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Interpolation and reconstruction of curves and surfaces

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1 Polynomial interpolation in one variable

Interpolation is an ubiquitous technique arising in Mathematics, specially in Numerical Analysis. The name interpolation itself means *to place something between selected sites* , in contrast to extrapolation which means *place something outside*. Interpolation is generally more accurate than extrapolation leading to better stability in computations and lower error bounds. Since most methods can be used for both interpolation and extrapolation, the word interpolation is preferred.

Simple interpolation methods can be dated back to babylonians and the greeks. The Mathematical Syntaxis of Ptolemy shows how to construct a table of chords as a function of its corresponding arcs, the first table of a trigonometric function $2\sin(x/2)$. John Napier and Joost Bürgi introduced in the 17th century tables of logarithms which were improved by Henry Briggs, who introduced new tables of functions. The tables of functions contained a lot of decimal places and computing them was a very laborious task. The idea of approximating a curve by its chord or a parabola leads to a first instance of linear or quadratic interpolation. Interpolation formulae were implicit or explicitly used for constructing and filling up the tables. Briggs propose to subtitute the true logarithmic function by a quadratic function in some parts of the table (see Chapter 1 of [22]) which can be regarded as an interpolation by a quadratic polynomial. The result obtained leads to very accurate results.

The final user of a table might need to access to a value not contained in the table without having to recompute a whole part of the table. If extremely accurate results are not required, tables can be filled up using linear interpolation. For better results, quadratic and cubic interpolation methods are preferred. This offers the possibility of

using short tables, which are more handy if not many significant digits are required. Only a subtraction, a multiplication and an addition are enough to obtain the value of a linear interpolant.

Assume that a table of a function f is known at two points x_0, x_1. The chord joining $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is a segment whose equation is $y = p(x)$, $x \in [x_0, x_1]$, where

$$p(x) := f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0).$$

Substituting the true value of the function $f(x)$ by the value of the first degree polynomial $p(x)$ might lead to big errors but if the function is smooth and the points x_0, x_1 are close enough, the error might be very small. In fact, the difference $f(x) - p(x)$ can be bounded in terms of the second derivative of f by the formula

$$|f(x) - p(x)| \leq \frac{M}{8}h^2, \quad M := \max_{x \in [x_0, x_1]} |f''(x)|, \quad h := x_1 - x_0. \tag{1.1}$$

Therefore, if the second derivative is bounded, the interpolation error tends to zero when $h \to 0$. Figure 1 shows how close is the linear interpolant p to the logarithm function at $x_0 = 1$, $x_1 = 2$. Figure 4 shows the linear interpolant of the log function on the intervals $[1, 3]$, $[3, 5]$ and $[5, 7]$. We observe that the second derivative of the log function $-x^{-2}$ is much lower on the interval $[5, 7]$ than on the interval $[1, 3]$ and so the interpolation error is lower on $[5, 7]$ than on $[1, 3]$.

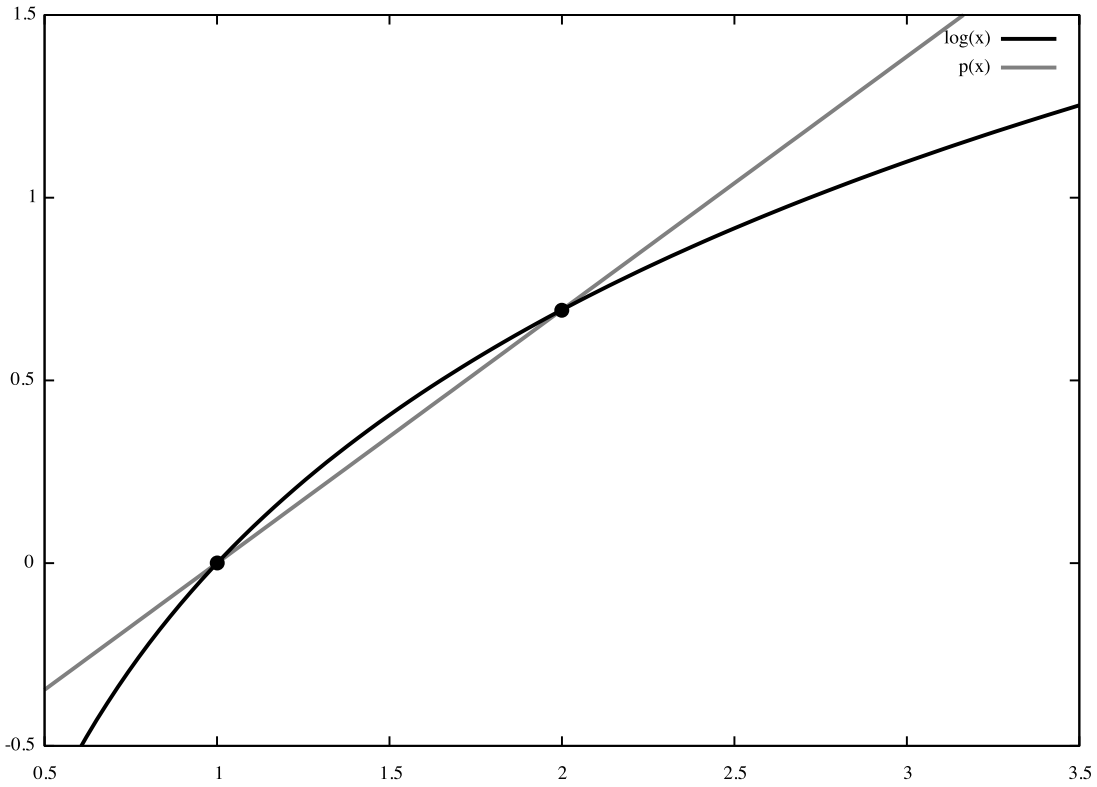

Figure 1. The natural logarithm and its linear interpolant at $x_0 = 1$, $x_1 = 2$.

While the error for linear interpolation is quadratic on h, the convergence order can be improved by using polynomials of higher degree.

Let us pose the Lagrange interpolation problem for polynomials of degree less than or equal to n. Given a function f, the Lagrange interpolation problem at $x_0 < \cdots < x_n$ consists of finding a polynomial p of degree not greater than n such that $p(x_i) = f(x_i)$, $i = 0, \ldots, n$. The solution of the problem is provided by the Lagrange formula

$$p(x) = \sum_{i=0}^{n} f(x_i) l_i(x), \quad l_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}.$$

Another possibility is to express the interpolant as a sum of polynomials of increasing degree by Newton's formula

$$p(x) = \sum_{k=0}^{n} f[x_0, \ldots, x_{k-1}] \pi_k(x), \quad \pi_k(x) = \prod_{i=0}^{k-1} (x - x_i),$$

where $f[x_i, \ldots, x_{i+k}]$ are the k-th order divided differences, which can be computed by the recurrence formula

$$f[x_i, \ldots, x_{i+k}] = \frac{f[x_{i+1}, \ldots, x_{i+k}] - f[x_i, \ldots, x_{i+k-1}]}{x_{i+k} - x_i},$$

starting from the initial values $f[x_i] := f(x_i)$, $i = 0, \ldots, n$. A third formula for the interpolant is the Aitken-Neville formula

$$p(x \mid x_i, \ldots, x_j) = \frac{x - x_i}{x_j - x_i} p(x \mid x_{i+1}, \ldots, x_j) + \frac{x_j - x}{x_j - x_i} p(x \mid x_i, \ldots, x_{j-1}),$$

where $p(x \mid x_0, \cdots, x_n)$ denotes the value at x of the interpolating polynomial of f at $x_0, \ldots, x_n$. The Aitken-Neville formula suggests an algorithm for computing the interpolant based on the idea of repeated linear interpolation.

A formula for the error of the interpolant is of the form

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i), \tag{2.1}$$

where n is the degree, ξ is any point of a compact interval I containing all the interpolation sites $x_0, \ldots, x_n$. A typical error bound for the interpolant can be expressed in the form

$$|f(x) - p(x)| \leq K \sup_{x \in I} |f^{(n+1)}(x)| \, h^{n+1},$$

where h is the length of a compact interval I containing all the interpolation sites and K is a constant which does not depend on f but might depend on the distribution of the interpolation sites in I. Let us observe, that even with very low degrees, powerful approximation methods can be devised using polynomial interpolation.

As an example, let us discuss how polynomial interpolation can be used to compute the functions $\sin x$ and $\cos x$ using only few arithmetic operations similarly as it is done

in scientific calculators, function libraries of programming languages or simple computer applications. The functions $\sin x$, and $\cos x$ are periodical of period 2π. Furthermore

$$\sin(x+\pi) = -\sin x, \quad \cos(x+\pi) = -\cos x$$

Therefore, if $\hat{x}$ is the value in $(-\pi/2, \pi/2]$ such that

$$n := \frac{x-\hat{x}}{\pi} \in \mathbf{Z},$$

then we can reduce the problem of computing the values of both trigonometric functions to the interval $(-\pi/2, \pi/2]$

$$\sin(x) = (-1)^n \sin(\hat{x}), \quad \cos(x) = (-1)^n \cos(\hat{x}).$$

The relations

$$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x), \quad \sin(\pi/2-x) = \cos(x), \quad \cos(\pi/2-x) = \sin(x),$$

allow us to reduce the problem to the interval $[0, \pi/4]$ for both functions. Taking into account that the cosine can be expressed in terms of the sine of the half angle

$$\cos x = 1 - 2\sin^2(x/2),$$

the problem is reduced to compute the value $\sin x$ for $x \in [0, \pi/4]$. There are many approches for computing the value. One might use the MacLaurin expansion

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - e_9(x), \quad x \in [0,1],$$

where $0 < e_9(x) < 1.8 \times 10^{-9}$. Computing the sine function using the above expansion has computational cost of 13 operations: 4 subtractions, 5 products and 4 divisions. Another approach is the use of tables. The function has been previously computed with high accuracy (for instance, taking enough terms in the MacLaurin expansion) at selected points obtaining a table of the function, which is kept in memory. These tables can be combined with the addition formulae to reduce the interval to be considered, allowing to reduce the number of terms in the MacLaurin expansion. Another possibility is to subtitute $\sin(x)$ by a polynomial approximation. The Approximation Theory shows that polynomials obtained by interpolation exhibit remarkable approximation properties.

A good approximation of $\sin(x)$ on $[0, \pi/4]$ can be obtained using the symmetry properties of this function. Let us introduce the cardinal sine function

$$\mathrm{sinc}(x) = \frac{\sin(x)}{x},$$

and consider the polynomial interpolant $p(x)$ of degree $2n+3$ at the symmetrically placed sites

$$-x_n < -x_{n-1} < \cdots < -x_0 < 0 < x_0 < \cdots < x_{n-1} < x_n$$

of $\mathrm{sinc}(x)$. Due to the fact that $\mathrm{sinc}(x)$ is an even function, it immediately follows that $p(x)$ is also an even function with $p(0) = 1$. Therefore the interpolant $p(x)$ can be written in the form

$$p(x) = 1 - x^2 q(x^2),$$

where $q(t)$ is the polynomial of degree n which interpolates the function

$$f(t) = \frac{1 - \mathrm{sinc}(\sqrt{t})}{t}$$

at the sites $t_i = x_i^2$, $i = 0, \ldots, n$. The error formulae of polynomial interpolation allow us to deduce that

$$|\sin(x) - x(1 - x^2 q(x^2))| \leq \frac{1}{(2n+4)!} \sin(x) x^3 |x^2 - x_0^2| \cdots |x^2 - x_n^2|, \quad x \in [0, \pi/4].$$

and taking

$$n = 2, \quad x_0 = 1/2, \quad x_1 = 5/8, \quad x_2 = 3/4,$$

we have the following bound for the relative error

$$e_r(x) = \frac{|\sin(x) - x(1 - x^2 q(x^2))|}{\sin(x)} \leq \frac{1}{40320} x^3 \Big|x^2 - \frac{1}{4}\Big| \Big|x^2 - \frac{25}{64}\Big| \Big|x^2 - \frac{9}{16}\Big|, \quad x \in [0, \pi/4].$$

We may bound $e_r(x)$ by the maximum value of the right hand side attained at $x = \pi/4$ to find that

$$e_r(x) \leq 5.42 \times 10^{-8} \tag{2.2}$$

which is lower than the maximal accuracy that can be ensured in simple precision arithmetic $\epsilon = 2^{-24} \approx 5.96 \times 10^{-8}$.

