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Nuclearity of certain vector-valued sequence spaces

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Abstract

In this note, we deal with the space of Λ-summable sequences from a locally convex space E, where Λ is a perfect sequence space. We make use of a result of M. Florencio and Pedro J. Paúl in [4] to give a characterization of the nuclearity of $\Lambda(E)$ in terms of that of Λ and E and the AK property.

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1 Introduction

A. Pietsch [10] in connection with the nuclearity of a locally convex space E introduced, for the first time, the spaces $\ell_p[E]$ and $\ell_p\{E\}$ respectively of weakly ℓ_p-summable and absolutely ℓ_p-summable sequences in E. This allowed him also to introduce and study the absolutely p-summing operators. Later, in case E is a normed space, J. S. Cohen [2] introduced the space $\ell_p\langle E\rangle$ of strongly p-summable sequences. He used this space together with the spaces $\ell_p[E]$ and $\ell_p\{E\}$ to define the strongly and the nuclear p-summing operators. The definition of $\ell_p\langle E\rangle$ was generalized to an arbitrary locally convex space E by H. Apiola [1] in order to get new conditions for nuclearity of E. H. Apiola studied the duality relations between the three spaces, namely $\ell_p[E]$, $\ell_p\{E\}$ and $\ell_p\langle E\rangle$. He also gave a characterization of nuclearity of E using these spaces. In [10], A. Pietsch introduced and studied also the space $\Lambda(E)$ of Λ-summable sequences in E, Λ being a perfect sequence space in the sense of Köthe endowed with its normal topology. Later M. Florencio and P. J. Paúl [3] considered the general case where Λ is no longer equipped with the normal topology, but with a general polar one. They obtained results on $\Lambda(E)$ such as the characterization of the AK property and then the relationship with the completion $\Lambda\widetilde{\otimes}_\epsilon E$ of the injective tensor product $\Lambda \otimes_\epsilon E$. In [7] and [8], L. Oubbi and M. A. Ould Sidaty

gave a definition of strongly Λ-summable sequences. They then reconsidered the space $\Lambda(E)$ and obtained some of its properties. In [7], they mainly describe the continuous dual space of $\Lambda(E)$ in terms of strongly Λ^*-summable sequences in E'; Λ^* being the α-dual of Λ and E' the dual of E. While in [8], they gave a characterization of the reflexivity of $\Lambda(E)$ in terms of that of Λ and E and the AK property, extending so the result stated in [9] for the normed case. In this note, we are concerned with the nuclearity of the locally convex space $\Lambda(E)$.
In section 1, we endow the space $\Lambda\langle E\rangle$ of all strongly Λ-summable sequences in E with a naturel topology in the spirit of [1] for $\ell_p\langle E\rangle$. We also show that the injections $\Lambda\langle E\rangle \subset \Lambda\{E\} \subset \Lambda(E) \subset \Lambda[E]$ are continuous. The section 2 is devoted to the nuclearity of $\Lambda(E)$. We show that if E is nuclear then all these spaces coincide, and that $\Lambda(E)_r = \Lambda(E)$, where $\Lambda(E)_r$ is the subspace of $\Lambda(E)$ consisting of the sequences which are the limit of their finite sections. Using a result of [3], we establish the theorem 3.3.

2 Preliminaries

Throughout this paper, Λ will be a perfect sequence space and E a sequentially complete Hausdorff locally convex space. The Köthe dual space of Λ will be denoted by Λ^*, while E' will stand for the topological dual of E. The collection of all absolutely convex, $\sigma(E',E)$-closed and equicontinuous subsets of E' will be denoted by $\mathcal{M}$, while $\mathcal{S}$ will denote a collection of closed, absolutely convex, normal and $\sigma(\Lambda^*,\Lambda)$-bounded subsets of Λ^* such that Λ^* is the union of the members of $\mathcal{S}$ and the latter is stable by homothety. We will then consider on Λ the polar topology $\tau_{\mathcal{S}}$ associated with the collection $\mathcal{S}$. This topology is generated by the seminorms

$$P_S(\alpha) := \sup\{\sum_n |\alpha_n\beta_n|, \beta = (\beta_n)_n \in S\},\ S \in \mathcal{S}.$$

