligned}\right\} \tag{2}$$

where $t \in I$, $x, x^* \in \mathbb{R}$, and $-1 < \alpha_n < 1$; $n = 1, 2, \ldots, N$. Define, $\omega_n(t, x) = (L_n(t), F_n(t, x))$ for all $n = 1, 2, \ldots, N$. The definition of a FIF originates from the following proposition:

Proposition 2.1. *[2] The above IFS $\{I \times \mathbb{R}; \omega_n, n = 1, 2, \ldots, N\}$ has a unique attractor G. G is the graph of a continuous function $f : I \to \mathbb{R}$ which satisfies $f(t_n) = x_n$ for $n = 0, 1, 2, \ldots, N$.*

The previous function f in Proposition 2.1 is called a fractal interpolation function (FIF) corresponding to the IFS $\{I \times \mathbb{R}; \omega_n, n = 1, 2, \ldots, N\}$. Let $\mathcal{G} = \{g : I \to \mathbb{R} \mid g \text{ is continuous}, g(t_0) = x_0 \text{ and } g(t_N) = x_N\}$. Define a metric on $\mathcal{G}$ by

$$\rho(g, h) = \|g - h\|_\infty = \max\{|g(t) - h(t)| : t \in I\} \ \forall g, h \in \mathcal{G}.$$

Then, $(\mathcal{G}, \rho)$ is a complete metric space. Define the Read-Bajraktarević operator T on $(\mathcal{G}, \rho)$ such that

$$Tg(t) = F_n(L_n^{-1}(t), g(L_n^{-1}(t))), \ t \in I_n, \ \ n = 1, 2, \ldots, N. \tag{3}$$

Using Eqs. (1)-(2), it can be verified that Tg is continuous on the interval $[t_{n-1}, t_n]$ for $n = 1, 2, \ldots, N$ and at each of the points $t_1, t_2, \ldots, t_{N-1}$. T is a contraction mapping on the metric space $(\mathcal{G}, \rho)$ i.e.

$$\|Tf - Tg\|_\infty \leq |\alpha|_\infty \|f - g\|_\infty, \tag{4}$$

where $|\alpha|_\infty = \max\{|\alpha_n| : n = 1, 2, \ldots, N\}$. Since $|\alpha|_\infty < 1$, T possesses a unique fixed point f (say) on $\mathcal{G}$. Hence, it follows from Eq. (3) that the FIF satisfies the following functional relation

$$f(L_n(t)) = F_n(t, f(t)), \ t \in I, \ \ n = 1, 2, \ldots, N. \tag{5}$$

The most widely studied FIFs so far are defined by the IFS

$$\left.\begin{aligned} L_n(t) &= a_n t + b_n, \\ F_n(t, x) &= \alpha_n x + q_n(t), \end{aligned}\right\} n = 1, 2, \ldots, N, \tag{6}$$

where α_n is called a vertical scaling factor of the transformation ω_n and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is the scale vector of IFS.

To construct an α-fractal function h^α for a given continuous function h through the IFS (6), we chose $q_n(t) = h \circ L_n(t) - \alpha_n b(t)$, where b is a real continuous function such that $b \neq h, b(t_0) = x_0, \ b(t_N) = x_N$ and h satisfies $h(t_i) = x_i, i = 0, 1, 2, \ldots, N$ [9]. An upper bound of the difference between a continuous function h and its α-fractal function h^α is given in the following proposition.

Proposition 2.2. *[12] The uniform distance between the α-fractal function h^α and h is given by*

$$\|h^\alpha - h\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|h - b\|_\infty, \tag{7}$$

where $|\alpha|_\infty = \max\{|\alpha_n| : n = 1, 2, \ldots, N\}$.

Remark 2.1. *For the particular case of $\alpha = 0$, the fractal function h^α agrees with the given function h.*

The existence of a differentiable or spline FIF is guaranteed by the following proposition:

Proposition 2.3. *[3] Let $\{(t_n, x_n)|n = 0, 1, 2, \ldots, N\}$ be the interpolation data with $t_0 < t_1 < t_2 < \cdots < t_N$. Let $L_n(t) = a_n t + b_n$ that satisfies Eq. (1) and $F_n(t, x) = \alpha_n x + q_n(t)$ for $n = 1, 2, \ldots, N$. Suppose for some integer $r > 0$, $|\alpha_n| < a_n^r$, and $q_n \in C^r[t_0, t_N]; n = 1, 2, \ldots, N$. Let*

$$F_{n,k}(t,x) = \frac{\alpha_n x + q_n^{(k)}(t)}{a_n^k},\ x_{0,k} = \frac{q_1^{(k)}(t_0)}{a_1^k - \alpha_1},\ x_{N,k} = \frac{q_N^{(k)}(t_N)}{a_N^k - \alpha_N},\ k = 1, 2, \ldots, r. \quad (8)$$

If $F_{n-1,k}(t_N, x_{N,k}) = F_{n,k}(t_0, x_{0,k})$ for $n = 2, 3, \ldots, N$ and $k = 1, 2, \ldots, r$, then $\{(L_n(t), F_n(t, x))\}_{n=1}^N$ determines a FIF $f \in C^r[t_0, t_N]$ and $f^{(k)}$, $k = 1, 2, \ldots, r$, is the FIF determined by $\{(L_n(t), F_{n,k}(t, x))\}_{n=1}^N$.