Now it remains to compute the polynomial q. First, we need to have the values of $\sin(1/2)$, $\sin(5/8)$ and $\sin(3/4)$, which can be obtained taking enough terms in the MacLaurin expansion. Then we compute the quadratic interpolation polynomial to $f(t)$ at $1/16, 1/4, 9/16$

$$q(t) \approx 0.16666652 - 0.0083930591\, t + 0.00019512189\, t^2.$$

and obtain an approximation of the sine function on $[0, \pi/4]$ using the formula

$$\sin(x) \approx x(1 - x^2 q(x^2)).$$

The evaluation of the sine then is reduced to 3 products and 1 sum and the computational cost of an evaluation of q, which needs 2 sums and 2 products. So the amount of computations to be done to compute the sine function on $[0, \pi/4]$ is 8 operations: 5 products and 3 sums. Taking into account that trigonometric functions might be evaluated many times in a computer program, the saving in computational cost implies a

substantial reduction in the computation time of many tasks in scientific computing. The maximal error occurs in $x = \pi/4$. Computations in double precision arithmetic lead to the value 0.7071067752634779 whose relative error with respect to the correct value $\sqrt{2}/2 \approx 0.7071067811865476$ is less than 10^{-8}, lower than the predicted error bound (2.2) and below the unit roundoff in single precision arithmetic.

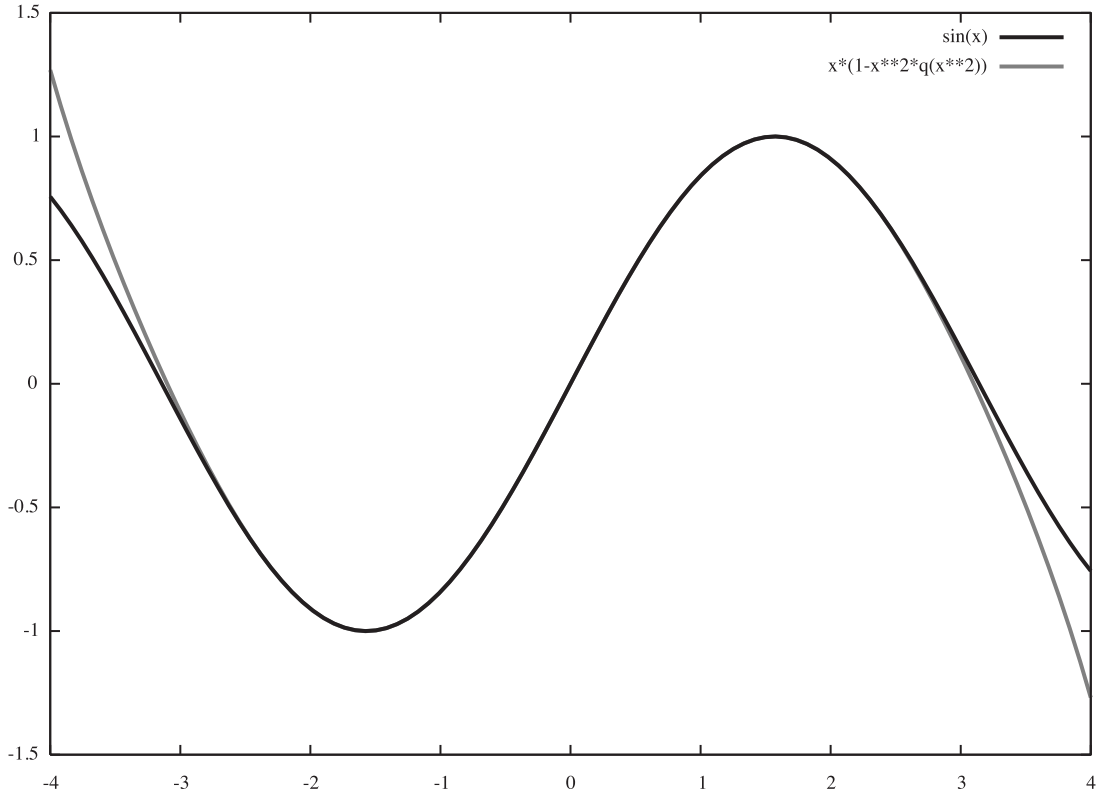

Figure 2. The sine function and the 7th degree polynomial approximant

Figure 2 shows that the approximant is very close to the sine function even in wider intervals than $[-\pi/4, \pi/4]$. The figure shows that the absolute error keeps very low even in the interval $[-\pi, \pi]$ which is 4 times wider. The absolute error in the approximation of $\sin \pi = 0$ is 4.7×10^{-2}.

2 High amplitude oscilations of polynomial interpolants

Polynomial interpolants may not look as expected. They might oscillate much more than the function to be interpolated. It often happens that, when increasing the degree, the error increases. In fact, the sequence of interpolants might not converge to the function. A first question which arises is to derive conditions which imply the convergence of the interpolant. Since the power series centered at c can be seen as an special case when all the interpolation sites tend to coincide with c, we deduce that the analyticity of f at some points might play an important role in the discussion of the convergence. Formula (2.1) allows us to deduce the following error bound for the interpolant p_n at sites

$x_{0,n}, \ldots, x_{n,n}$ in a compact interval $[a, b]$ of length $h = b - a$

$$|f(x) - p_n(x)| \leq \frac{M_{n+1}}{(n+1)!} h^n, \quad M_{n+1} := \max_{x \in I} |f^{(n+1)}(x)|.$$

We observe that if

$$\frac{M_{n+1} h^n}{(n+1)!} \to 0,$$

then p_n converges to f uniformly on the interval I. This condition is satisfied if f is analytical in $[a-h/2, b+h/2]$ and the power series centered at $c = (a+b)/2$ has convergence radius $R > 3h/2$. In general convergence of the interpolant might depend on the function f and the distribution of the sites in the interval $[a, b]$. In [26], Carl Runge studied the convergence in the case of equidistant sites $x_i = a + ih/n$, $i = 0, \ldots, n$. He showed that even in the simple case of interpolation of a C^∞ differentiable function with equidistant sites, the interpolant might not converge. For his example, he chose the function $f(x) = (1 + x^2)^{-1}$ on the interval $[-5, 5]$. Observe that the function can be expanded in a power series centered at the origin

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}, \quad |x| < 1$$

with convergence radius $R = 1 < 15 = 3h/2$. An analysis of the error formula (2.1) for even degree $n = 2k$ leads to

$$\Big| \frac{1}{1+x^2} - p_{2k}(x) \Big| = \frac{1}{1+x^2} \prod_{j=0}^{k} \frac{|x^2 - (5j/k)^2|}{1 + (5j/k)^2}.$$

Taking into account that

$$\lim_{k \to \infty} \frac{5}{k} \log \Big(\prod_{j=0}^{k} \frac{|x^2 - (5j/k)^2|}{1 + (5j/k)^2} \Big) = q(x) := \int_0^5 \log \Big(\frac{|x^2 - t^2|}{1 + t^2} \Big) dt$$

the convergence depends on the sign of $q(x)$. Let $\xi \approx 3.63$ be the zero of q on $[0, 5]$. It can be shown that, if $|x| < \xi$, then the sequence of interpolating polynomials converges, while if $|x| > \xi$ the sequence diverges. Figure 3, shows then interpolants corresponding to degrees 2, 8 and 14.

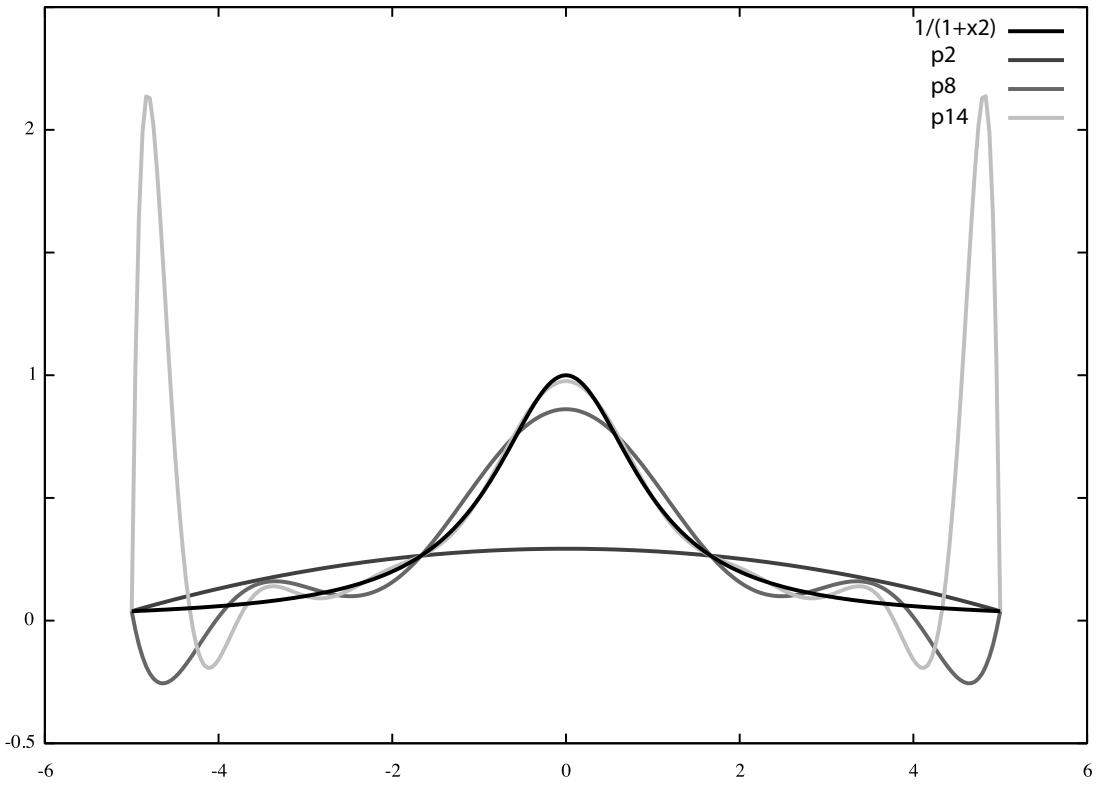

Figure 3. Runge's example

The previous example shows that interpolation by polynomials of high degree leads to interpolants of high amplitude oscillations. The same can be said for the basic Lagrange polynomials

$$l_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}, \quad i = 0, \ldots, n$$

whose maximal absolute value increases when the number of points increases. This fact has an important consequence: the bad stability and poor condition of polynomial interpolation with high degree polynomials. Assuming that the given data y_i are not exact but very close to the function values

$$|y_i - f(x_i)| \leq \epsilon$$

then we find that the perturbed interpolant $\tilde{p}(x) := \sum_{i=0}^{n} y_i l_i(x)$ may differ considerably from the true interpolant $p(x) := \sum_{i=0}^{n} f(x_i) l_i(x)$

$$|\tilde{p}(x) - p(x)| \leq \epsilon \Lambda(x),$$

where

$$\Lambda(x) := \sum_{i=0}^{n} |l_i(x)|$$

is the Lebesgue function corresponding to the sequence of nodes $x_0, \ldots, x_n$. The norm (and condition) of the interpolation operator is given by $\max_{x \in [a,b]} \Lambda(x)$ and depends on

the degree and the distribution of the nodes on the interval. In general we can say that the norm of the operator tends to infinity and satisfies

$$\max_{x\in[a,b]} \Lambda(x) > \frac{2}{\pi}\log(n) + 0.5212.$$

In the case of Chebyshev nodes

$$x_j = \frac{a+b}{2} + \frac{b-a}{2}\cos\Big(\frac{2j-1}{2n}\pi\Big), \quad j = 0,\ldots,n,$$

we find an almost optimal behaviour and the norm of the interpolation operator satisfies

$$\max_{x\in[a,b]} \Lambda(x) < \frac{2}{\pi}\log(n) + 1.$$

For equidistant nodes, the norm of the interpolation operator exhibits an exponential growth

$$\max_{x\in[a,b]} \Lambda(x) \sim \frac{2^n}{en\log(n)}.$$

Therefore equidistant nodes are not recomended for interpolation problems of high degree. For more information see Chapter 3 of [18].