For an absolutely convex bounded subset A of a Hausdorff topological vector space F, let us denote by F_A the subspace of F generated by A. When no topology is specified on F_A, it will be endowed with the gauge $||\cdot||_A$ of A as a norm. We will then consider without any further mention the spaces E_B, E'_M, Λ_R and Λ^*_S, where B is a bounded subset of E, $M \in \mathcal{M}$, $S \in \mathcal{S}$ and R is a bounded absolutely convex subset of Λ. For every $M \in \mathcal{M}$, consider on E the seminorm P_M defined by

$$P_M(x) = \sup\{|a(x)|, a \in M\}.$$

A sequence $(x_n)_n \subset E$ is said to be Λ-summable (absolutely Λ-summable) if the series $\sum \alpha_n x_n$ converges (absolutely) in E for all $(\alpha_n)_n$ in Λ^*. It is weakly Λ-summable if $(a(x)_n)_n \in \Lambda$, for all $a \in E'$. The space of all Λ-summable (absolutely Λ-summable) sequences from E will be denoted by $\Lambda(E)$ ($\Lambda\{E\}$), while that of the weakly Λ-summable

ones will be designated by $\Lambda[E]$. Similarly, $\Lambda_S^*[E'_M]$ will stand for the weakly Λ_S^*-summable sequences from E'_M, $S \in \mathcal{S}$ and $M \in \mathcal{M}$. Following [7], we will then say that the sequence $(x_n)_n$ is strongly Λ-summable if for every $M \in \mathcal{M}$, the series $\sum a_n(x_n)$ converges for all $(a_n)_n \in \Lambda^*[E'_M]$. The space of all such sequences will be denoted by $\Lambda\langle E\rangle$.
Following [3], $\Lambda(E)$ will be equipped with the topology $\epsilon_{\mathcal{M},\mathcal{S}}$ generated by the family $(\epsilon_{S,M})_{S\in\mathcal{S},M\in\mathcal{M}}$ of seminorms, where

$$\epsilon_{S,M}(x) = \sup\left\{\sum_{n=1}^{\infty} |\alpha_n a(x_n)|, a \in M, \alpha = (\alpha_n)_{n\in\mathbb{N}} \in S\right\}, \ \forall x = (x_n)_n \in \Lambda(E).$$

These seminorms turn out to be defined also on $\Lambda[E]$ so that $\Lambda(E)$ is a closed topological subspace of $\Lambda[E]$. Both spaces will henceforth be equipped with this topology. Notice that, if E and $(\Lambda, \tau_{\mathcal{S}})$ happen to be metrizable, then so is also $\Lambda[E]$. Hence, if E and $(\Lambda, \tau_{\mathcal{S}})$ are Fréchet spaces, then so are also $\Lambda[E]$, $\Lambda(E)$ and their closed subspaces $\Lambda[E]_r$ and $\Lambda(E)_r$.
We refer the reader to Section 30 of [6] and Chapter 2 of [13] for details concerning Köthe theory of sequence spaces and to [5] for the terminology and notations concerning the general theory of locally convex spaces.

In order to define a locally convex topology on $\Lambda\langle E\rangle$, we need the following

Proposition 2.1. *For all $S \in \mathcal{S}$ and $M \in \mathcal{M}$, $\sigma_{S,M}$ defined by*

$$\sigma_{S,M}(x) = \sup\left\{\sum_{n=1}^{\infty} |a_n(x_n)|, a = (a_n)_n \in \Lambda_S^*[E'_M], \varepsilon_{S^\circ,M^\circ}(a) \leq 1\right\},$$

for $x = (x_n)_n \in \Lambda\langle E\rangle$, is a seminorm on $\Lambda\langle E\rangle$.

Proof: It is enough to show that, for all $x \in \Lambda\langle E\rangle$, $\sigma_{S,M}(x)$ is finite. Define the linear mapping T_x from $\Lambda_S^*[E'_M]$ to ℓ_1 by $T_x((a_n)_n) = (a_n(x_n))_n$. Suppose that $(f^i)_i \subset \Lambda_S^*[E'_M]$ is a sequence which converges to $f := (f_n)_n$ and $(T_x(f^i))_i$ converges in ℓ_1 to $(\alpha_n)_n$.
By the continuity of the projections, $(f_n^i)_i$ converges to f_n for all $n \in \mathbb{N}$ and then $(f_n^i(x_n))_{i\in\mathbb{N}}$ converges to $f_n(x_n)$ as well. Indeed, for every $n \in \mathbb{N}$, if

$$j_n : E'_M \longrightarrow \mathbb{K}, \ x' \longrightarrow x'(x_n),$$

then $|j_n(x')| \leq P_M(x_n)\|x'\|_M$, wherby j_n is continuous on E'_M. It follows that $(f_n(x_n))_n = (\alpha_n)_n$. This shows that the graph of T_x is closed. So, T_x is continuous and bounbed on the unit ball of $\Lambda_S^*[E'_M]$ endowed with the norm $\varepsilon_{S^\circ,M^\circ}$. So, $\sigma_{S,M}(x)$ is finite. $\square$.