Based on Proposition 2.3, we have constructed generalized Lagrange FIF in the construction of generalized Hermite FIF.

3 Generalized Hermite FIF

We construct generalized Hermite fractal interpolation functions in Section 3.1 and obtain the upper bounds of the error between the Hermite FIF and the original function in Section 3.2.

3.1 Construction of generalized Hermite FIF

Walsh [16] proved the following uniqueness theorem in the complex plane.

Proposition 3.1. *Let the distinct points $z_0, z_1, \ldots, z_N \in \mathbb{C}$ and values $w_n^{(0)}, w_n^{(1)}, \ldots, w_n^{(\eta_n - 1)} \in \mathbb{C}$, $n = 0, 1, \ldots, N$ be given. Then there exists a unique polynomial $p(z)$ of degree*

$$r = -1 + \sum_{n=0}^{N} \eta_n \quad (1)$$

which satisfies the conditions

$$p^{(\nu)}(z_n) = w_n^{(\nu)}, \quad \nu = 0, 1, \ldots, \eta_n - 1, n = 0, 1, \ldots, N.$$

The above proposition was generalized by Spitzbart [14] in the case of real functions giving an explicit expression. With the modern notations, we have chosen the following construction for the generalized Hermite interpolation [15].

Let us consider the real numbers $t_n, x_n^{(k)}$; $k = 0, 1, \ldots, \eta_n - 1$, $n = 0, 1, \ldots, N$ such that $t_0 < t_1 < \cdots < t_N$. Then there exists a unique polynomial $H(t)$ whose degree does not exceed r, where r satisfies Eq. (1) and the interpolation conditions:

$$H^{(\nu)}(t_n) = x_n^{(\nu)}, \quad \nu = 0, 1, \ldots, \eta_n - 1, n = 0, 1, \ldots, N. \quad (2)$$

The above classical generalized Hermite interpolation $H(t)$ is given by [15]

$$H(t) = \sum_{n=0}^{N} \sum_{k=0}^{\eta_n - 1} x_n^{(k)} H_{n,k}(t). \tag{3}$$

The polynomials $H_{n,k} \in \Pi_r$ are generalized Lagrange polynomials, where Π_r is the set of all real polynomials whose degree does not exceed r. To define $H_{n,k}$,we have the auxiliary polynomials

$$l_{nk}(t) = \frac{(t - t_n)^k}{k!} \prod_{j=0, j\neq n}^{N} \Big(\frac{t - t_j}{t_n - t_j}\Big)^{\eta_j}, \quad 0 \le n \le N, \quad 0 \le k < \eta_n, \tag{4}$$

put $H_{n,\eta_n-1} = l_{n,\eta_n-1}, \quad n = 0, 1, \ldots, N$, and recursively $k = \eta_n - 2, \eta_n - 3, \ldots, 0$,

$$H_{n,k}(t) = l_{nk}(t) - \sum_{s=k+1}^{\eta_n - 1} l_{nk}^{(s)}(t_n) H_{n,s}(t). \tag{5}$$

Now, we wish to construct a class of fractal functions such that each member of this class will satisfy Eq. (2). For a special case of parameters, we will obtain the classical Hermite polynomial. This fact may be useful whenever some additional condition is imposed to interpolation, for instance the solution of an optimization problem. We need the following result for the construction of a generalized Hermite FIF.

Theorem 3.1. *Let a finite set of equidistant data: $t_0 < t_1 < \cdots < t_N$ and $\{x_n^{(k)};\ k = 0, 1, \ldots, \eta_n - 1,\ n = 0, 1, \ldots, N\}$ be given. Let the fixed vertical scaling factor $\alpha_m, m = 1, 2, \ldots, N$ to be chosen in the following way:*

$$|\alpha_m| < \frac{1}{N^p}, \quad p = \max\{\eta_n; n = 0, 1, \ldots, N\} - 1. \tag{6}$$

Then, for a fixed n and any $k = 0, 1, \ldots, \eta_n - 1$, there exists a fractal function H_{nk}^{α} such that

$$(H_{nk}^{\alpha})^{(\nu)}(t_m) = (H_{nk})^{(\nu)}(t_m),\ m = 0, 1, \ldots, N;\ \nu = 0, 1, \ldots, p. \tag{7}$$