3 Spline interpolation

Piecewise interpolation has a long history. The linear interpolation for function tables mentioned above defines a continuous piecewise linear interpolant, due to the fact that different linear polynomials are used to represent the interpolant depending on the relative position of the point with respect to the data. Geometrically, this means that the graph of the function in approximated by the polygon joining the points $(x_i, f(x_i))$, $i = 0,\ldots,n$. If we increase the number of interpolation sites on an interval in such a way that the mesh size $h := \max_{i=0,\ldots,n-1} |x_{i+1} - x_i|$ tends to 0, the convergence of the piecewise linear interpolant to f is ensured by (1.1).

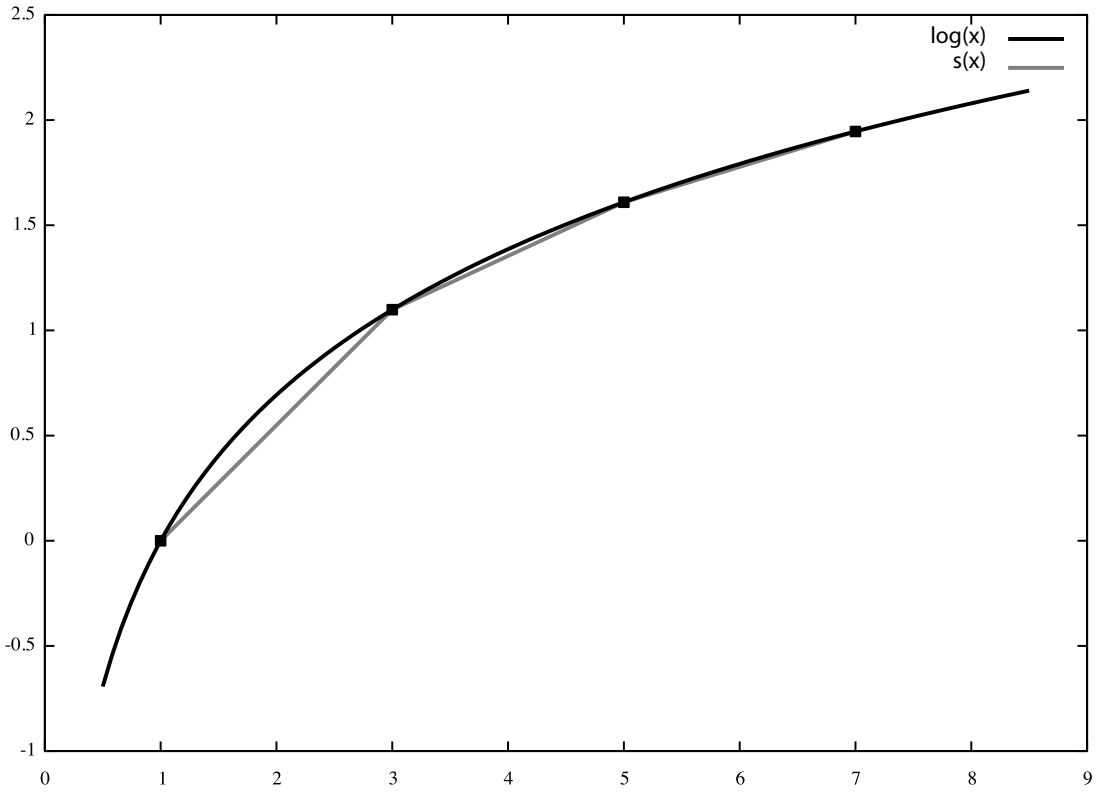

Figure 4. Piecewise linear interpolation

The inflexibility of polynomials can be avoided using piecewise polynomial functions also called spline functions. Piecewise polynomial functions arise in a natural way in elasticity theory of beams and flexible bars. The Bernoulli-Euler law relates the deflection of a beam and the bending moment. Assume that we are given a long and thin beam and that the cross-section is constant along its axis and composed of an isotropic material. The neutral axis is the locus of the barycenters of each cross-section. Assume that forces (loads) are applied orthogonally to the neutral axis and act in a unique plane causing the beam to bend. Then the curvature of the neutral axis is proportional to the bending moment,

$$M = \kappa/R,$$

where M represents the bending moment at a given point of the beam, κ the curvature of the neutral axis and R is a constant called flexural rigidity.

In order to formulate mathematically these ideas, we first choose coordinates to represent the neutral axis on the plane x, y, where $(0, 1, 0)$ is the direction of the forces bending the beam and (0,0,1) is the direction on the load plane orthogonal to the beam. Let us assume that the deflections are sufficiently small, so that we may represent the position of the neutral axis in $\mathbb{R}^3$ by $(x, y(x), 0)$, where $y(x)$ is a function of x such that $|y'(x)| << 1$ and the force density (load) at $(x, y(x), 0)$ is given by $(0, f(x), 0)$, $x \in [0, L]$.

The curvature at the point $(x, y(x), 0)$ is given by

$$\kappa(x) = \frac{y''(x)}{(1 + (y'(x))^2)^{3/2}} \approx y''(x)$$

The flexural rigidity $R = EI$ depends on the material and on the geometry of the cross section D, E represents the Young's elasticity modulus and I is the area inertia moment

$$I = \int_D z^2 dydz.$$

Then we can express the Euler-Bernoulli formula by the following approximate equation

$$y''(x) = \frac{1}{R} M(x).$$

The bending moment at the point $(x, y(x), 0)$ is given by

$$M(x) = M_0 + R_0 x + \int_0^x f(t)(x - t)dt = M_0 + R_0 x + \int_0^L f(t)(x - t)_+ dt,$$

where $R_0 = M'(0)$ is the force exerted at the left end $(0, y(0), 0)$ and $M_0 = M(0)$ is the moment applied to the left end of the beam. The force R_0 is usually due to the reaction of the pivot where it is supported or of the wall where it is clamped. The moment M_0 is usually due to the clamping moment and elastic supports. We use the simbol $(x - t)_+$ to indicate the positive part of $x - t$. For more general purposes the truncated power function can be used

$$(x - t)_+^k = \begin{cases} (x - t)^k, & \text{if } t \leq x, \\ 0, & \text{if } t > x. \end{cases}$$

Differentiating with respect to x we deduce that

$$M''(x) = f(x).$$

So, for a given distribution of the force density f, the deflection y of a beam can be approximately found by solving the differential equation

$$y^{(4)}(x) = \frac{1}{R} f(x).$$

An interesting case not covered by the above reasoning is the case in which the forces acting on the beam are concentrated at points $\tau_0, \ldots, \tau_n \in (0, L)$. Assume that a force $(0, F_i, 0)$ is applied at $(\tau_i, y(\tau_i), 0)$, $i = 0, \ldots n$, and no forces act on the other points. Additional terms must be included to take into account the reaction forces $R_0 = M'(0)$, $R_L = M'(L)$ and bending moments $M_0 = M(0)$, $M_L = M(L)$ acting at the ends $\tau_{-1} = 0$ and $\tau_{n+1} = L$, respectively, to meet static equilibrium conditions

$$M'(0) + \sum_{i=0}^{n} F_i + M'(L) = 0, \quad M(0) + \sum_{i=1}^{n} \tau_i F_i + LM'(L) + M(L) = 0$$

and to fit boundary conditions. Then we may represent the bending moment by the formula

$$M(x) = M_0 + R_0 x + \sum_{i=1}^{n} F_i (x - \tau_i)_+,$$

and, by the Bernoulli-Euler law, the deflection of the beam satisfies the differential equation

$$R y''(x) = M_0 + R_0 x + \sum_{i=0}^{n} F_i (x - \tau_i)_+.$$

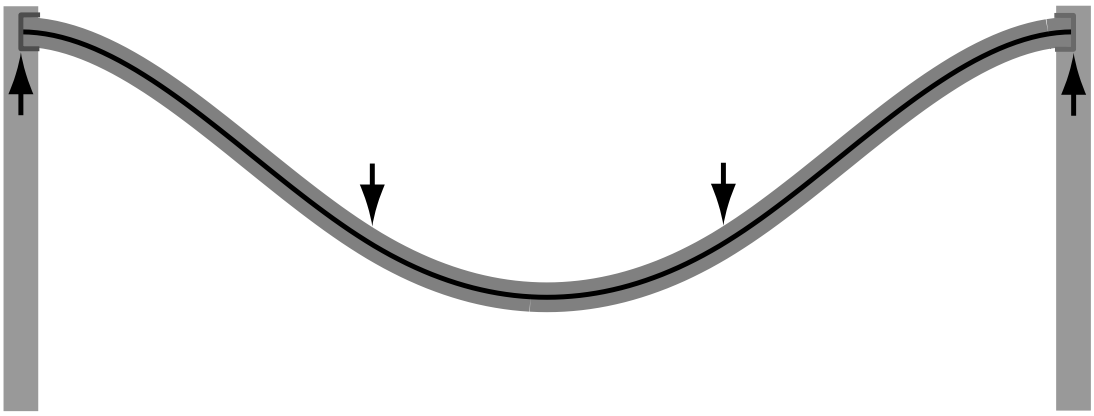

Figure 5. A beam clamped between two walls with two concentrated loads

We deduce that y must be a C^2 piecewise cubic function, because the second derivative is a continuous piecewise linear function. Figure 5 shows the deflection of a beam clamped to the ends with two symmetrically placed loads in the places indicated by the arrows. Two supporting forces at the end are necessary to fit the boundary conditions.

In each subinterval $[\tau_i, \tau_{i+1}]$, $i = -1, \ldots, n$, the deflection y is represented by a cubic polynomial. The points τ_i, $i = 0, \ldots, n$, are called the knots of the cubic spline function.

Beams, wooden rods and bars have been used for the design of ships based on the fact that boundary conditions at the ends and concentrated loads on the interior knots τ_i, $i = 0, \ldots, n$, can be selected so that the deflection attained at the knots can have prescribed values y_i. This experimental fact can be confirmed analyzing the interpolation problem at the knots

$$y(\tau_i) = y_i, \quad i = 0, \ldots, n,$$

with boundary conditions

$$y(a) = y_{-1}, \quad y'(a) = m_{-1}, \quad y(b) = y_{n+1}, \quad y'(b) = m_{n+1}.$$

by cubic spline functions in $S_3(\tau_0, \ldots, \tau_n)[a, b]$, the space of all C^2 piecewise cubic functions at knots $a < \tau_0 < \cdots < \tau_n < b$ on the interval $[a, b]$. A simple discussion of a related linear system of equations shows that this interpolation problem has always a unique solution.

