Rev. Real Academia de Ciencias. Zaragoza. **60:** 107–115, (2005).

The Arzéla-Ascoli theorem for non-locally convex weighted spaces

L.A. Khan

Department of Mathematics, King Abdul Aziz University

P.O. Box 80203, Jeddah 21589, (Saudi Arabia)

E-mail: akliaqat@hotmail.com

and

L. Oubbi

Department of Mathematics. École Normale Supérieure de Rabat

BP 5118, Takaddoum, 10105 Rabat (Morocco)

E-mail: l_oubbi@hotmail.com

Abstract

We deal with Arzéla-Ascoli type theorems in non-locally convex weighted spaces $CV_0(X, E)$ and $CV_p(X, E)$.

Key words: Arzéla Ascoli theorem, Weighted spaces of continuous functions

1 Introduction and Preliminaries

In this paper we characterize the precompact subsets of the weighted spaces $CV_0(X, E)$ and $CV_p(X, E)$ for an arbitrary topological vector space (TVS) E. This extends to the non-locally convex setting Arzéla-Ascoli type theorems given in [12]. Whenever the space E happens to be quasicomplete, this turns out to be characterizations of relative compactness. The importance of Arzéla-Ascoli type theorems is evident. Their applications already in the case of scalar-valued functions are numerous, namely in differential equations, in finding extremal curves, in Mazure-Orlicz criterion for the consistency of systems involving certain inequalities etc. For example, by the Arzéla-Ascoli theorem, every bounded equicontinuous sequence in C(X), with X compact, has a uniformly convergent subsequence. This observation is very useful in the existence of solutions of differential equations. In [12], W.M. Ruess and W.H. Summers used effectively the Arzéla-Ascoli theorem for the locally convex weighted spaces to obtain a solution of the Cauchy problem

concerning the asymptotic almost periodic behavior of motion solutions. Our results may then provide a framework for further applications in the non-locally convex setting.
Throughout this paper, unless stated otherwise, X will denote a completely regular Hausdorff space and E a non-trivial Hausdorff topological vector space (TVS) with a base $\mathcal{W}$ of closed balanced neighborhoods of 0. A *Nachbin family* V on X is a set of non-negative upper semicontinuous functions on X, called *weights*, such that given $u,\ v \in V$, $t \geq 0$ and $x \in X$, there exists $w \in V$ with $tu,\ tv \leq w$ (pointwise) and $w(x) > 0$. Let $C(X,E)$ be the vector space of all continuous E-valued functions on X, and $C_b(X,E)$ (resp. $C_p(X,E),\ C_0(X,E)$) the subspace of $C(X,E)$ consisting of those functions which are bounded (resp. have precompact range, vanish at infinity). Further, let

$$\begin{aligned} CV_b(X,E) &= \{f \in C(X,E) : vf(X) \text{ is bounded in } E \text{ for all } v \in V\}, \\ CV_p(X,E) &= \{f \in C(X,E) : vf(X) \text{ is precompact in } E \text{ for all } v \in V\}, \\ CV_0(X,E) &= \{f \in C(X,E) : vf \text{ vanishes at infinity on } X \text{ for all } v \in V\} \end{aligned}$$

Clearly, $CV_0(X,E) \subset CV_p(X,E) \subset CV_b(X,E)$. The first inclusion is due to the upper semicontinuity of the weights. When $E = I\!K$, the above spaces are denoted by $C(X)$, $C_b(X)$, $C_p(X)$, $C_0(X)$, $CV_b(X)$, and $CV_0(X)$. If $\varphi \in C(X)$ and $a \in E$, then $\varphi \otimes a$ is the function in $C(X,E)$ defined by $(\varphi \otimes a)(x) = \varphi(x)a; x \in X$. The *weighted topology* w_V [11, 4] on $CV_b(X,E)$ is defined as the linear topology which has a base of neighborhoods of 0 consisting of all sets of the form

$$N(v,G) = \{f \in CV_b(X,E) : vf(X) \subset G\},$$

where $v \in V$ and $G \in \mathcal{W}$. We mention that, in the non-locally convex setting, the weighted function spaces $CV_0(X,E)$ and $CV_b(X,E)$ have been studied by several authors in recent years for a variety of problems; see e.g. [4, 5, 7, 8, 10, 13].

The following are some instances of weighted spaces.

1. If $V = K^+(X) = \{\lambda\chi_X : \lambda > 0\}$, the set of all non-negative constant functions on X, then $CV_b(X,E) = C_b(X,E)$, $CV_p(X,E) = C_p(X,E)$, $CV_0(X,E) = C_0(X,E)$, and w_V is the *uniform topology* σ.

2. If $V = S_0^+(X)$, the set of all non-negative upper semi-continuous functions on X which vanish at infinity, then $CV_b(X,E) = CV_p(X,E) = CV_0(X,E) = C_b(X,E)$ and w_V is the *strict topology* β_0.