Proof. For the given equidistant data, we define

$$a_m = \frac{t_m - t_{m-1}}{t_N - t_0} = \frac{1}{N},\ m = 1, 2, \ldots, N.$$

We will define a suitable FIF $H_{nk}^{\alpha}(t)$ satisfying the conditions of Proposition 2.3. This FIF will agree at the nodes t_n (up to order p, where $p \ge \eta_n - 1$) with $H_{nk}(t)$. Consider the IFS $\{(L_m, F_m^{nk});\ m = 1, 2, \ldots, N\}$, where $L_m(t) = \frac{t}{N} + b_m$ satisfying Eq. (1) and $F_m^{nk}(t, x) = \alpha_m x + q_m^{nk}(t)$ such that

$$q_m^{nk}(t) = H_{nk} \circ L_m(t) - \alpha_m b_{nk}(t). \tag{8}$$

We will choose $b_{nk}(t)$ so that Eq. (7) is true and $b_{nk} \in C^p(I)$. From Eq. (8),

$$x^{nk}_{N,\nu} = \frac{(q^{nk}_N)^{(\nu)}(t_N)}{a^\nu_N - \alpha_N} = \frac{{H_{nk}}^{(\nu)}(t_N) - N^\nu \alpha_N b^{(\nu)}_{nk}(t_N)}{1 - N^\nu \alpha_N}$$

and

$$x^{nk}_{0,\nu} = \frac{(q^{nk}_1)^{(\nu)}(t_0)}{a^\nu_1 - \alpha_1} = \frac{{H_{nk}}^{(\nu)}(t_0) - N^\nu \alpha_1 b^{(\nu)}_{nk}(t_0)}{1 - N^\nu \alpha_1}.$$

Using these end points in the join-up conditions $F^{n,k}_{m-1,\nu}(t_N, x^{nk}_{N,\nu}) = F^{n,k}_{m,\nu}(t_0, x^{nk}_{0,\nu})$ of Proposition 2.3, we have

$$\begin{aligned} &\alpha_{m-1}\Big[\frac{{H_{nk}}^{(\nu)}(t_N) - N^\nu \alpha_N b^{(\nu)}_{nk}(t_N)}{1 - N^\nu \alpha_N} - b^{(\nu)}_{nk}(t_N)\Big] \\ &\quad = \alpha_m \Big[\frac{{H_{nk}}^{(\nu)}(t_0) - N^\nu \alpha_1 b^{(\nu)}_{nk}(t_0)}{1 - N^\nu \alpha_1} - b^{(\nu)}_{nk}(t_0)\Big],\ m = 1, 2, \ldots, N. \end{aligned}$$

If all these vertical scaling factors are same i.e. $\alpha_m = \alpha^*, m = 1, 2, \ldots, N$, the above condition gives

$${H_{nk}}^{(\nu)}(t_N) - b^{(\nu)}_{nk}(t_N) = {H_{nk}}^{(\nu)}(t_0) - b^{(\nu)}_{nk}(t_0). \tag{9}$$

The Eq. (9) is true if

$$\left.\begin{aligned} H^{(\nu)}_{nk}(t_N) = b^{(\nu)}_{nk}(t_N), \\ H^{(\nu)}_{nk}(t_0) = b^{(\nu)}_{nk}(t_0), \end{aligned}\right\} \quad \nu = 0, 1, \ldots, p, \tag{10}$$

and Eq. (10) is true if we consider Hermite polynomial $b_{nk}(t)$ with respect to nodes t_0 and t_N with contact order p. Hence, $b_{nk}(t)$ is given by

$$b_{nk}(t) = \sum_{\nu=0}^{p} H^{(\nu)}_{nk}(t_0)\tilde{H}_{0,\nu}(t) + \sum_{\nu=0}^{p} H^{(\nu)}_{nk}(t_N)\tilde{H}_{N,\nu}(t), \tag{11}$$

where $\tilde{H}_{0,\nu}$ and $\tilde{H}_{N,\nu}$ are generalized Lagrange polynomials defined as follows. Starting with the auxiliary polynomials

$$\tilde{l}_{0,\nu}(t) = \frac{(t - t_0)\nu}{\nu!}\Big(\frac{t - t_N}{t_0 - t_N}\Big)^{p+1},\ \tilde{l}_{N,\nu}(t) = \frac{(t - t_N)\nu}{\nu!}\Big(\frac{t - t_0}{t_N - t_0}\Big)^{p+1};\ 0 \le \nu \le p,$$

put $\tilde{H}_{i,p}(t) = \tilde{l}_{i,p}(t)$ for $i = 0, N$, and recursively for $\nu = p-1, p-2, \ldots, 0$ with $i = 0, N$, we have

$$\tilde{H}_{i,\nu}(t) = \tilde{l}_{i,\nu}(t) - \sum_{s=\nu+1}^{p} \tilde{l}^{(s)}_{i,\nu}(t_i)\tilde{H}_{i,s}(t). \tag{12}$$

Suppose $H^\alpha_{nk}(t)$ be the fractal function with the choice of $b_{nk}(t)$ given by Eq. (11). The IFS associated with $(H^\alpha_{nk})^\nu(t)$ is given by (cf. Proposition 2.3)

$$\left\{\begin{aligned} L_m(t) &= \frac{1}{N}t + b_m, \\ F^{nk}_{m\nu}(t, x) &= N^\nu \alpha^* x + N^\nu (q^{nk}_m)^{(\nu)}(t) = N^\nu \alpha^* x + H^{(\nu)}_{nk}(L_m(t)) - N^\nu \alpha^* {b_{nk}}^{(\nu)}(t) \end{aligned}\right. \tag{13}$$