In the sequel $\Lambda\langle E\rangle$ will be equipped with the locally convex topology generated by the seminorms $(\sigma_{S,M})_{S\in\mathcal{S},M\in\mathcal{M}}$, while (See e.g. [12]) the topology of $\Lambda\{E\}$ will be defined by the seminorms $\pi_{S,M}, S \in \mathcal{S}, M \in \mathcal{M}$, where

$$\pi_{S,M}(x) = \sup\left\{\sum_{n=1}^{\infty} P_M(\alpha_n x_n), \alpha = (\alpha_n)_{n\in\mathbb{N}} \in S\right\}, \ \forall x = (x_n)_n \in \Lambda\{E\}.$$

Proposition 2.2. *The spaces* Λ *and* E *are closed subspaces of* $\Lambda[E]$ *and* $\Lambda(E)$.

Proof: 1. Let $k \in \mathbb{N}$. The mapping $I : E \longrightarrow \Lambda[E], t \longrightarrow te_k$, where t is at k^{th} place. It is clear that I is linear and injective. Let $S \in \mathcal{S}$ and $M \in \mathcal{M}$. Since $\varepsilon_{S,M}(te_k) = P_S(e_k)P_M(t)$, I is continuous. In other hand, if we choose S with $e_k \in S$ we get $P_M(t) = \varepsilon_{S,M}(te_k)$. So I is an isomorphism from E to the subspace $I(E) = \{te_k, t \in E\}$ of $\Lambda(E)$.
Let us prove that this subspace is closed in $\Lambda[E]$. Let $(t_ie_k)_i$ is a net in $I(E)$ which converges in $\Lambda[E]$ to $x = (x_n)_n$. For all $n \in \mathbb{N}$, $n \neq k$, the net $(I_n(t_ie_k))_i$ converges to x_n, so $x_n = 0$ and $x = x_ke_k \in I(E)$.
2. Now, fix $x_0 \in E, x_0 \neq 0$ and consider the mapping

$$J : \Lambda \longrightarrow \Lambda[E],\ \alpha = (\alpha_n)_n \longrightarrow \alpha x_0 = (\alpha_n x_0)_n.$$

Then $\varepsilon_{S,M}((\alpha_n x_0)_n) = P_S(\alpha)P_M(x_0)$, which proves the continuity of J. Let $M \in \mathcal{M}$ such that $P_M(x_0) \neq 0$, then $P_S(\alpha) = \frac{1}{P_M(x_0)}\varepsilon_{S,M}((\alpha_n x_0)_n)$. So, J is un isomorphism from Λ to $J(\Lambda)$. To prove that it is closed, let $(\alpha^i x_0)_i = ((\alpha_n^i x_0)_n)_i$ is a net in $J(\Lambda)$ which converge in $\Lambda[E]$ to $x = (x_n)_n$. For all $n \in \mathbb{N}$, the net $(\alpha_n^i x_0)_i$ converges to x_n, by the continuity of the projections. There exists $\alpha_n \in \mathbb{K}$, such that $(\alpha_n^i x_0)_i$ converges to $\alpha_n x_0$. Then, $x_n = \alpha_n x_0$. Let $a \in E'$ such that $a(x_0) = 1$, then $(\alpha_n)_n = (a(x_n))_n \in \Lambda$. □

Proposition 2.3. *Let* $\mathcal{B} \subset \Lambda(E)$*. The following conditions are equivalent:*
1. $\mathcal{B}$ *is bounded in* $\Lambda(E)$.
2. For all $M \in \mathcal{M}, A_M = \{(a(x_n))_n,\ (x_n)_n \in \mathcal{B},\ a \in M\}$ *is bounded in* Λ.
3. For all $S \in \mathcal{S}, A_S = \{\sum_{n=1}^{\infty} \alpha_n x_n,\ (x_n)_n \in \mathcal{B},\ (\alpha_n)_n \in S\}$ *is bounded in* E.

