Rev. Real Academia de Ciencias. Zaragoza. **62:** 75–79, (2007).

Which spheres admit a topological group structure?

Ignacio Santa-María Megía

Depto. de Geometría y Topología. Facultad de Ciencias Matemáticas.

Universidad Complutense de Madrid

Abstract

It is well known since the 1940's that $\mathbb{S}^0$, $\mathbb{S}^1$ and $\mathbb{S}^3$ are the only spheres admitting a topological group structure. In this short note we provide an easy and direct proof (without using Lie group theory nor dimension theory) of the fact that $\mathbb{S}^{2n}$ does not admit such a structure for any $n > 0$. The proof is based upon the notion of group actions on a topological space; loosely speaking what makes possible this argument is that there are more self-homeomorphisms of a topological group than of an even sphere.

2000 MS Classification: 22-02.

Key Words: Topological group, sphere.

1 Introduction

The proof of the fact that the only Euclidean spheres that can be made into topological groups are $\mathbb{S}^0$, $\mathbb{S}^1$ and $\mathbb{S}^3$ is given in [7, p. 144], it refers to a result from Cartan [2, p. 179] and uses Lie group theory and dimension theory for the proof.

Other proofs found in the literature use the fact that the only H-spaces among $\mathbb{S}^n$, $n \geq 0$, are those given by $n = 0, 1, 3, 7$ (see [1, Theorem 1.1.1]). Since a topological group must be an H-space, the only ones to test are for those values of n. Moreover, in [4, Theorem 1] it can be found a direct proof of the fact that $\mathbb{S}^7$ is not a Lie group and therefore not a topological group, because *"all locally euclidean groups are isomorphic to Lie groups"* [5, p. 213].

None of these proofs are elementary. The goal of this note is to present a more simple proof of the fact that no even dimensional sphere admits a topological group structure.

In order to make the paper self-contained we define the tools required for its development.

2 Preliminaries

In this section we provide the definition, some examples and some properties of topological groups. For a deeper comprehension of the subject read [6].

Definition 1. *Let G be a group (with multiplicative notation) and a topological space at the same time. Then G is said to be a* topological group *if the group operation and the inverse operation are continuous mappings:*

$$\begin{array}{rcl} G\times G & \to & G \\ (f,g) & \mapsto & f\cdot g \end{array} \qquad\qquad \begin{array}{rcl} inv: G & \to & G \\ g & \mapsto & g^{-1} \end{array}.$$

Example 1. *We now give the topological group structure for $\mathbb{S}^n$, $n \in \{0,1,3\}$.*

$(n=0)$ *As a topological space it is the discrete space $\{-1,1\}$ and if it is given a group structure we get $\mathbb{Z}_2$, so $\mathbb{S}^0 \approx \mathbb{Z}_2$.*

$(n=1)$ *Let us consider $\mathbb{S}^1$ as the unitary complex numbers, endowed with the complex product and with the topology induced from the usual one of $\mathbb{C}$. Clearly, $\mathbb{S}^1$ is a topological group and it is usually denoted by $\mathbb{T}$.*

$(n=3)$ *We manage to see $\mathbb{S}^3$ in $\mathbb{C}\times\mathbb{C}$ by means of the following product:*

Let us represent each element (z,w) of $\mathbb{C}\times\mathbb{C}$ by a matrix of the form

$$\begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}$$

and define the product of two elements as the matrix multiplication.

This representation permits us to write the Euclidean norm of such a (z,w) as

$$||(z,w)|| = \sqrt[2]{\det(z,w)}.$$

When $\mathbb{C}\times\mathbb{C}$ is seen in this fashion it is called the quaternionic space *and denoted by $\mathbb{H}$. Now it is clear that $\mathbb{S}^3$ is the set of unitary quaternions and that this product operation makes it a topological group since, as a complex function, it is continuous and the inversion map (which is the matrix inversion): $(z,w)^{-1} = (\overline{z},-w)$ is also a continuous complex function in $\mathbb{S}^3$.*

Proposition 1. *Let G be a topological group. Then for each $a\in G$ the left and right translations, given by a, are homeomorphisms. Furthermore, the inversion map is also an homeomorphism.*

Proof. Let $a \in G$ and $L_a : G \to G$ (with $L_a(g) = a \cdot g$) be the left translation given by a. Then L_a is a composition of continuous functions and its inverse function is $L_{a^{-1}}$. Hence L_a is a homeomorphism (it follows analogously for the right translation $R_a : G \to G$, $R_a(g) = g \cdot a$).