Now, according to Eq. (8) and (10)

$$\begin{aligned}
H^{\alpha}_{nk}{}^{(\nu)}(t_0) = x^{nk}_{0\nu} = \frac{(q^{nk}_1)^{(\nu)}(t_0)}{a^{\nu}_1 - \alpha^*} &= \frac{1}{a^{\nu}_1 - \alpha^*}\Big(\frac{H^{(\nu)}_{nk}(L_1(t_0))}{N^{\nu}} - \alpha^* b_{nk}{}^{(\nu)}(t_0)\Big) \\
&= \frac{1}{1-\alpha^* N^{\nu}}\Big(H^{(\nu)}_{nk}(t_0) - \alpha^* N^{\nu} b_{nk}{}^{(\nu)}(t_0)\Big) = H^{(\nu)}_{nk}(t_0).
\end{aligned}$$

Similarly, we have $H^{\alpha}_{nk}{}^{(\nu)}(t_N) = H^{(\nu)}_{nk}(t_N)$. For intermediate points $m = 1, 2, \ldots, N-1$,

$$\begin{aligned}
H^{\alpha}_{nk}{}^{(\nu)}(t_m) &= F^{nk}_{m\nu}(L^{-1}_m(t_m), H^{\alpha}_{nk}{}^{(\nu)} \circ L^{-1}_m(t_m)) \\
&= N^{\nu}\alpha^* H^{\alpha}_{nk}{}^{(\nu)} \circ L^{-1}_m(t_m) + H^{(\nu)}_{nk}(t_m) - N^{\nu}\alpha^* b_{nk}{}^{(\nu)} \circ L^{-1}_m(t_m) \\
&= N^{\nu}\alpha^* H^{\alpha}_{nk}{}^{(\nu)}(t_N) + H^{(\nu)}_{nk}(t_m) - N^{\nu}\alpha^* b_{nk}{}^{(\nu)}(t_N) = H^{(\nu)}_{nk}(t_m).
\end{aligned}$$

Hence, H^{α}_{nk} is the required generalized Lagrange fractal function. □

Definition 3.1. *With these Lagrange fractal functions $H^{\alpha}_{n,k}$ defined in Theorem 3.1, the generalized Hermite FIF H^{α} is given by*

$$H^{\alpha}(t) = \sum_{n=0}^{N}\sum_{k=0}^{\eta_n - 1} x^{(k)}_n H^{\alpha}_{n,k}(t). \tag{14}$$

Remark 3.1. *If $\alpha = 0$, $H^{\alpha}_{nk} = H_{nk}$ and $H^{\alpha} = H$ (cf. Remark 2.1) and we obtain the classical Hermite polynomial case.*

For a fixed partition Δ, a fixed scaling vector α and fixed set of $\{\eta_0, \eta_1, \ldots, \eta_N\}$, We can define an operator $\mathcal{H}$ on $C^p(I)$ such that $\mathcal{H}(g)$ is the generalized Hermite FIF (cf. Definition 3.1) for a function $g \in C^p(I)$.

Theorem 3.2. *$\mathcal{H}$ is a linear and bounded operator of $C^p(I)$.*

Proof. For given $g, h \in C^p(I)$ and real scalar λ, Definition 3.1, we have

$$\begin{aligned}
\mathcal{H}(g+h) &= \sum_{n=0}^{N}\sum_{k=0}^{\eta_n - 1} (g+h)^{(k)}(t_n) H^{\alpha}_{n,k}(t) = \mathcal{H}(g) + \mathcal{H}(h), \\
\mathcal{H}(\lambda g) &= \sum_{n=0}^{N}\sum_{k=0}^{\eta_n - 1} (\lambda g)^{(k)}(t_n) H^{\alpha}_{n,k}(t) = \lambda \mathcal{H}(g).
\end{aligned}$$

We consider $C^p(I)$ endowed with the norm $\|g\|_{C^p(I)} = \sup\limits_{0 \le k \le p} \|g^{(k)}\|_{\infty}$. Now,

$$\begin{aligned}
\|(\mathcal{H}(g))^{(\nu)}\|_{\infty} &= \sup_{t \in I} |\sum_{n=0}^{N}\sum_{k=0}^{\eta_n - 1} g^{(k)}(t_n)(H^{\alpha}_{n,k})^{(\nu)}(t)| \\
&\le \|g\|_{C^p(I)} \sum_{n=0}^{N}\sum_{k=0}^{\eta_n - 1} \|(H^{\alpha}_{n,k})^{(\nu)}\|_{\infty}
\end{aligned} \tag{15}$$

From the IFS (13) of $(H^\alpha_{n,k})^{(\nu)}$, the vertical scaling factor is $N^\nu\alpha^*$. Using Proposition 2.2, for $\nu = 0, 1, \dots, p$,

$$\begin{aligned}\|(H^\alpha_{n,k})^{(\nu)} - (H_{n,k})^{(\nu)}\|_\infty &\le \frac{N^\nu|\alpha^*|}{1 - N^\nu|\alpha^*|}\|(H_{n,k})^{(\nu)} - (b_{n,k})^{(\nu)}\|_\infty \\ &\le \frac{N^p|\alpha^*|}{1 - N^p|\alpha^*|}\|H_{n,k} - b_{n,k}\|_{C^p(I)}\end{aligned}$$

this gives

$$\|(H^\alpha_{n,k})^{(\nu)}\|_\infty \le \frac{N^p|\alpha^*|}{1 - N^p|\alpha^*|}\|H_{n,k} - b_{n,k}\|_{C^p(I)} + \|H_{n,k}\|_{C^p(I)} \tag{16}$$