Proof: The result follows from the equalities

$$\begin{aligned}
\epsilon_{S,M}(x) &= \sup\left\{\sum_{n=1}^{\infty} |\alpha_n a(x_n)|, a \in M, \alpha = (\alpha_n)_{n\in\mathbb{N}} \in S\right\} \\
&= \sup\{P_S((a(x_n))_n), a \in M\} \\
&= \sup\left\{P_M(\sum_n \alpha_n a(x_n)), \alpha = (\alpha_n)_n \in S\right\}.
\end{aligned}$$

which hold for all $S \in \mathcal{S}, M \in \mathcal{M}$ and $x = (x_n)_n \in \mathcal{B}$. □

Proposition 2.4. *Every sequence in* E *which is strongly* Λ*-summable in* E *is absolutely* Λ*-summable. Moreover, the inclusion* $\Lambda\langle E\rangle \subset \Lambda\{E\}$ *is continuous.*

Proof: Let $x = (x_n)_n \in \Lambda\langle E\rangle, M \in \mathcal{M}$ and $\alpha = (\alpha_n)_n \in \Lambda^*$. Let us prove that the series $\sum \alpha_n P_M(x_n)$ is absolutely convergent.
Let $\varepsilon > 0$ and $n \in \mathbb{N}$. Since $P_M(\alpha_n x_n) = \sup\{|a(\alpha_n x_n)|,\ a \in M\}$, there exists $a_n \in M$

such that $P_M(\alpha_n x_n) \leq \frac{\varepsilon}{2^n} + |a_n(\alpha_n x_n)|$. So, $(\alpha_n a_n)_n \in \Lambda^*[E'_M]$. Indeed, if $f \in (E'_M)'$, we get

$$|\alpha_n f(a_n)| \leq |\alpha_n| \, |f(a_n)| \leq |\alpha_n| \, \|f\| \, \|a_n\|_M \leq \|f\| \, |\alpha_n|$$

By the normality of Λ^*, $(\|f\| \, |\alpha_n|)_n \in \Lambda^*$ and

$$(f(\alpha_n a_n))_n = (\alpha_n f(a_n))_n \in \Lambda^*.$$

The series $\sum \alpha_n P_M(x_n)$ is absolutely convergent since $x = (x_n)_n \in \Lambda\langle E\rangle$ and

$$\sum_{n=1}^{k} |\alpha_n| \, P_M(x_n) \leq \varepsilon + \sum_{n=1}^{\infty} |\alpha_n f(a_n)| \, , \text{ for all } k \in \mathbb{N},$$

This last inequality shows that for all $\alpha \in \Lambda^*, x = (x_n)_n \in \Lambda\langle E\rangle$ and $M \in \mathcal{M}$,

$$\sum_{n=1}^{\infty} |\alpha_n P_M(x_n)\| \leq \sigma_{S,M}(x).$$

So, the inclusion of $\Lambda\langle E\rangle$ in $\Lambda\{E\}$ is continuous. □

Proposition 2.5. *Every sequence in E which is absolutely Λ-summable in E is Λ-summable. Moreover, the inclusion $\Lambda\{E\} \subset \Lambda(E)$ is continuous.*

Proof: Let $x = (x_n)_n \in \Lambda\{E\}, M \in \mathcal{M}$, $\alpha = (\alpha_n)_n \in \Lambda^*$ and $\varepsilon > 0$. Since the series $\sum \alpha_n P_M(x_n)$ is absolutely convergent, there exists $N \in \mathbb{N}$ such that for all $p \in \mathbb{N}$,

$$p \geq q \geq N \Rightarrow \sum_{n=q+1}^{p} |\alpha_n| \, P_M(x_n) \leq \varepsilon.$$

So,

$$P_M(\sum_{n=q+1}^{p} \alpha_n x_n) \leq \sum_{n=q+1}^{p} |\alpha_n| \, P_M(x_n) \leq \varepsilon.$$

Since E is sequentially complete, the series $\sum \alpha_n x_n$ is convergent. Moreover, for all $S \in \mathcal{S}$ and $M \in \mathcal{M}$, $\varepsilon_{S,M}((x_n)_n) \leq \pi_{S,M}((x_n)_n)$, which gives the continuity. □

3 Nuclearity of $\Lambda(E)$

The following lemma will be needed to prove Theorem 3.2:

Lemma 3.1. *For every $M \in \mathcal{M}$ and $(a_n)_n \in \Lambda^*[E'_M]$, the set $S = \{(a_n(t))_n, t \in M^\circ\}$ is bounded in Λ.*