General spline function spaces of arbitrary degree can be constructed. If $\tau_0 < \cdots < \tau_n$, then we denote by $S_k(\tau_0, \ldots, \tau_n)[a, b]$ the space of C^{k-1} functions which coincide with polynomials of degree k on each subinterval $[\tau_i, \tau_{i+1}]$, $i = -1, \ldots, n$ where $\tau_{-1} := a$ and $\tau_{n+1} := b$. When knots tend to coincide smoothness properties of the spline functions are lost. If $m(\tau_i)$ is the number of times that the knot τ_i appears in the sequence $\tau_0, \ldots, \tau_n$, then we only require that the spline function is of class $C^{k+1-m(\tau_i)}$ in a neighbourhood of τ_i

$$S_k(\tau_0, \ldots, \tau_n)[a, b] : = \{s : s|_{(\tau_i, \tau_{i+1})} \in P_k,\ i = -1, \ldots, n,\ s \in C^{k+1-m(\tau_i)} \text{ on a neighbourhood of } \tau_i\}.$$

It follows from the definition that $S_k(\tau_0, \ldots, \tau_n)[a, b]$ is a vector space of dimension $n+k+2$. The Schoenberg-Whitney theorem [28] shows that a necessary and sufficient condition on the interpolation sites for the Lagrange interpolation problem

$$s(x_i) = y_i, \quad i = 0, \ldots, n + k + 1,$$

in $S_k(\tau_0, \ldots, \tau_n)[a, b]$ to have always a unique solution is that

$$x_0 \in [a, \tau_0), \quad x_i \in (\tau_{i-k-1}, \tau_i),\ i = 1, \ldots, n + k, \quad x_{n+k+1} \in (\tau_n, b].$$

We use the convention that $\tau_{-k-1} = \cdots = \tau_{-1} = a$ and $\tau_{n+1} = \cdots = \tau_{n+k+1} = b$.

As a consequence of the Schoenberg-Whitney theorem, we can say that there are many possibilities for choosing the degree and the knot positions for spline interpolation problems. This freedom gives rise to different interpolation problems with the same data whose solutions are interpolants with different smoothness properties.

4 Shape-preserving interpolation

Spline interpolants are often used for curve design. They also arise in the numerical analysis of differential equations. They are a fundamental tool in experimental data fitting because smoothness and order approximation can be adjusted in spline interpolation.

Although spline interpolants do not present oscillations of much higher amplitude than the data, they may present oscillations not suggested by the data. A typical example is when a rapid transition between two states needs to be simulated. This often arises in examples from biology, chemistry and physics. An increasing function changes its rate of increase. First the function increases slowly, then in the transition zone it increases quickly and after this transition has been crossed the function turns again to present a slow increasing behaviour.

The use of a non-monotonic interpolating function might be problematic. For instance assume that we are evaluating the relative concentration of a substance arising in an

non-reversible chemical reaction. Monotonicity is a fundamental feature of the process. Furthermore relative concentrations below 0 and greater than 1 do not make sense.

$x:$	0.00	0.50	0.90	1.10	1.50	2.00
$y:$	0.00	0.05	0.15	0.85	0.95	1.00

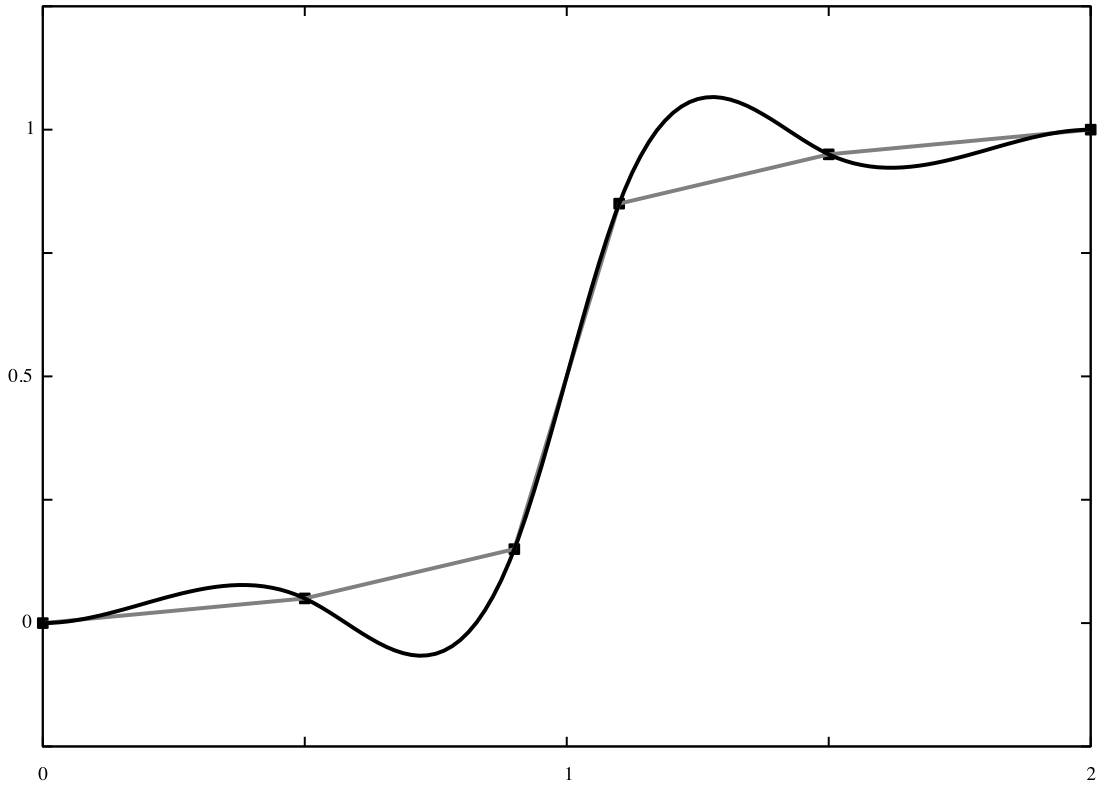

Figure 6. A rapid transition between two states

Figure 6 depicts the data, the piecewise linear interpolant and a cubic spline interpolant. The cubic spline interpolant oscillates taking values beyond the minimum and maximum of the data while the piecewise linear interpolant preserves monotonicity. In this case, the piecewise linear interpolant reflects important features of the solution although it is not smooth and presents lower degree of approximation.

Another shape property which might be preserved is convexity. Again the piecewise linear interpolant is convexity preserving but it is not smooth. Monotonicity and convexity provide a control on the oscillation properties of the interpolant. The search of smooth shape-preserving interpolants is a difficult task. The problem is non-linear in its nature. One reason is that the set of increasing or convex functions is not a vector space. Another problem arises when trying to establish a linear correspondence between convex data and convex functions. Assume that we want to find a convex interpolant to the data

$$p(0) = 0, \quad p'(0) = -1, \quad p(1) = 0, \quad p'(1) = m,$$

with $m > 0$. Since the data are convex then we deduce

$$-\min(x, m(1-x)) \leq p(x) \leq 0.$$

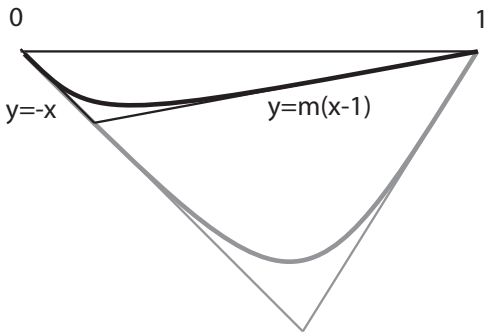

Figure 7. When $m \to 0^+$, the convexity preserving interpolant must converge to the zero function

Taking into account that $\lim_{m\to 0^+} \min(x, m(1-x)) = 0$, we deduce that the solution of the problem converges to the zero function when $m \to 0+$. But the zero function does not satisfy the condition $p'(0) = -1$. This indicates that, if we define a convexity preserving interpolant to data, continuous dependence on the parameters will be lost and then the method will be nonlinear.

Tension methods are a simple and versatile way of solving shape-preserving interpolation problems. The name tension refers to a free parameter and goes back to a paper by Schweikert [29], considering the shape of a beam or flexible bar under tension as a way of solving shape-preserving interpolation problems.

If the deflections are sufficiently small and all tension forces and bending moments lie on the same plane, we can represent the position of the neutral axis in $\mathbb{R}^3$ by $(x, y(x), 0)$, where $y(x)$ is a function of x such that $|y'(x)| \ll 1$. The force density at $(x, y(x), 0)$ is given by $(0, f(x), 0)$, $x \in [0, L]$.

The tension acting at the ends of a flexible bar is transmitted to all the points in the bar and is exerted in a direction tangent to the bar. Therefore the tensile stress $\mathbf{T}(x)$ has constant modulus T_0 and can be described by

$$\mathbf{T}(x) = T_0 \left(\frac{1}{(1+y'(x))^2}, \frac{y'(x)}{(1+y'(x))^2}, 0 \right) \approx (T_0, T_0 y'(x), 0).$$

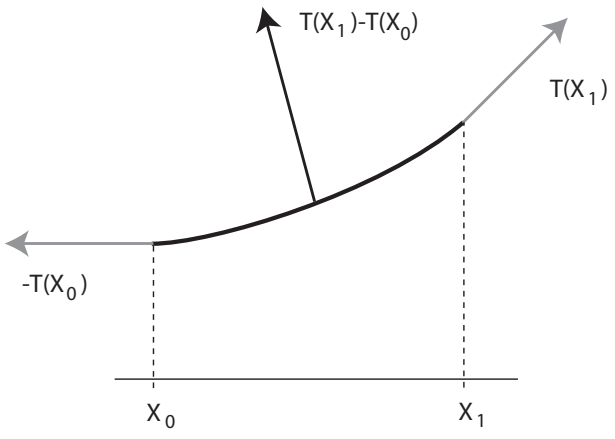

Figure 8. A beam in tension

We observe that if the bar has been curved by the effect of the loads the resultant of all tensile stresses acting on the piece of the bar between $(x_0, y(x_0), 0)$ and $(x_1, y(x_1), 0)$ is given by $\mathbf{T}(x_1) - \mathbf{T}(x_0)$. When $x_1 \to x_0$, a force density arises due to tension

$$\lim_{x \to x_0} \frac{1}{x_1 - x_0}(\mathbf{T}(x_1) - \mathbf{T}(x_0)) = \mathbf{T}'(x_0) \approx (0, T_0 y''(x_0), 0).$$

Now we can introduce the effect of tension in the computation of the bending moment

$$M(x) \approx M_0 + R_0 x + \int_0^x (f(t) + T_0 y''(t))(x - t)dt$$

and applying the Bernoulli-Euler law, $M(x) \approx R y''(x)$, we deduce that

$$R y^{(4)}(x) \approx M''(x) = f(x) + T_0 y''(x).$$

In the case of concentrated loads $(0, F_i, 0)$ applied at $(\tau_i, y(\tau_i), 0)$, $i = 0, \ldots, n$, the deflection of the beam can be described by the solution of the differential equation

$$y''(x) - \frac{T_0}{R}(y(x) - y(0) - xy'(0)) = \frac{1}{R}\Big(M_0 + R_0 x + \sum_{i=0}^{n} F_i (x - \tau_i)_+\Big).$$

Therefore y is a C^2 function satisfying in each subinterval (τ_i, τ_{i+1}), $i = -1, \ldots, n$ the fourth order differential equation

$$y^{(4)}(x) - \alpha^2 y''(x) = 0,$$

where $\alpha = \sqrt{T_0/R}$. It immediately follows that the restriction of $y(x)$ to each subinterval (τ_i, τ_{i+1}) must be of the form

$$y(x) = c_0 + c_1 x + c_2 \cosh(\alpha x) + c_3 \sinh(\alpha x).$$

The space of hyperbolic functions

$$H_\alpha = \langle 1, x, \cosh(\alpha x), \sinh(\alpha x) \rangle = \langle 1, x, e^{\alpha x}, e^{-\alpha x} \rangle,$$

depending on the tension parameter α, can be used to introduce hyperbolic spline function spaces.

$$H_\alpha(\tau_0, \ldots, \tau_n)[a, b] := \{s \in C^2[a, b] : s|_{(\tau_i, \tau_{i+1})} \in H_\alpha,\ i = -1, \ldots, n\}$$