Combining Eq. (15) and Eq. (16), an upper bound of $\mathcal{H}$ is given by

$$\|\mathcal{H}\| \le \frac{N^p|\alpha^*|}{1 - N^p|\alpha^*|}\sum_{n=0}^{N}\sum_{k=0}^{\eta_n-1}\|H_{n,k} - b_{n,k}\|_{C^p(I)} + \sum_{n=0}^{N}\sum_{k=0}^{\eta_n-1}\|H_{n,k}\|_{C^p(I)}$$

□

3.2 Upper bounds of the error

We need the following result from classical numerical analysis [15].

Proposition 3.2. *Consider the real numbers $t_n, g^{(k)}(t_n)$, $k = 0, 1, \dots, \eta_n - 1$, $n = 0, 1, \dots, N$ such that $t_0 < t_1 < \cdots < t_N$ where g is r-times differentiable with $r+1 = \sum_{n=0}^{N}\eta_n$. If the polynomial $H(t)$ is of degree at most r and satisfies the following interpolation conditions*

$$H^{(k)}(t_n) = g^{(k)}(t_n), \quad k = 0, 1, \dots, \eta_n - 1, n = 0, 1, \dots, N,$$

then to every $\bar{t} \in [t_0, t_N]$, there exists $\xi \in [t_0, t_N]$ such that

$$g(\bar{t}) - H(\bar{t}) = \frac{\omega(\bar{t})g^{(r+1)}(\xi)}{(r+1)!}, \tag{17}$$

where $\omega(t) = (t - t_0)^{\eta_0}(t - t_1)^{\eta_1}\dots(t - t_N)^{\eta_N}$.

Let H^α be the Hermite FIF to the original function f. To compute an upper bound of uniform distance between f and H^α, we use Proposition 3.2, Proposition 2.2 and the definitions of H and H^α:

$$\begin{aligned}\|f - H^\alpha\|_\infty &\le \|f - H\|_\infty + \|H - H^\alpha\|_\infty \\ &\le \frac{K\|f^{(r+1)}\|_\infty}{(r+1)!} + \|H - H^\alpha\|_\infty, \quad K = \sup_{t\in[t_0,t_N]}|\omega(t)| \\ &\le \frac{K\|f^{(r+1)}\|_\infty}{(r+1)!} + \sum_{n=0}^{N}\sum_{k=0}^{\eta_n-1}\|f^{(k)}\|_\infty\|H_{n,k} - H^\alpha_{n,k}\|_\infty \\ &\le \frac{K\|f^{(r+1)}\|_\infty}{(r+1)!} + \sum_{n=0}^{N}\sum_{k=0}^{\eta_n-1}\|f^{(k)}\|_\infty\frac{|\alpha^*|}{1 - |\alpha^*|}\|H_{n,k} - b_{n,k}\|_\infty\end{aligned} \tag{18}$$

In order to compute $\|H_{n,k} - b_{n,k}\|_\infty$, we need the following proposition.

Proposition 3.3. *[6, 15] Let* $x(t) \in C^u[t_0, t_N]$ *with* $u \geq 2s+2$*, let* Δ *be any partition of* $[t_0, t_N]$*,* $\Delta : t_0 < t_1 < \cdots < t_N$*, and let* $\phi(t)$ *be the unique Hermite interpolation of* $x(t)$ *such that* $x^{(\nu)}(t_n) = \phi^{(\nu)}(t_n)$ *for all* $0 \leq n \leq N,\ 0 \leq \nu \leq s$*. Then, for all* l *with* $0 \leq l \leq s$

$$\|x^{(l)} - \phi^{(l)}\|_\infty \leq \frac{\|\Delta\|^{2s+2-l}}{2^{2s+2-2l} s!(2s+2+2l)!}\|x^{(2s+2)}\|_\infty \tag{19}$$

Taking $x(t) = H_{n,k}(t)$ and $\phi(t) = b_{n,k}(t)$ with $l = 0$ and $s = 0$ in Eq. (19), we have

$$\|H_{n,k} - b_{n,k}\|_\infty \leq \frac{|I|^2}{8}\|H^{(2)}_{n,k}\|_\infty, \tag{20}$$

where I is the length of the interval $I = [t_0, t_N]$. Set $L = \max\{\|f^{(k)}\|_\infty : k = 0, 1, \ldots, \eta_n - 1; n = 0, 1, \ldots, N\}$ and $\Theta = \max\{\|H^{(2)}_{n,k}\|_\infty : k = 0, 1, \ldots, \eta_n - 1; n = 0, 1, \ldots, N\}$. From Eq. (18) and Eq. (20), an upper bound of uniform error bound between the original function and generalized Hermite FIF is given by

$$\|f - H^\alpha\|_\infty \leq \frac{K\|f^{(r+1)}\|_\infty}{(r+1)!} + \frac{|I|^2(r+1)}{8}L\Theta. \tag{21}$$