3. If $V = K_c^+(X) = \{\lambda\chi_K : \lambda > 0 \text{ and } K \subset X,\ K \text{ compact}\}$, then $CV_b(X,E) = CV_p(X,E) = CV_0(X,E) = C(X,E)$ and w_V is the *compact-open topology* k.

4. If $V = K_f^+(X) = \{\lambda\chi_A : \lambda > 0 \text{ and } A \subset X,\ A \text{ finite }\}$, then $CV_b(X,E) = CV_p(X,E) = CV_0(X,E) = C(X,E)$ and w_V is the *pointwise topology* p.

Clearly $p \leq k$ on $C(X,E)$ and $k \leq \beta_0 \leq \sigma$ on $C_b(X,E)$. Moreover, the conditions on a Nachbin family V imply that $p \leq w_V$.

Let $B(E,F)$ denote the algebra of all continuous linear mappings T from a TVS E into another F. For any collection $\mathcal{A}$ of subsets of E, $B_{\mathcal{A}}(E,F)$ denotes the subspace of $B(E,F)$ consisting of those T which are bounded on the members of $\mathcal{A}$ together with the topology $\tau_{\mathcal{A}}$ of uniform convergence on the elements of $\mathcal{A}$. This topology has a base of neighborhoods of 0 consisting of all sets of the form

$$M(A,U) := \{T \in B(E,F) : T(A) \subset U\},$$

where $A \in \mathcal{A}$ and U is a neighborhood of 0 in F.

Finally, we will say that a net $\{x_\alpha : \alpha \in I\}$ is a V-*net* if it is contained in $S_{v,1} := \{x \in X : v(x) \geq 1\}$ for some $v \in V$. Following Bierstedt [1], X is said to be a $V_{\mathbb{R}}$-space if a function $f : X \to \mathbb{R}$ is continuous whenever, for each $v \in V$, the restriction of f to $S_{v,1}$ is continuous. If $V = K(X)$, then X is a $V_{\mathbb{R}}$-space means that X is a $k_{\mathbb{R}}$-space.

2 Main Results

For any $x \in X$, let $\delta_x : CV_b(X,E) \to E$ denote the evaluation map $\delta_x(f) = f(x)$ at x. Clearly, $\delta_x \in B(CV_b(X,E),E)$. Next, define the evaluation map $\Delta : X \to B(CV_b(X,E),E)$ by $\Delta(x) = \delta_x$, $x \in X$. If the subscript c in $B_c(CV_b(X,E),E)$ stands for the topology of uniform convergence on precompact subsets of $CV_b(X,E)$, then one has the following lemma given in [10], see also [1, 12]

Lemma 2.1 *The evaluation map $\Delta : X \to B_c(CV_b(X,E),E)$ is continuous if and only if every precompact subset of $CV_b(X,E)$ is equicontinuous. In particular, if X is $V_{\mathbb{R}}$-space, then every precompact subset of $CV_b(X,E)$ is equicontinuous.*

The following theorem extends Theorem 2.1 of [12] to the general setting of TVS's.

Theorem 2.2 *Let A be a subset of $CV_0(X,E)$. Then A is precompact whenever the following conditions hold.*

i. A is equicontinuous.

ii. $A(x) = \{f(x) : f \in A\}$ is precompact in E for each $x \in X$.

iii. vA vanishes at infinity on X for each $v \in V$. This is for each $v \in V$ and $G \in \mathcal{W}$, there exists a compact set $K \subset X$ such that $v(y)f(y) \in G$ for all $f \in A$ and $y \in X \setminus K$.
If X happens to be a $V_{\mathbb{R}}$-space, then the converse is also true.

Proof. Suppose $i. - iii.$ hold. Since $CV_0(X,E) \subset C(X,E)$, by $i.$, A is an equicontinuous subset of $C(X,E)$. Further, since, by $ii.$, A is p-precompact, it follows from ([14],

p. 289) that A is a precompact subset of $(C(X,E),k)$. To show that A is precompact in $CV_0(X,E)$, let $v \in V$ and $G \in \mathcal{W}$. Choose a balanced $H \in \mathcal{W}$ such that $H + H \subset G$. By *iii.*, there exists a compact $K \subset X$ such that

$$v(y)f(y) \in H \text{ for all } f \in A \text{ and } y \in X \setminus K. \tag{1}$$