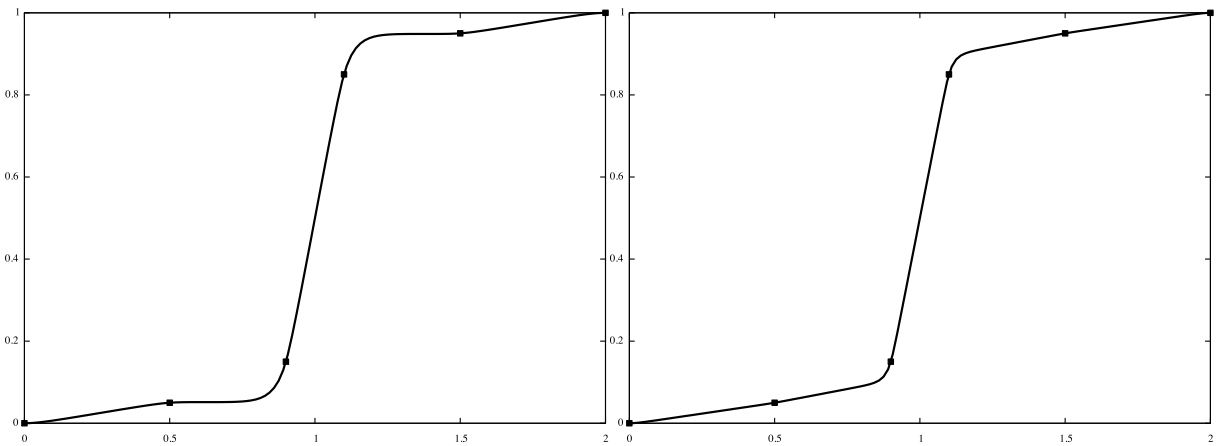

Figure 9. A rapid transition between two states simulated with hyperbolic splines the tension parameters are $\alpha = 10$ (left) and $\alpha = 20$ (right)

When the tension parameter α increases, the solution of the problem converges to the piecewise linear interpolant which is shape preserving. On the other hand if the tension parameter tends to 0, the solution converges to the usual cubic spline. For more flexibility, different tension parameters can be used in each subinterval. If we choose large tension parameters, the shape requirements will be satisfied but smoothness properties become worse. The choice of the tension parameters might be a difficult task and other methods for obtaining shape preserving interpolants have been considered. Most of them can be regarded as tension methods, where the tension parameters are automatically selected.

The minimal amount of work necessary to take a beam from the equilibrium position to a stressed state due to tension forces and bending moments caused by transversal loads can be summarized by the energy integral

$$\mathcal{E} = \frac{1}{2T_0}\int_0^L \|\mathbf{T}\|^2 ds + \frac{1}{2}\int_0^L M\kappa ds,$$

where s denotes the arc-length parameter of the neutral axis. In the case of small deflections this energy can be described by a functional

$$\mathcal{E} = RJ_\alpha[y], \quad J_\alpha[y] := \frac{1}{2}\int_0^L y''(x)^2 dx + \frac{T_0}{2R}\int_0^L y'(x)^2 dx = \frac{1}{2}\int_0^L (y''(x)^2 + \alpha^2 y'(x)^2) dx.$$

The hyperbolic spline in tension can be regarded as a solution of the following minimization problem: *find s minimizing the energy functional J_α subject to interpolation and boundary conditions*. Observe that the interpolation by cubic spline functions corresponds to $\alpha = 0$, and cubic spline interpolants minimize the integral $J_0[y] = \int_a^b y''(x)^2 dx$.

Several methods for the construction of shape-preserving interpolants use the solution of a minimization functional subject to the shape constraints. Let us describe the approach given in [1, 2, 3, 4, 5]. We want to obtain a solution of the interpolation problem

$$u(x_i) = y_i, \quad i = 0, 1, \ldots, n,$$

among all strictly convex functions with bounded second derivative

$$K = \{u \in C^1[a,b] \mid u'' \in L^\infty[a,b], \operatorname{ess\,inf} u > 0\}.$$

Assuming that the data are strictly convex

$$\frac{y_{i+1} - y_i}{x_{i+1} - x_i} < \frac{y_{i+2} - y_{i+1}}{x_{i+2} - x_{i+1}}, \quad i = 0, \ldots, n-2,$$

it can be shown that the set of all interpolants in K is an infinite convex set

$$L = \{u \in K \mid u(x_i) = y_i, \quad i = 0, 1, \ldots, n\}.$$

Then we can choose the unique interpolant minimizing a penalty functional

$$J[u] = \int_a^b F_p(u''),$$

where $p \in [0, 1]$ and

$$F_p(t) := \begin{cases} t \log t, & \text{if } p = 1, \\ -t^p/p, & \text{if } 0 < p < 1, \\ -\log t, & \text{if } p = 0. \end{cases}$$

In [1, 2], it was shown that there exists a unique interpolant u characterized by the fact that in each subinterval $[x_i, x_{i+1}]$ satisfies the fourth order nonlinear differential equation

$$u^{(4)}(x) - (2-p)\frac{(u^{(3)}(x))^2}{u''(x)} = 0.$$

If we solve the above equation we find that in each subinterval the function u is of the form

$$u(x) = c_0 + c_1 x + a(x+b)^{(1-2p)/(1-p)}, \quad 0 < p < 1/2, \text{ or } 1/2 < p < 1.$$

There are three special cases for $p = 0, 1/2, 1$

$$\begin{aligned} u(x) &= c_0 + c_1 x + a\exp(bx), && (p = 1) \\ u(x) &= c_0 + c_1 x + a\log(bx), && (p = 1/2) \\ u(x) &= c_0 + c_1 x + a(x+b)\log(x+b), && (p = 0) \end{aligned}$$

We remark that the interpolation problem is nonlinear. So, in order to find the unique solution of the problem, a nonlinear system of equations must be solved.

Figure 10 shows a convex interpolant obtained with $p = 1$. The second derivative of the interpolant is a continuous piecewise exponential function and minimizes the entropy functional $\int_a^b f(x) \log f(x) dx$.

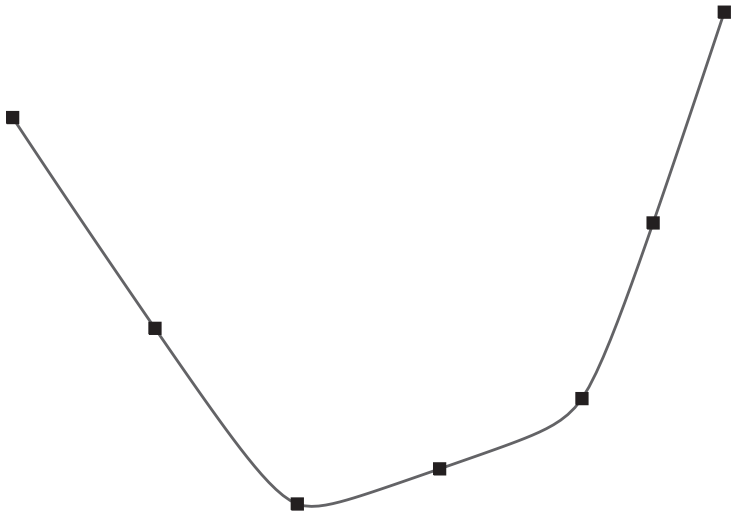

Figure 10. Convex interpolant minimizing the entropy of the second derivative

5 Tensor-product constructions in multivariate interpolation

Let U be a vector space of continuous functions defined on the set $\Omega \subset \mathbb{R}^d$ and let X be a finite set of points (usually called nodes). Given a function $f : X \to \mathbb{R}$, the Lagrange interpolation problem on the set of nodes X consist of finding a function $u \in U$ such that

$$u(\mathbf{x}) = f(\mathbf{x}), \quad \forall \mathbf{x} \in X.$$

We say that the set X is *correct* for the Lagrange interpolation problem by functions in U if it is *unisolvent*, that is, there exists a unique solution to the Lagrange problem in U for any f. Multivariate interpolation is a more difficult task than the problem in one variable because even the existence of interpolants is an interesting problem by itself.

In 1918, Haar [23] discussed the problem of finding the best uniform approximation to a function. The uniqueness of the best approximation $u_0 \in U$ to a function f by functions in a space U on a compact domain Ω in the sense of Chebyshev

$$\max_{x \in \Omega} |f - u_0| = \min_{u \in U} \max_{x \in \Omega} |f - u|$$

is equivalent to the fact that the Lagrange interpolation problem is unisolvent on X for any set X with $\#X = \dim U$. In one variable, there exists function spaces (called Haar spaces) U such that there exists a unique interpolant $u \in U$ for any function f and any set X with $\#X = \dim U$. However, in [23] it was shown that there exist no Haar spaces if Ω contains interior points, $d > 1$ and the set X contains more than one point. In 1955, Mairhuber [25] solved a problem suggested by I. J. Schoenberg, showing that U is a Haar space with $\dim U \geq 2$ if and only if Ω is homeomorphic to a closed subset of the circumference of a circle. For more information on this problem see the book of Zielke [30].

A consequence of the above result is that the Lagrange interpolation problem on a given space U of functions defined on a multivariate domain Ω will not be unisolvent for certain sets X.

The use of polynomial interpolants give rise to local function representations with high order of approximation. Interpolation by multivariate polynomials is a powerful approximation technique used in the *finite element method* for finding a numerical solution to partial differential equations. The domain is subdivided into small subdomains called elements. This is a usual technique used in engineering analysis of structures. In many practical cases elements can be taken as rectangles or triangles in two variables (boxes or simplices in more than two variables) and the space of interpolants can be chosen any subspace of the space of polynomials containing all polynomials up to a given total degree. The use of a interpolants to represent or approximate the solution has the advantage that

we can add or manipulate easily physical constraints or boundary conditions and that we can interprete easily the values of the polynomials as approximation of the values to the true solution of the differential equation. Continuity conditions for relating different patches of the global solution can easily be imposed and low order smoothness condition are not hard to incorporate. The approximation power of polynomial representations makes it easy the design of *finite elements* with high order of approximation which might imply fast convergence properties of *finite element method*.

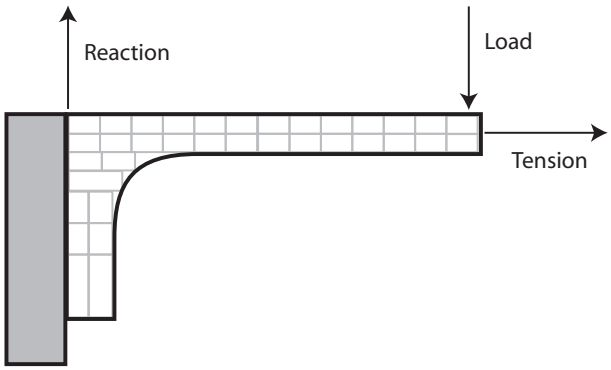

Figure 11. Analysis of the mechanical properties of a structure with finite elements

On the other hand, bivariate interpolation techniques are fundamental in the representation of surface patches by polynomials. The use of bivariate polynomial interpolants give rise to local surface representations for reproducing surfaces accurately.

In surface design other representations than the representations by interpolation are preferred. However, in the design process it is usual to impose conditions for restricting the freedom of the surface. Many restrictions can be formulated as interpolation or approximation conditions leading to bivariate interpolation problems.

In most of the above applications, the interpolation nodes and the space of interpolants can be chosen. If we are free to select the nodes and the space, tensor product constructions, based on simpler univariate problems can be easily described, analyzed and implemeted. In order to simplify our discussion, we shall draw our attention on bivariate problems.

The interpolation problems on rectangular grids

$$X = \{(x_i, y_j) \mid i = 0, \ldots, n, j = 0, \ldots, m\} = \{x_i \mid i = 0, \ldots n\} \times \{y_j \mid j = 0, \ldots, m\},$$

$$x_0 < x_1 < \cdots < x_n, \quad y_0 < y_1 < \cdots < y_m,$$

are usually called tensor product constructions because the space of polynomial interpolants

$$P_{n,m} = \{p(x, y) \mid \deg_x p \leq n, \deg_y q \leq m\} = \langle x^i y^j \mid i = 0, \ldots, n, j = 0, \ldots, m \rangle$$

can be regarded as a tensor product of univariate polynomial function spaces $P_n \otimes P_m$. We observe that $\dim P_{n,m} = (n+1)(m+1) = \#X$. Furthermore, the Lagrange interpolation problem is unisolvent and a Lagrange formula can be derived immediately

$$p(x,y) = \sum_{i=0}^{n} f(x_i, y_j) l_i^x(x) l_j^y(y), \quad l_i^x(x) = \prod_{k \neq i} \frac{x - x_k}{x_i - x_k}, \quad l_j^y(y) = \prod_{k \neq j} \frac{y - y_j}{y_k - y_j}.$$

Multivariate extensions for the Newton and Aitken-Neville formulae can also be easily derived in this case.