For convergence of H^α and its derivatives towards the original function f, consider the following for $\nu = 0, 1, \ldots, p$.

$$\begin{aligned}\|H^{(\nu)} - (H^\alpha)^{(\nu)}\|_\infty &\leq \sum_{n=0}^{N}\sum_{k=0}^{\eta_n-1}\|f^{(k)}\|_\infty \frac{N^\nu|\alpha^*|}{1-N^\nu|\alpha^*|}\|H^{(\nu)}_{n,k} - b^{(\nu)}_{n,k}\|_\infty \\ &\leq \sum_{n=0}^{N}\sum_{k=0}^{\eta_n-1}\|f\|_{C^p(I)} \frac{N^p|\alpha^*|}{1-N^p|\alpha^*|}\|H_{n,k} - b_{n,k}\|_{C^p(I)}\end{aligned}$$

which gives that

$$\|H - H^\alpha\|_{C^p(I)} \leq \frac{N^p|\alpha^*|\|f\|_{C^p(I)}}{1-N^p|\alpha^*|}\sum_{n=0}^{N}\sum_{k=0}^{\eta_n-1}\|H_{n,k} - b_{n,k}\|_{C^p(I)} \tag{22}$$

Hence, we have the following result.

Theorem 3.3. *Let* $f(t) \in C^p(I)$ *be the original function approximated by a generalized Hermite FIF* $H^\alpha(t)$ *such that* $|\alpha^*| < \frac{1}{N^p}$*, where* $p = \max\{\eta_n; n = 0, 1, \ldots, N\} - 1$*. Then,*

$$\|f - H^\alpha\|_{C^p(I)} \leq \|f - H\|_{C^p(I)} + \frac{N^p|\alpha^*|\|f\|_{C^p(I)}}{1-N^p|\alpha^*|}\sum_{n=0}^{N}\sum_{k=0}^{\eta_n-1}\|H_{n,k} - b_{n,k}\|_{C^p(I)}$$

Proof. Using the triangle inequality,

$$\|f - H^\alpha\|_{C^p(I)} \leq \|f - H\|_{C^p(I)} + \|H - H^\alpha\|_{C^p(I)}$$

and Eq. (22), Theorem 3.3 follows. For the first term of the sum, one can apply a classical theorem of Hermite interpolation error. □

4 Examples

Suppose $N = 2$, $\eta_0 = 3, \eta_1 = 4, \eta_2 = 2$ and

$$\begin{aligned} &t_0 = 0, \quad x_0 = 1, \quad x_0^{(1)} = 4, \quad x_0^{(2)} = 9; \\ &t_1 = 2, \quad x_1 = -1, \quad x_1^{(1)} = -3, \quad x_1^{(2)} = -8, \quad x_1^{(3)} = -5; \\ &t_2 = 4, \quad x_2 = 3, \quad x_2^{(1)} = 7. \end{aligned}$$

From the given data, we have $a_m = 1/2$ for $m = 1, 2$ and

$$L_1(t) = \frac{t}{2}, \quad L_1(t) = \frac{t}{2} + 2.$$

According to Eq. (6), $\alpha^* < \frac{1}{2^3}$.
For $n = 0$, using Eq. (5) , we have

$$\begin{aligned} H_{0,0}(t) &= \frac{(1 + \frac{5}{2}t + \frac{59}{16}t^2)(t-2)^4(t-4)^2}{256}, \quad H_{0,2}(t) = \frac{t^2(t-2)^4(t-4)^2}{512}, \\ H_{0,1}(t) &= \frac{(2t + 5t^2)(t-2)^4(t-4)^2}{512}. \end{aligned}$$

Similarly, for $n = 1$, we have

$$\begin{aligned} H_{1,0}(t) &= \frac{t^3(64 - 64t + 24t^2 - 3t^3)(t-4)^2}{256}, \quad H_{1,2}(t) = \frac{t^3(t-2)^2(4-t)^3}{128}, \\ H_{1,1}(t) &= \frac{t^3(20 - 14t + 3t^2)(t-2)(t-4)^2}{128}, \quad H_{1,3}(t) = \frac{t^3(t-2)^3(t-4)^2}{192} \end{aligned}$$

and for $n = 2$,

$$H_{2,0}(t) = \frac{(48 - 11t)t^3(t-2)^4}{4096}, \quad H_{2,1}(t) = \frac{t^3(t-4)(t-2)^4}{1024}.$$

For the construction of b_{nk}, we need the following Hermite polynomials of order 3 (cf. Eq. (12)).