Proof: For $(\alpha_n)_n \in \Lambda$, define the linear mapping $\varphi : (E'_M)' \to \ell_1$ by $\varphi(f) = (\alpha_n f(a_n))_n$. It is not hard to show that the graph of φ is closed and then φ is continuous. So, $\sum_n |\alpha_n f(a_n)| \leq \|\varphi\| \|f\|$, for all $f \in (E'_M)'$. If $t \in M^\circ$, then the mapping f_t defined on E'_M by $f_t(a) = a(t)$ satisfies $\sum_n |\alpha_n f_t(a_n)| \leq \|\varphi\| \|f_t\|$, that is $\sum_n |\alpha_n a_n(t)| \leq \|\varphi\|$. So, S is bounded in Λ^*.

Theorem 3.2. *For every nuclear locally convex space E and a perfect sequence space Λ, $\Lambda\{E\} = \Lambda(E) = \Lambda[E]$, algebraically and topologically and $\Lambda\{E\} = \Lambda\langle E\rangle$, algebraically. Moreover, if $\mathcal{S}$ is the family of all bounded sets of Λ^*, then this equality is topological.*

Proof: 1. Let us prove the equality $\Lambda(E) = \Lambda\{E\}$. By Proposition 2.5, we only have to show that $\Lambda(E) \subset \Lambda\{E\}$ with continuous injection. Since E is nuclear then, by 21.2.1 of [5], we have

$$\ell_1\{E\} = \ell_1(E) = \ell_1[E]. \tag{1}$$

Let $x = (x_n)_n \in \Lambda(E)$. For all $\alpha = (\alpha_n)_n \in \Lambda^*$, $(\alpha_n x_n)_n \in \ell_1(E) = \ell_1\{E\}$. Whereby, $x \in \Lambda\{E\}$ and $\Lambda\{E\} = \Lambda(E)$ algebraically. Now, let $M \in \mathcal{M}$. By (1), there exists $K > 0$ such that for all $x = (x_n)_n \in \ell_1(E)$,

$$\sum_{n=1}^{\infty} P_M(x_n) \leq K P_M(\sum_{n=1}^{\infty} x_n).$$

If $S \in \mathcal{S}$ and $\alpha = (\alpha_n)_n \in S$ then, $(\alpha_n x_n)_n \in \ell_1(E)$, whereby

$$\sum_{n=1}^{\infty} |\alpha_n| \, P_M(x_n) \leq K P_M(\sum_{n=1}^{\infty} \alpha_n x_n).$$

We then have, for all $x = (x_n)_n \in \Lambda(E)$,

$$\begin{aligned}
\pi_{S,M}(x) &= \sup\left\{\sum_{n=1}^{\infty} |\alpha_n| \, P_M(x_n),\ \alpha = (\alpha_n)_n \in S\right\} \\
&\leq K \sup\left\{\sum_{n=1}^{\infty} |\alpha_n a(x_n)|\,,\ \alpha = (\alpha_n)_n \in S,\ a \in M\right\} \\
&= K\varepsilon_{S,M}(x).
\end{aligned}$$

2. To prove the equality $\Lambda\langle E\rangle = \Lambda\{E\}$, it is enough, by Proposition 2.4, to show that $\Lambda\{E\} \subset \Lambda\langle E\rangle$ with continuous injection. Let $x = (x_n)_n \in \Lambda\{E\}$, $M \in \mathcal{M}$ and $a = (a_n)_n \in \Lambda^*[E'_M]$. Since E is nuclear there exists $N \in \mathcal{M}$ such that $N \supset M$ and the canonical injection from E'_M to E'_N is nuclear. Let $\lambda = (\lambda_i)_i \in \ell_1$ with $\|\lambda\|_{\ell_1} \leq 1$, an equicontinuous sequence $(f_i)_i$ in the topological dual of E'_M and a bounded sequence $(x'_i)_i \subset E'_N$ such that

$$\forall x' \in E'_M,\ x' = \sum_{n=1}^{\infty} \lambda_i f_i(x') x'_i.$$

We may and do assume that $(x'_i)_i \subset N$ and that $\sum_{i=1}^{\infty} |\lambda_i| \, \|f_i\| = 1$. We have

$$a_n(x_n) = \sum_{n=1}^{\infty} \lambda_i f_i(a_n) x'_i(x_n), \text{ for all } n \in \mathbb{N}.$$