Tensor product interpolants provide excellent approximations properties and give rise to low errors. However they share the inflexibility properties of the univariate polynomials and are not able to represent global surfaces in large areas of the domain and they should rather be used to represent local surface patches. It is not recommended to use surface interpolants of very high degree because the complexity of the formulae representing the interpolant increases. Since the dimension of $P_{n,n}$ is $(n+1)^2$, we may expect to obtain the interpolant as the sum of $(n+1)^2$. This gives rise to 441 terms to be considered for $n = 20$ and 961 terms for $n = 30$. By increasing the degree, the condition of the problem measured by the Lebesgue functions also grows very fast. The higher complexity of the formula implies more computation time and an increased risk of error propagation.

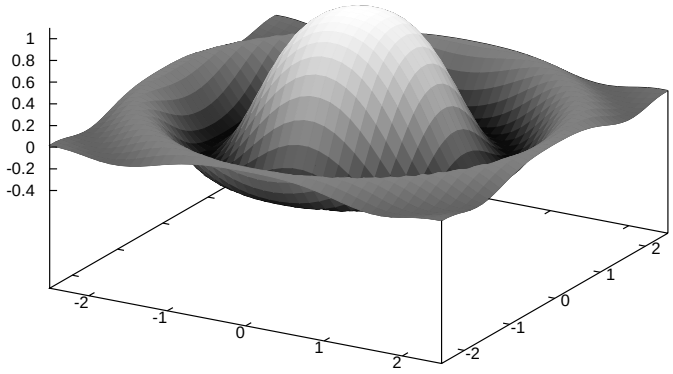

Figure 12. The surface $z = \exp(x^2 + y^2)/3 \cos(x^2 + y^2)$

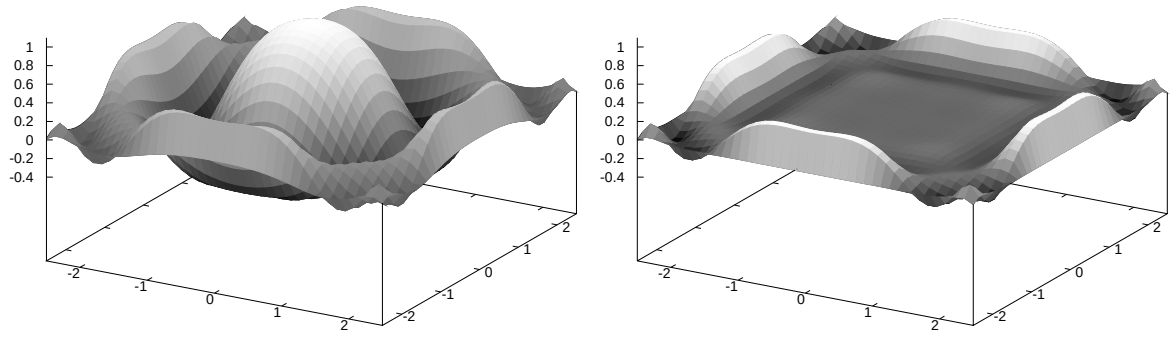

Figure 13. The $P_{8,8}$ tensor product interpolant and its error

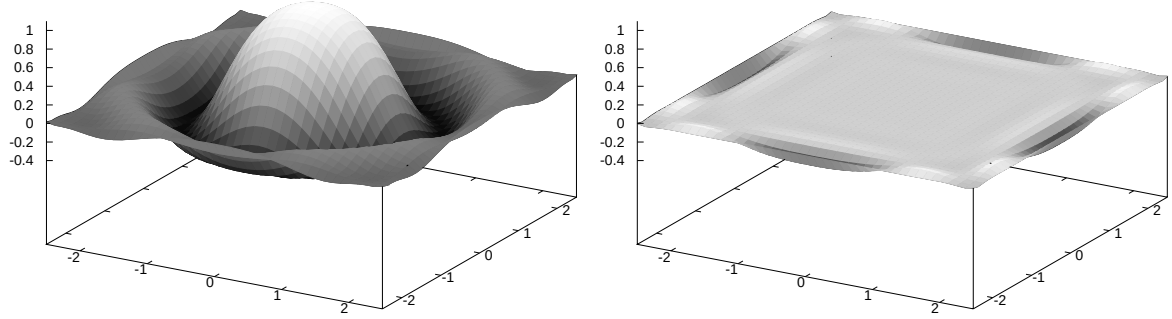

Figure 14. The $P_{12,12}$ tensor product interpolant and its error

Figure 12 shows a moderately oscillating function whose behaviour can only be imitated by considering a large number of data

$$f(x,y) = \exp\Big(-\frac{x^2+y^2}{3}\Big)\cos(x^2+y^2), \quad x,y \in [-5/2, 5/2].$$

We consider tensor product interpolants in $P_{n,n}$ at the grid of points

$$\Big(-\frac{5}{2}+\frac{5i}{n}, -\frac{5}{2}+\frac{5j}{n}\Big), \quad i,j = 0,\ldots,n.$$

When $n = 8$ (91 data points) the behaviour of the interpolant is only very well imitated at the center of the square $[-5/2, 5/2] \times [-5/2, 5/2]$ (see Figure 13). By increasing a little the degree $n = 12$ (169 data points) , we obtain more acceptable results in the boundary (the error is lower than the 10% of the range) and very good approximation properties at the center of the square (see Figure 14).

6 Multivariate interpolation by polynomials of bounded degree

Lagrange interpolation in more than one variable in a finite dimensional space of polynomial functions will not be unisolvent for certain sets X as a consequence of Haar's results in [23]. In order to face this problem two different but complementary approaches have been considered. If the set of nodes $X \subset \Omega$ cannot freely be chosen, search for a space of polynomial functions such that the Lagrange interpolation problem at X is unisolvent. The second possibility is that the space of interpolants is fixed. The space of polynomials with total degree not greater than n is considered in most cases

$$P_n^d = \{p(x_1, \ldots, x_d) \mid \deg p \leq n\} = \langle x_1^{i_1} \cdots x_d^{i_d} \mid i_1 + \cdots + i_d \leq n\}$$

because of this space has optimal approximation order properties with respect other polynomial spaces of the same dimension. Then the problem consists in identifying or constructing correct sets for the given space and provide efficient algorithms to find the interpolant.

A polynomial interpolant to any set of data can be obtained by increasing the total degree n according to the data. A simple construction of a polynomial of degree not greater than $n = \#X - 1$ similar to the univariate construction can be obtained as follows:

$$p(x) = \sum_{y \in X} f(y) l_y(x), \quad l_y(x) := \prod_{z \in X \setminus \{y\}} \frac{(y-z)^T (x-z)}{\|y-z\|^2}. \tag{6.1}$$

Reducing the degree as much as possible allows us to fully exploit the power of approximation of multivariate polynomials. Another reason is that the growth and oscillation properties of the polynomials might be better controlled by keeping the degree as low as possible. A *minimal degree interpolation space* for a set of nodes X is any interpolation space U such that $U \subseteq P_n$ such that no interpolation space for the set of nodes X is contained in the space P_{n-1}. The space generated by the polynomials l_y, $y \in X$, in (6.1) is a minimal degree interpolation space if and only if all nodes lie on the same line. For most distributions of points lower degree interpolants can be obtained. For instance, if X consists of three noncollinear points on the plane, then a linear interpolant in P_1 can be always provided, whereas formula (6.1) provides quadratic interpolants. For most sets X, the degree can be considerably reduced and attains its minimal possible value n with $\dim P_{n-1} < \#X \leq \dim P_n$.

A related question is degree reduction. Given a set X, we can define the restriction operator which associates to each function f its restriction $f|_X$ to the set X. For any space U of polynomials such that the Lagrange interpolation problem is unisolvent the restriction operator is a bijection between U and $\mathbb{R}^X$. The inverse of the restriction operator is the

interpolation operator which associates to each function on X, a polynomial function defined on a wider domain $L_U : \mathbb{R}^X \to U$. The interpolation at points X by functions in U is *degree reducing* if for any polynomial p, we have that $\deg L_U[p] \leq \deg p$. Any degree reducing space is a minimal degree interpolation space but the converse is not true in general. The construction of degree reducing and minimal degree spaces for any sets of nodes has been investigated by T. Sauer among others (see [27]).

In the second approach we fix the interpolation space P_n^d and search for correct sets. A set $X \subset \mathbb{R}^d$ is correct if and only if

$$\#X = \dim P_n^d = \binom{n+d}{d}$$

and all the points in X are not contained in an algebraic hypersurface of degree n. This geometric condition is not practical to check because it requires almost the same effort than trying to solve the interpolation problem and find that there exist no unique solution.

We should take into account that, for $d > 1$, the number of nodes and the dimension of the interpolation space P_n^d grows considerably with the degree. This implies additional complexity in the formulae. For instance, if we perform bivariate interpolation of degree 15 we would need to manipulate, in general, 105 different terms because $\dim P_{15}^2 = 105$. This also means that, apart from the condition of the problem, stability of computations should be carefully considered to avoid propagation of the roundoff errors.

The Lagrange polynomials $l_y \in P_n^d$, $y \in X$, are implicitly defined by the conditions

$$l_y(y) = 1, \quad l_x(y) = 0, \quad \forall x \in X \setminus \{y\}.$$

and give a representation of the interpolant by means of the Lagrange formula

$$p(x) = \sum_{y \in X} f(y) l_y(x).$$

The Lagrange formula is often used in the finite element method because its coefficients are directly the values of the solution and further evaluation of the formula might be avoided if we are dealing with a sufficiently fine grid.

On the other hand Newton-like formulae

$$p(x) = \sum_{k=0}^{N} c_k \pi_k(x)$$

are more versatile than the Lagrange formulae and can be applied to a wider set of problems. A Newton-like basis $\pi_k(x)$, $k = 0, \ldots, N$, is usually formed by polynomial whose degree gradually increases with k and vanishes on subsets X_k of X whose size gradually increases

$$0 = X_0 \subseteq \cdots \subseteq X_N \subset X,$$

so that, the Lagrange interpolation problem leads to a triangular (or block triangular) system of equations. The coefficients of a Newton-like formula can be regarded as generalizations of the divided diferences. The computation of the coefficients is usually unstable, in contrast to Lagrange formula whose coefficients are the interpolation data $f(y)$, $y \in X$.

The complexity of the formulae representing the Lagrange polynomials must also be considered in the choice of a Lagrange formula to represent the interpolant. Each of the Lagrange polynomials will be expressed in terms of a basis of P_n^d and might have $\dim P_n^d$ terms. Taking into account that the Lagrange formula express the solution as a sum of $\dim P_n^d$ terms, the final value is the sum of $(\dim P_n^d)^2$ terms. For $n = 15$, $d = 2$, this gives a huge number of terms 10125 making the evaluation of the interpolant a high demanding computational task leading to a unreliable computed value. Chung and Yao suggested to use sets X leading to simple Lagrange formulae, in the sense that the Lagrange polynomials can be completely factored into first degree polynomial factors. These sets can be characterized by the following property

Geometric characterization by Chung and Yao. *For each node $x \in X$, there exists a set of n hyperplanes $\Gamma_{x,X} = \{H_1^x, \ldots, H_n^x\}$ such that $x \notin \bigcup_{i=1}^n H_i^x$ and $X \setminus \{x\} \subset \bigcup_{i=1}^n H_i^x$. We say that X is a GC_n set for short.*

For GC_n sets the Lagrange formula can be written as follows

$$p(x) = \sum_{y \in X} f(y) l_y(x), \quad l_y(x) = \prod_{i=1}^{n} \frac{h_i^y(x)}{h_i^y(y)},$$

where h_i^y is a first degree polynomial such that $h_i^y(x) = 0$ is the equation of the hyperplane $H_i^y \in \Gamma_{y,X}$ (determined up to a constant factor). The formula retains many features of the univariate one and has a similar computational complexity.