To show that the inversion map is a homeomorphism, just notice that it is its own inverse (it is continuous by hypothesis). □

Next, we define homogeneity in the context of topological spaces and we show that every topological group is homogeneous.

Definition 2. *A topological space X is said to be* homogeneous *if for any $x, y \in X$, there exists a homeomorphism $f : X \to X$ such that $f(x) = y$.*

Observation 1. *Every topological group is homogeneous. This follows by proposition 1, since given $a, b \in G$ we can define the homeomorphism $f = L_{a^{-1}} \circ R_b$, which verifies $f(a) = b$.*

But not every homogeneous topological space can be made into a topological group, the Sorgenfrey line is a counterexample.

Example 2. *The Sorgenfrey line is a homogeneous topological space but not a topological group.*

It is homogeneous: given a, b in the Sorgenfrey line, the map $f(x) = x + b - a$ sends a to b and is an homeomorphism since it is continuous, and its inverse, $f^{-1}(y) = y - b + a$, is also continuous.

It is not a topological group with the addition of real numbers because the inverse operation fails to be continuous.

Furthermore, it does not admit a topological group structure at all: every first countable topological group is metrizable but the Sorgenfrey line is first countable and not metrizable. Therefore, there is no group operation in $\mathbb{R}$ that could turn the Sorgenfrey line into a topological group.

The two dimensional sphere is another example of an homogeneous topological space (think of rotations) but not a topological group, as we shall see.

3 $\mathbb{S}^{2n}$ is not a topological group.

First of all, we recall some notions about group actions on topological spaces and the concept of degree of a continuous map from a topological space to itself. Let us also fix some notation: X is a topological space, G a group and $\mathrm{Homeo}(X)$ the group of self-homeomorphisms of X.

Definition 3. *We say that* G acts on X *if there exists an homomorphism* $\Theta : G \to Homeo(X)$*. For each* $g \in G$ *we write* Θ_g *instead of* $\Theta(g)$*.*

If, in addition, $\Theta_g : X \to X$ *has no fixed points for any* $g \in G$ *but for the identity (i.e.,* $\Theta_g(x) \neq x$, $\forall\, x \in X$ *and* $\forall\, g \in G \setminus \{e\}$*), then the action is said to be* free *(or that* G acts freely *on* X*).*

Observation 2. *Every topological group acts freely on itself. Indeed, since translations are homeomorphisms we can define the map* $L : G \to Homeo(G)$ *such that* $g \mapsto L_g$*, which is an homomorphism. Moreover,* L_g *is a translation, so if* $g \neq e$ *then* L_g *has no fixed points.*

The following definition, properties and proposition can be found in [3, pp. 134-135]. They are the tools we use to reach our goal. First, let us fix $n \in \mathbb{N} \setminus \{0\}$.

Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a continuous map. Now consider the induced endomorphism of the n^{th}-homology group of the n-sphere, $f_* : H_n(\mathbb{S}^n) \to H_n(\mathbb{S}^n)$. Since $H_n(\mathbb{S}^n) \simeq \mathbb{Z}$, there exists a unique integer d such that $f_*(\alpha) = d\alpha$ for every $\alpha \in H_n(\mathbb{S}^n)$.

Definition 4. *We define the* degree of f *as* $\deg(f) = d$*.*

Some properties of the degree that we shall use further on are:

(i) $\deg(\mathrm{id}) = 1$ where $\mathrm{id} : \mathbb{S}^n \to \mathbb{S}^n$ is the identity map;

(ii) if $f, g : \mathbb{S}^n \to \mathbb{S}^n$ are continuous maps, then $\deg(f \circ g) = \deg(f) \cdot \deg(g)$;

(iii) let f and g be as above. Then, they are homotopic if and only if $\deg(f) = \deg(g)$;

(iv) if f is the reflection of $\mathbb{S}^n$ that interchanges the hemispheres determined by a fixed equator, then $\deg(f) = -1$;

(v) let id^- be the antipodal map, then $\deg(\mathrm{id}^-) = (-1)^{n+1}$;

(vi) if $f : \mathbb{S}^n \to \mathbb{S}^n$ is a continuous map without fixed points, then $\deg(f) = (-1)^{n+1}$.