Whereby, for all $k \in \mathbb{N}$,

$$\sum_{n=1}^{k} a_n(x_n) = \sum_{n=1}^{k}\left(\sum_{n=1}^{\infty}\lambda_i f_i(a_n)x_i'(x_n)\right) = \sum_{n=1}^{\infty}\lambda_i f_i\left(\sum_{n=1}^{k} x_i'(x_n)a_n\right),$$

and

$$\begin{aligned}\left|\sum_{n=1}^{k} a_n(x_n)\right| &= \left|\sum_{n=1}^{\infty}\lambda_i f_i\left(\sum_{n=1}^{k} x_i'(x_n)a_n\right)\right| \leq \sum_{n=1}^{\infty}|\lambda_i|\left|f_i(\sum_{n=1}^{k} a_n x_i'(x_n))\right| \\ &\leq \sum_{n=1}^{\infty}|\lambda_i|\,\|f_i\|\left\|\sum_{n=1}^{k} a_n x_i'(x_n)\right\|_M .\end{aligned}$$

But,

$$\begin{aligned}\left\|\sum_{n=1}^{k} a_n x_i'(x_n)\right\|_M &= \sup\left\{\left|\sum_{n=1}^{k} x_i'(x_n)a_n(t)\right|,\ t \in M^\circ\right\} \\ &\leq \sup\left\{\sum_{n=1}^{k}|x_i'(x_n)a_n(t)|,\ t \in M^\circ\right\} \\ &\leq \sup\left\{\sum_{n=1}^{k} P_M(x_n)\,|a_n(t)|,\ t \in M^\circ\right\} \\ &\leq \varepsilon_{S^\circ,M^\circ}((a_n)_n),\end{aligned}$$

for all $S \in \mathcal{S}$, such that $(P_M(x_n))_n \in S^\circ$.
So,

$$\left|\sum_{n=1}^{k} a_n(x_n)\right| \leq \left(\sum_{i=1}^{\infty}|\lambda_i|\,\|f_i\|\right)\varepsilon_{S^\circ,M^\circ}((a_n)_n).$$

If $(\varepsilon_n)_n$ is such that $|a_n(x_n)| = \varepsilon_n a_n(x_n)$, for all $n \in \mathbb{N}$, then

$$\sum_{n=1}^{k}|a_n(x_n)| \leq \left(\sum_{i=1}^{\infty}|\lambda_i|\,\|f_i\|\right)\varepsilon_{S^\circ,M^\circ}((\varepsilon_n a_n)_n) = \varepsilon_{S^\circ,M^\circ}((a_n)_n).$$

This shows that the series $\sum|a_n(x_n)|$ is convergent and that $(x_n)_n \in \Lambda\langle E\rangle$.
We also have,

$$\sum_{n=1}^{\infty}|a_n(x_n)| \leq \sup\left\{\sum_{n=1}^{\infty} P_M(x_n)|a_n(t)|,\ t \in M^\circ\right\}.$$

Let $S_1 = \{(a_n(t))_n,\ t \in M^\circ\}$. By lemma 3.1, S_1 is bounded in Λ^*. If we choose $S \in \mathcal{S}$ such that $S_1 \subset S$, we will obtain $\sum_{n=1}^{\infty}|a_n(x_n)| \leq \pi_{S,M}(x)$.
So, $\sigma_{S,M}(x) \leq \pi_{S,M}(x), \forall x = (x_n)_n \in \Lambda\langle E\rangle$.

3. If E is nuclear then $\Lambda(E) = \Lambda[E]$ algebraically and topologically by (1) and the very definition of the topology of these spaces. □.

We now give the main result:

Theorem 3.3. *Let E be a complete locally convex space and Λ be a perfect sequence space. Then $\Lambda(E)$ is nuclear if, and only if, Λ and E are nuclear.*

Proof: If $\Lambda(E)$ is nuclear, so are Λ and E by proposition 2.2. Inversely, suppose Λ and E are nuclear. Since E is nuclear, every weakly convergent sequence in E is convergent to the same limit, by 3.2.6(a) of [14]. Now, an application of ([3], Teorema) gives the equality

$$\Lambda(E) = \Lambda(E)_r.$$

But $\Lambda(E)_r = \Lambda\widetilde{\otimes}_\varepsilon E$, by ([3], Proposición 2), so

$$\Lambda(E)_r = \Lambda\widetilde{\otimes}_\varepsilon E = \Lambda\widetilde{\otimes}_\pi E,$$

which is nuclear by 4.6.4 of [14]. □

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