A *lattice* is determined by a set of affine/projective manifolds and its incidence relations. A lattice ca be described as a set of points X equipped with a *lattice structure*, which means sets of lines, planes, hyperplanes, containing relevant subsets of X. An important feature of the lattice structure is the inclusion and the incidence relations between the different affine/projective manifolds of the structure (points contained in lines, lines in planes, planes intersecting in lines, lines intersecting at points, etc). In the analysis of GC_n sets X, it will be important the lattice structure provided by the hyperplanes in $\Gamma_{x,X}$, $x \in X$ and other auxiliary manifolds used to understand its structure. For this reason, structured GC_n sets are also called lattices.

Principal lattices of degree n can be defined as a set of points of the form

$$x_\alpha = \sum_{r=0}^{d} \frac{\alpha_r}{n} V_r, \quad |\alpha| = n$$

where $V_0, \ldots, V_d$ are the vertices of a nondegenerate simplex $[V_0, \ldots, V_n]$, $\alpha = (\alpha_0, \ldots, \alpha_d)$ is a multiindex (a vector whose components are nonnegative integers) and $|\alpha| = \sum_{r=0}^{d} \alpha_i$.

Let us observe that all the points x_α with $\alpha_r = i$, lie on the same hyperplane

$$H_i^r = \{x \in \mathbb{R}^d \mid \lambda_r(x) = i/n\},$$

where $\lambda_r(x)$ denotes the r-th barycentric coordinate of the simplex $[V_0, \ldots, V_d]$, defined to be the unique linear function such that

$$\lambda_r(V_r) = 1, \quad \lambda_r(V_j) = 0 \quad j \neq r.$$

On the other hand x_α can be regarded as the unique intersection of the hyperplanes H_i^r. Therefore we have

$$H_i^r = \text{aff hull}\{x_\alpha \mid \alpha_r = i, |\alpha| = n\}, \quad \{x_\alpha\} = \bigcap_{r=0}^{d} H_{\alpha_r}^r,$$

where aff hull denotes the affine hull of a set of points. The relation between points and hyperplanes gives rise to a characteristic lattice structure. Let us observe that the set of n hyperplanes

$$\Gamma_{x_\alpha, X} = \{H_i^r \mid i < \alpha_r, r \in \{0, \ldots, d\}\}$$

contains all points in $X \setminus \{\alpha\}$ but not x_α. Therefore any principal lattice of degree n is a GC_n set.

Principal lattices in the plane have a triangular structure. The points are points regularly distributed on a triangle. There are three pencils of parallel lines and each point is the intersection of the three concurrent lines each from one different pencil (see Figure 15). An analogous structure can be described for more than two variables: the points are regularly distributed on a simplex. The hyperplanes belong to $d + 1$ different pencils of parallel hyperplanes and each point is the intersection of $d+1$ hyperplanes each belonging to a different pencil.

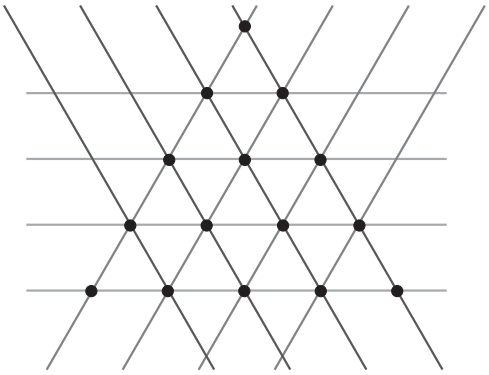

Figure 15. A planar principal lattice

All the interpolation nodes of a principal lattice lie inside the simplex $[V_0, \ldots, V_d]$. This fact is fundamental for the construction of piecewise interpolants on a simplicial partition.

Assuming that a multidimensional domain is subdivided into simplices, we can find an interpolant to given data on each simplex. Since interpolants defined on adjacent simplices agree on the common facet, we deduce that the piecewise polynomial interpolant is at least continuous. Unfortunately, smoothness conditions are far from being trivial require additional work. Principal lattices are often used for surface reconstruction and in the finite element method. The subdivision of a bidimensional domain into triangles (also called triangulation) is a more flexible tool than subdivision into rectangular pieces. For higher dimensional domains, we can say that a partition into boxes suggests regular grids while simplicial partitions offer more possibilities for constructing nonuniform partitions. The use of nonuniform partitions is crucial for solving certain differential equations by the finite element method. Using a more detailed representation in selected regions of the domain (like corners), improves considerably the degree of approximation of the numerical solution.

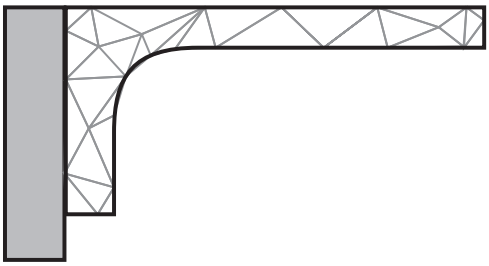

Figure 16. A triangulation of a domain for finite element analysis

Generalizations of principal lattices have been described. One may replace the condition that all hyperplanes in $H_0^r, \ldots, H_n^r$ are parallel for $r = 0, \ldots, d$ by the more general condition

$$H_0^r, \ldots, H_n^r \in \Lambda_r,$$

where Λ_r denotes a linear pencil of hyperplanes (see [24]). Lattices generated by polynomial pencils of hyperplanes have also described.

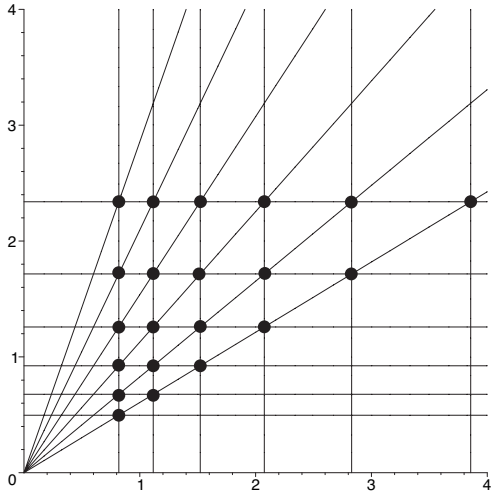

Figure 17. A three-pencil lattice

Lattices generated by polynomial pencils of hyperplanes have also described [13]. This motivates the following definition

Definition 6.1. *A generalized principal lattice of degree n is a set X that can be so indexed as $X = \{x_\alpha : \alpha \in \Gamma_n^0\}$ that, for $d+1$ families of hyperplanes*

$$(H_i^r, i = 0, \ldots, n), \quad r \in \{0, \ldots, d\},$$

containing altogether $(d+1)(n+1)$ distinct hyperplanes,

$$\{x_\alpha\} = \bigcap_{r=0}^{d} H_{\alpha_r}^r = \bigcap_{r\in\{0:d\}\setminus\{l\}}^{d} H_{\alpha_r}^r, \quad \forall \alpha \in \Gamma_n^0, \quad \forall l \in \{0, \ldots, d\},$$

and

$$\bigcap_{r=0}^{d} H_{\alpha_r}^r \cap X \neq \emptyset \Longrightarrow \alpha \in \Gamma_n^0.$$

It has been recently shown that all lines defining a generalized principal lattice in the plane must belong to the same cubic pencil [15]. In [10, 11] the generation of generalized principal lattices from a cubic pencil of lines was discussed. In[12] a complete classification of all lattices generated by cubic pencils was provided. Figure 18 shows some lattices obtained by lines in the same reducible cubic pencil formed by a linear pencil and a quadratic pencil. Figure 19 illustrates lattices obtained by lines in the same irreducible cubic pencil. Multivariate constructions of generalized principal lattices using polynomial pencils have been discussed in [13].