$$\begin{aligned} \tilde{H}_{0,0}(t) &= \frac{(1 + t + \frac{5}{8}t^2 + \frac{5}{16}t^3)(t-4)^4}{256}, \quad \tilde{H}_{0,2}(t) = \frac{t^2(1+t)(t-4)^4}{512}, \\ \tilde{H}_{0,1}(t) &= \frac{(1 + t + \frac{5}{8}t^2)t(t-4)^4}{256}, \quad \tilde{H}_{0,3}(t) = \frac{t^3(t-4)^4}{1536}, \\ \tilde{H}_{2,0}(t) &= \frac{(35 - 21t + \frac{35}{8}t^2 - \frac{5}{16}t^3)t^4}{256}, \quad \tilde{H}_{2,2}(t) = \frac{t^4(t-4)^2(5-t)}{512}, \\ \tilde{H}_{2,1}(t) &= \frac{(15 - 6t + \frac{5}{8}t^2)(t-4)t^4}{256}, \quad \tilde{H}_{2,3}(t) = \frac{t^4(t-4)^3}{1536}. \end{aligned}$$

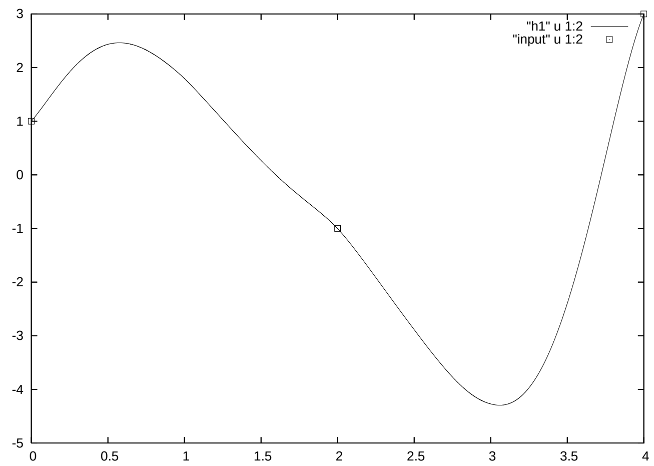

Figure 1.— Hermite FIF with $\alpha^* = 0.1$.

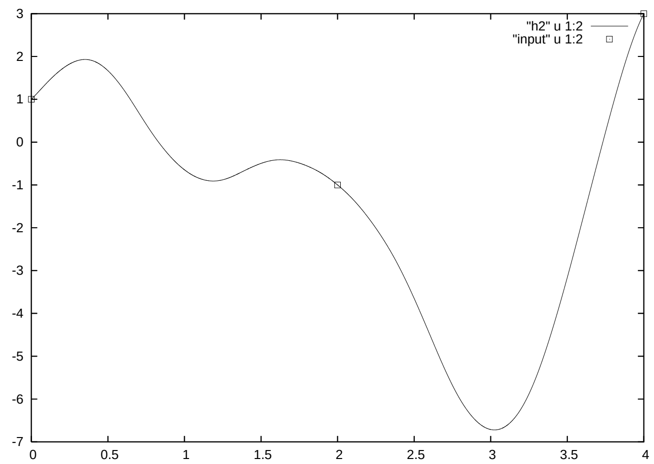

Figure 2.— Hermite FIF with $\alpha^* = -0.1$.

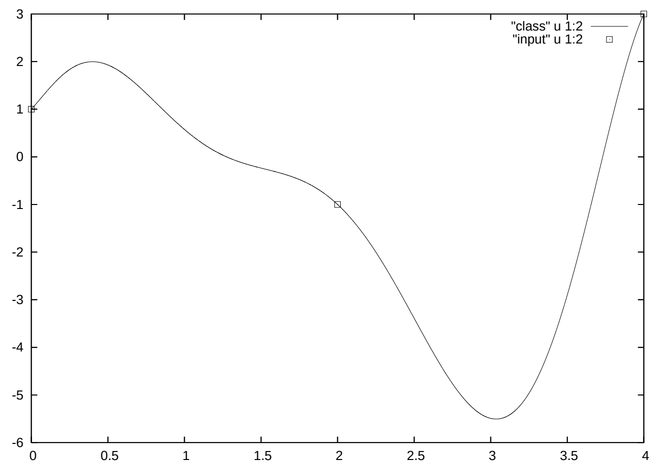

Figure 3.— Classical Hermite Interpolation with $\alpha^* = 0$.

From Eq. (11), the desired b_{nk} are computed as follows.

$$b_{0,0}(t) = \frac{(32+16t+6t^2-132t^3+21t^4+5t^5)(t-4)^2}{512}, \quad b_{0,2}(t) = \frac{(2-4t+t^2)t^2(t-4)^2}{64},$$
$$b_{0,1}(t) = \frac{(16+8t-56t^2+12t^3+t^4)t(t-4)^2}{256},$$
$$b_{1,0}(t) = \frac{(4-t)^3t^3}{16}, \qquad b_{1,1}(t) = \frac{(-10+t^2)(t-4)^2t^3}{32},$$
$$b_{1,2}(t) = \frac{(4-t)^3t^3}{32}, \qquad b_{1,3}(t) = \frac{(-4-2t+t^2)(t-4)^2t^3}{96},$$
$$b_{2,0}(t) = \frac{(96+138t-164t^2+51t^3-5t^4)t^3}{512}, \quad b_{2,1}(t) = \frac{(4+8t-6t^2+t^3)(t-4)t^3}{256}.$$