Hence from **(iii)**, **(v)** and **(vi)** we get that every continuous map from the n-sphere in itself without fixed points is homotopic to the antipodal map. The converse statement is false, for if n is odd then **(i)**, **(iii)** and **(v)** imply that the antipodal map is homotopic to the identity; and if n is even then **(iv)**, **(v)** and **(vi)** imply that a reflection as in **(iv)** is homotopic to the antipodal map.

As a direct consequence of the definition of degree and its properties we get the next result [3, Proposition 2.29].

Proposition 2. $\mathbb{Z}_2$ *is the only nontrivial group that can act freely on* $\mathbb{S}^n$ *if* n *is even.*

Proof. Let n be even and let G act freely on $\mathbb{S}^n$. Then there exists a homomorphism $\Theta : G \to \text{Homeo}(\mathbb{S}^n)$. Hence for each $g \in G$ we can compute the degree $\deg(\Theta_g)$. Furthermore, since Θ is an homomorphism, from **(i)** and **(ii)** we have the following equalities:

$$\deg(\Theta_g) \cdot \deg(\Theta_{g^{-1}}) = \deg(\Theta_g \circ \Theta_{g^{-1}}) = \deg(\Theta_e) = \deg(\text{id}) = 1.$$

So $\deg(\Theta_g) = \deg(\Theta_{g^{-1}}) = \pm 1$, because the degree must be an integer.

We can now construct a degree function $d : G \to \mathbb{Z}_2 = \{\pm 1\}$ as $d(g) = \deg(\Theta_g)$, which is an isomorphism. Indeed, d is an homomorphism by **(ii)** and because G acts freely on S^n we know that Θ_g has no fixed points for any $g \neq e$. So **(vi)** implies $\deg(\Theta_g) = -1$ and from **(i)** we have $\deg(\Theta_e) = 1$. Hence d is onto and has trivial kernel. ☐

Now it is clear that $\mathbb{S}^{2n}$ is not a topological group for any $n > 0$. Otherwise it would act freely on itself and by proposition 2 it would be isomorphic to $\mathbb{Z}_2$, which is clearly a contradiction.

This statement can be set in a naive way: "If an even-dimensional sphere were a topological group, it would be isomorphic to the zero-dimensional one".

Acknowledgments. I wish to thank Dr. Elena Martín Peinador for her valuable advice and guidance.

References

[1] Adams, J.F., *On the non-existence of elements of Hopf invariant one,* Ann. Math., 2nd Ser., **72** No. 1, 20-104. (JSTOR).

[2] Cartan, E., *La topologie des espaces répresentatives des groupes de Lie,* L'Enseignement Math., **35**, 177-200.

[3] Hatcher, A., *Algebraic Topology,* (Cambridge University Press). Also in the webpage http://www.math.cornell.edu/ hatcher/AT/AT.pdf.

[4] Jian, Z. and Senlin, X., $\mathbb{S}^7$ *without any construction of Lie group,* International Center for Theoretical Physics, unpublished preprint found at http://streaming.ictp.trieste.it/preprints/P/88/404.pdf.

[5] Montgomery, D. and Zippin, L. *Small subgroups of finite-dimensional groups,* Ann. Math., 2nd Ser., **56**, No. 2, 213-241. (JSTOR).

[6] Morris, Sidney A., *Pontryagin duality and the structure of locally compact Abelian groups,* London Math. Soc. Lecture Notes Series, **29**, (Cambridge University Press).

[7] Samelson, H., *Über die Sphären, die als Gruppenräume auftreten,* Comm. Math. Helvetici, **13** 145-155. Accessible from the website http://retro.seals.ch/digbib/home.