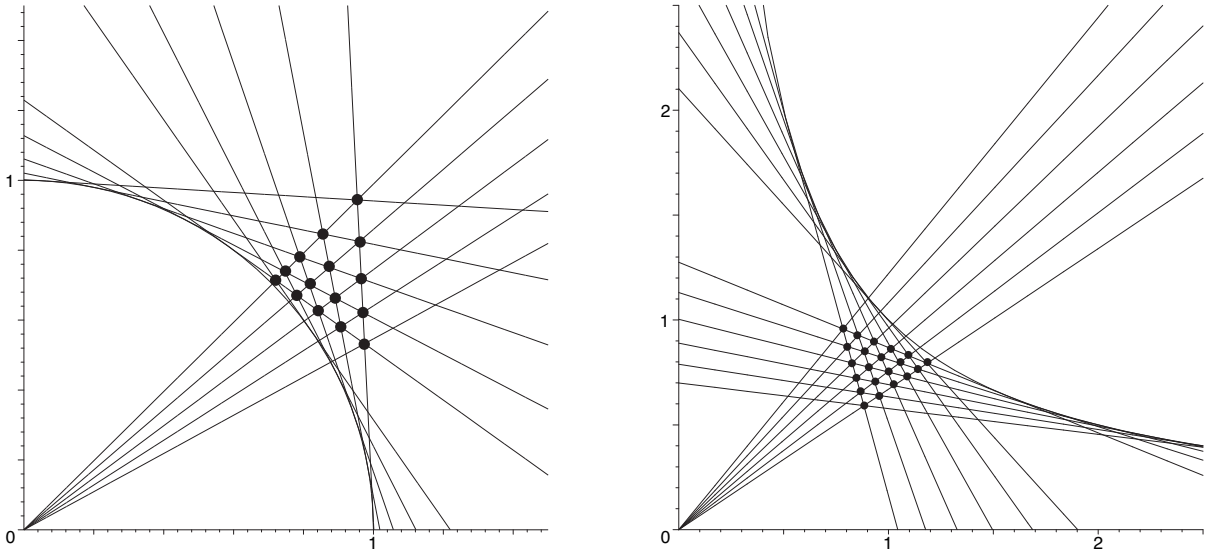

Figure 18. Lattices generated by linear pencil and a quadratic pencil

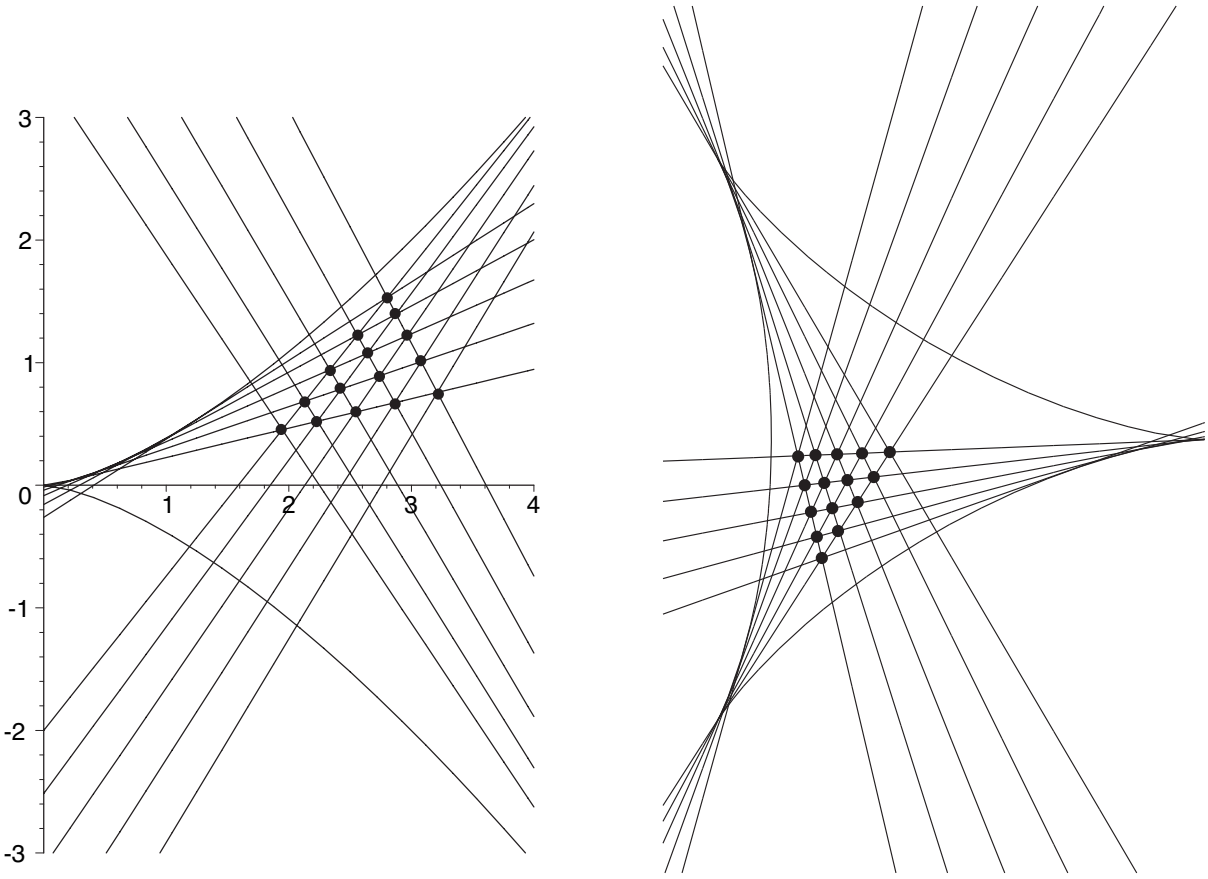

Figure 19. Lattices generated by cubic pencils of lines

Apart from generalized principal lattices some other kinds of GC_n sets can be described. For general dimension, the description is complicated and many questions remain unsolved. So, we shall limit our classification to planar GC_n sets. As a criterion of classification we shall use the number of lines containing $n+1$ points of X

$$\mathcal{K}_X := \{L \in \Gamma_X \mid \#(L \cap X) = n+1\}, \quad \Gamma_X := \bigcup_{x \in X} \Gamma_{x,X}.$$

Definition 6.2. *We say that a* GC_n *set* X *has defect* d *if the number of lines containing* $n+1$ *points of* X *is* $\#\mathcal{K}_X = n+2-d$. *We also say that* X *is a* $\mathrm{GC}_{n,d}$ *set for short.*

It is easy to deduce that $\#\mathcal{K} \leq n+2$. Therefore, the defect of a GC_n set is a nonnegative number less than or equal to $n-2$. However, Gasca and Maeztu [20] conjectured that for any GC_n set, there exists a line in Γ_X containing $n+1$ points of X, in other words $\mathcal{K}_X \neq \emptyset$.

Conjecture GM. *The defect* d *of any* $\mathrm{GC}_{n,d}$ *set of degree* n *is less than* $n+2$, *that is,* $d < n+2$.

The conjecture has been proved for $n \leq 4$ by J. R. Busch (see [6]) but even for $n = 5$, this conjecture has not been settled. Let us introduce the greatest degree for which we

can ensure that conjecture GM holds for a GC_n set and all possible GC subsets

$$\nu := \max\{n \mid \text{Conjecture GM holds for all GC sets of degree } <= n\} \geq 4.$$

In [8], it was shown that, for every planar $\mathrm{GC}_{n,d}$ set X with $n \leq \nu$, then there exists at least three lines in Γ_X containing $n+1$ points $\#\mathcal{K} \geq 3$, that is, the defect is $\leq n-1$. Since $\mathrm{GC}_{n,n-1}$ sets with $n \leq \nu + 2$ are just generalized principal lattices (see [14]), the classification of GC_n sets, which are not generalized principal lattices is reduced to examine the defects $d < n-1$.

In [21], it was shown that there exist no $\mathrm{GC}_{n,d}$ sets with $d = 4 < n-1$. From this fact it can be deduced that higher defects than 3 are not possible. The following result (see [21]) summarizes the above mentioned results on classification of GC_n sets

Theorem 6.3. *Let $X \subset \mathbf{R}^2$ be a $\mathrm{GC}_{n,d}$ set with $d \leq \nu + 2$. Then either $0 \leq d \leq 3$ or $d = n-1$. If $d = n-1$, then X is a generalized principal lattice.*

In [19], [7] and [16] $\mathrm{GC}_{n,d}$ sets with defects $0, 1, 2, 3$ were described. In order to provide a general description we recall that the lines in $\mathcal{K}_X$ are in general position: any two lines in $\mathcal{K}_X$ meet at a point in X and no triple of lines are concurrent at the same point.

Theorem 6.4. *A set $X \subset \mathbb{R}^2$ is a $\mathrm{GC}_{n,d}$ set, $d \in \{0,1,2,3\}$ if and only if $X = X_0 \cup X_1 \cup X_2$, where*

$$X_0 := \{x_{ij} \mid \{x_{ij}\} = K_i \cap K_j \mid i \neq j \in \{0, 1, \ldots, n+1-d\}\}$$

is the set of all intersection points of $n+2-d$ lines in general position $K_0, \ldots, K_{n+1-d}$, $\#X_0 = (n+1-d)(n-d)/2$,

$$X_1 := \bigcup_{i=0}^{n+1-d} X_{i,1}, \quad X_{i,1} := K_i \cap X \setminus \bigcup_{j \neq i} K_j$$

is the set of points belonging to exactly one of the lines K_i, $i = 0, \ldots, n+1-d$, with $\#X_{i,1} = d$ and $\#X_1 = d(n+1-d)$ and

$$X_2 := X \setminus \bigcup_{i=0}^{n+1-d} K_i$$

is the set of points in X not belonging to any line K_i, $i = 0, \ldots, n+1-d$, containing $\#X_2 = d(d-1)/2$ points and,

- *(a) if $d = 0$, then $X_1 = X_2 = \emptyset$;*
- *(b) if $d = 1$, then $X_2 = \emptyset$ and not all points in X_1 lie on the same line;*

- *(c) if $d = 2$, then there exist three lines L_0, L_1, L_2 such that $X_2 = L_0 \cap L_1 \cap L_2$, $L_i \cap K_i \cap X = \emptyset$, $i = 0, 1, 2$, and $X_1 \subset L_0 \cup L_1 \cup L_2$;*

- *(d) if $d = 3$, then there exist three lines L_0, L_1, L_2, in general position, such that*

$$X_2 = \{z_{01}, z_{12}, z_{02}\}, \quad \{z_{ij}\} := L_i \cap L_j, \quad i \neq j \in \{0, 1, 2\}$$
$$X_{i,1} = K_i \cap \bigcup_{j \neq i} L_j \cup \{z_i\}, \quad \{z_i\} := X_{i,1} \setminus \bigcup_{j \neq i} L_j, \quad i = 0, 1, 2,$$
$$X_{i,1} = K_i \cap (L_0 \cup L_1 \cup L_2), \quad i = 3, \ldots, n + 1 - d,$$

and the points z_i, z_j, z_{ij} are collinear for $i \neq j \in \{0, 1, 2\}$.

Figures 20 and 21 illustrate the different types of line configurations associated with $\mathrm{GC}_{n,d}$ sets, $d = 0, 1, 2, 3$.

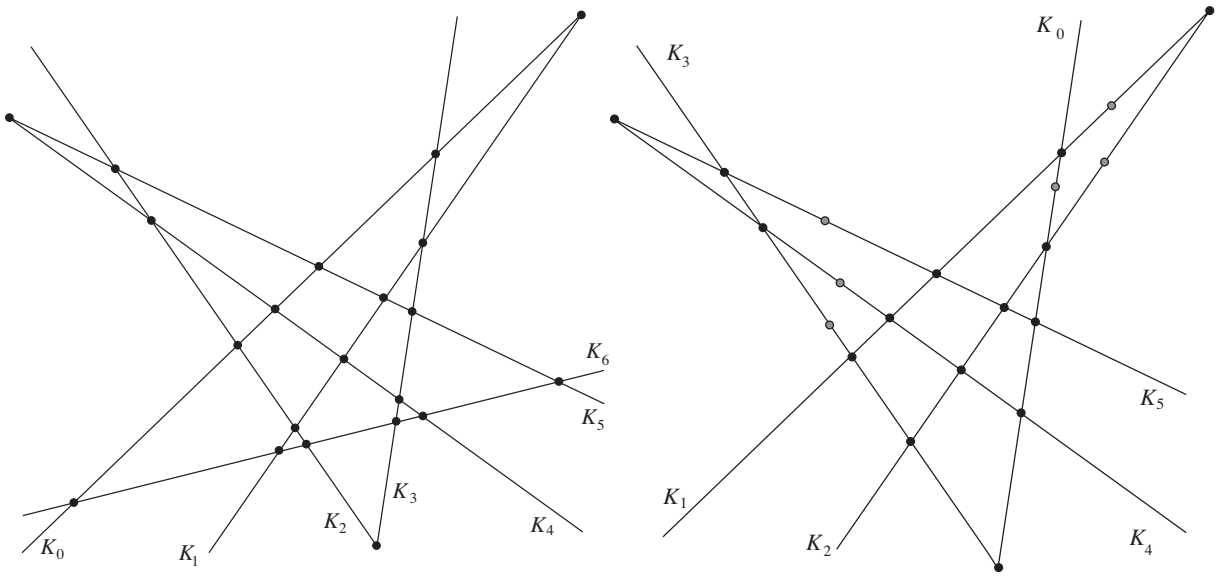

Figure 20. A $\mathrm{GC}_{5,0}$ set (left) and a $\mathrm{GC}_{5,1}$ set (right)

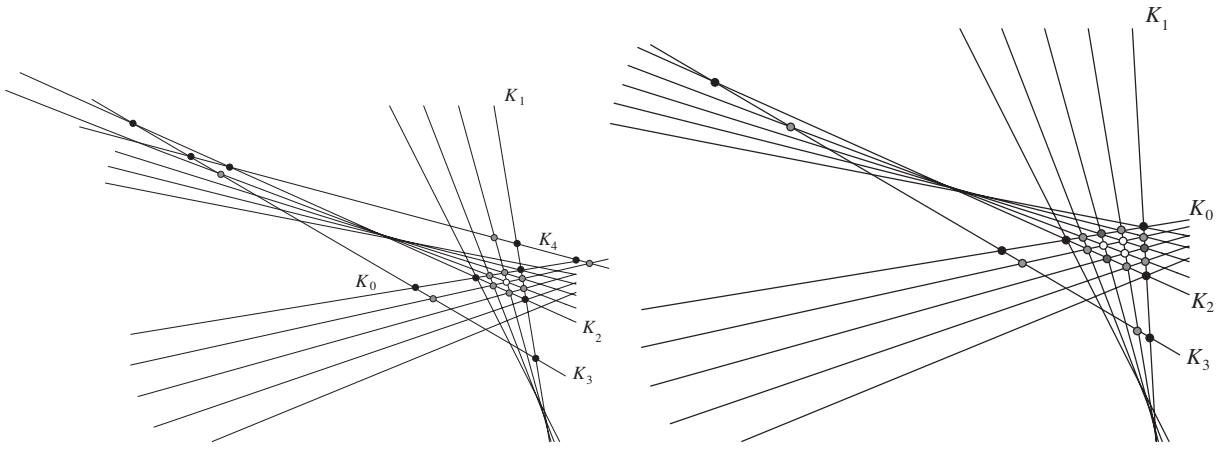

Figure 21. A $\mathrm{GC}_{5,2}$ set (left) and a $\mathrm{GC}_{5,3}$ set (right)

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