From Theorem 3.1, the IFS for the generalized Lagrange FIF corresponding to $H^{\alpha}_{n,k}$ is given by

$$\begin{cases} L_m(t) = \dfrac{t}{N} + b_m, \\ F^{nk}_m(t,x) = \alpha_m x + H_{nk} \circ L_m(t) - \alpha_m b_{nk}(t). \end{cases} \tag{1}$$

In view of Eq. (6), we have chosen $\alpha^* = 0.1$ and $\alpha^* = -0.1$ in the construction of the generalized Hermite FIFs. With the above choices for the vertical scaling factors, and the mappings H_{nk} and b_{nk}, the iteration of the resulting IFS code Eq. (1) generates the generalized Lagrange FIFs $H^{\alpha}_{n,k}$. Using these $H^{\alpha}_{n,k}$, we have plotted the generalized Hermite FIFs of Eq. (14) (see Figure 1 and Figure 2). Finally, we have chosen $\alpha^* = 0$. Since $F^{nk}_m(t,x)$ reduces to function of t only in this case, $H^{\alpha}_{n,k}(t) = H_{n,k}(t)$ and consequently $H^{\alpha}(t)$ agrees with the classical Hermite interpolation function $H(t)$ (see Figure 3). With the effective use of scaling factors, our construction of Hermite FIFs offers additional advantage over the classical Hermite interpolation functions. The use of Hermite FIF may be exploited in computer graphics, radial basis functions, and smooth object approximation in scientific problems as one can have infinite number of Hermite FIFs depending on the scaling factors, giving thereby a large flexibility in the choice of Hermite FIFs according to the need of an experiment.

5 Conclusion

In this paper, we generalize the classical Hermite interpolation by a class of fractal interpolants. This problem prescribes at each support mesh points not only the value of a function but also its derivative up to any finite order. The order of derivatives at each node is non-constant and generalizes the case considered in the reference [11]. In the simulation of smooth curves, our construction offers a large flexibility in the choice of a suitable interpolating Hermite fractal curves. This fact may be useful whenever some additional condition is imposed to the interpolation, for instance the solution of an optimization problem. Some upper bounds for the error between the original function f

and the generalized Hermite FIF are deduced and these bounds depend on the scaling vector α.

Acknowledgments

The work was supported by the project No: SB 2005-0199, Spain. The first author is thankful to the Institute of Mathematics and Applications, Bhubaneswar, India for its support after this project.

References

[1] M. F. Barnsley, Fractal functions and interpolation, Constr. Approx., 2, pp. 303-329 (1986).

[2] M. F. Barnsley, Fractals Everywhere, Academic Press, Orlando, Florida 1988.

[3] M. F. Barnsley and A. N. Harrington, The calculus of fractal interpolation functions, J. Approx. Theory, 57, pp. 14-34 (1989).

[4] G. D. Birkhoff, General mean value and remainder theorems with applications to mechanical differentiation and integration, Trans. Amer. Math. Soc., 7, pp. 107-136 (1906).

[5] A. K. B. Chand and G. P. Kapoor, Generalized cubic spline fractal interpolation functions, SIAM J. Num. Anal., 44(2), pp. 655-676 (2006).

[6] P. G. Ciarlet, M. H. Schultz and R. S. Varga, Numerical methods of high-order accuracy for nonlinear boundary value problems. I. One dimensional problem, Numer. Math., 9, pp 394-430 (1967).

[7] G. Fasshauer, Hermite interpolation with radial basis functions on spheres, Advances in Computational Mathematics, 10(1), pp. 81-96 (1999).

[8] C. Hermite, Sur la formule d'interpolation de Lagrange, J. Reine Angew. Math., 84, pp. 70-79 (1878).

[9] M. A. Navascués, Fractal polynomial interpolation, Z. Anal. Anwend, 25(2), pp. 401-418 (2005).

[10] M. A. Navascués and A. K. B. Chand, Fundamental sets of fractal functions, Acta Applicandae Mathematicae, 100, pp. 247-261 (2008).

[11] M. A. Navascués and M. V. Sebastián, Generalization of Hermite functions by fractal interpolation, J. Approx. Theory, 131(1), pp. 19-29 (2004).

[12] M. A. Navascués and M. V. Sebastián, Fitting curves by fractal interpolation: an application to the quantification of cognitive brain processes, Thinking in Patterns: Fractals and Related Phenomena in Nature. M. M. Novak, (ed.), pp. 143-154, World Scientific, 2004.

[13] D. Salomon, Curves and Surfaces for Computer Graphics, Springer Verlag, 2005.

[14] A. Spitzbart, A generalization of Hermite's interpolation formula, Amer. Math. Monthly, 67, pp. 42-46 (1960).

[15] J. Stoer and R. Bulirsch, Introduction to Numerical Analysis, Springer, New York, 1980.

[16] J. L. Walsh, Interpolation and approximation by rational functions in the complex domain, Amer. Math. Society Colloquium Publications, 20, pp. 49-54 (1935).