Since v is upper-semicontinuous, $\|v\|_K = \sup\{v(y) : y \in K\} < \infty$. But A is precompact in $(C(X,E),k)$. Then there exist $h_1, \ldots, h_n \in A$ such that

$$A \subset \bigcup_{i=1}^{n}(h_i + N(\chi_K, (\|v\|_k + 1)^{-1}H)). \tag{2}$$

We claim that

$$A \subset \bigcup_{i=1}^{n}\{h_i + N(v,G)\}.$$

Let $f \in A$ and $y \in X$. If $y \in K$, by (2), there exists $j \in \{1, ..., n\}$ such that

$$v(y)(f(y) - h_j(y)) \in v(y)(\|v\|_k + 1)^{-1}H \subset H.$$

If $y \in X \setminus K$, then, for any $i \in \{1, ..., n\}$, (1) gives

$$v(y)(f(y) - h_i(y)) = v(y)f(y) - v(y)h_i(y) \in H - H \subset G.$$

This establishes our claim, and so A is precompact in $CV_0(X,E)$. Now, Suppose X is a $V_{\mathbb{R}}$-space and that A is a precompact subset of $CV_0(X,E)$. Let us verify $i. - iii.$ Since X is a $V_{\mathbb{R}}$-space, by (2.1), A is equicontinuous whereby i. On the other hand, since $p \leq w_V$, H is p-precompact. Hence, for each $x \in X$, $A(x)$ is precompact in E. Whence ii. Finally, let $v \in V$ and $G \in \mathcal{W}$ and choose a balanced $H \in \mathcal{W}$ with $H + H \subset G$. Since A is precompact, there exist $h_1, \ldots, h_n \in A$ such that

$$A \subset \bigcup_{i=1}^{n}(h_i + N(v,H)). \tag{3}$$

Put $K = \cup_{i=1}^{n}\{y \in X : v(y)h_i(y) \notin H\}$. Since $h_i \in CV_0(X,E)$ for each i, K is compact. Now, let $f \in A$ and $y \in X \setminus K$. By (3), there exists $i \in \{1, ..., n\}$ such that $f \in h_i + N(v,H)$. Hence

$$v(y)f(y) = v(y)(f(y) - h_i(y)) + v(y)h_i(y) \in H + H \subset G.$$

Thus vA vanishes at infinity on X.

The following result is an extension of Theorem 2.2 of [12]. Following W.M. Ruess and W.H. Summers, we shall set, for any $A \subset CV_p(X,E)$, $v \in V$ and $G \in \mathcal{W}$,

$$T_x(A,v,G) = \{y \in X : v(y)f(y) - v(x)f(x) \in G \text{ for all } f \in A\}, \ x \in X.$$

Theorem 2.3 *Consider the following assertions:*

a. *(i) A is equicontinuous;*

(ii) $A(x)$ is precompact in E for each $x \in X$,

(iii) given $v \in V$ and $G \in \mathcal{W}$, there exists a compact set $K \subset X$ such that $\{T_x(A, v, G) : x \in K\}$ covers X.

b. *(i) $vA(X) = \{v(x)f(x) : x \in X,\ f \in A\}$ is precompact in E for each $v \in V$;*

(ii) given $v \in V$ and $G \in \mathcal{W}$, there exists a finite set $F \subset X$ such that $\{T_x(A, v, G) : x \in F\}$ covers X.

c. *(i) $A(x)$ is precompact in E for each $x \in X$;*

(ii) given $v \in V$ and $G \in \mathcal{W}$, there exists a finite set $F \subset X$ such that $\{T_x(A, v, G) : x \in F\}$ covers X.

d. *A is precompact.*

Then a. $\Longrightarrow$ b. $\Longrightarrow$ c. $\Longrightarrow$ d. If X is in addition a $V_{\mathbb{R}}$-space, then also c. $\Longrightarrow$ d.

Proof. $a. \Longrightarrow b.$ Suppose $a.$ holds. We first note that (i) and (ii) together imply, as in the proof of (2.2), that A is a precompact subset of $(C(X, E), k)$. We now verify $b.$ (i) and b (ii).

For $b.$ (i), let $v \in V$, $G \in \mathcal{W}$ and choose a balanced $H \in \mathcal{W}$ such that $H+H+H+H \subset G$. By $a.$ (iii), there exists a compact $K \subset X$ such that

$$X = \bigcup\{T_x(A, v, H) : x \in K\}. \tag{4}$$

Since A is precompact in $(C(X, E), k)$, there exist $h_1, ..., h_n \in A$ such that

$$A \subset \bigcup_{i=1}^{n}(h_i + N(\chi_k, (\|v\|_k + 1)^{-1}H). \tag{5}$$

Moreover, since each $vh_i(K)$ is precompact in E, there exist $\{x_{ij}\}_{j=1}^{n_i} \subset K$ such that

$$vh_i(K) \subset \bigcup_{j=1}^{n_i}(v(x_{ij})h(x_{ij}) + H). \tag{6}$$

Now, fix any $y \in X$ and $f \in A$. By (4), $y \in T_x(A, v, H)$ for some $x \in K$ and so

$$v(y)f(y) - v(x)f(x) \in H \text{ for all } f \in A. \tag{7}$$

By (5), there exists $i \in \{1, ..., n\}$ such that

$$(f - h_i)(K) \subset (\|v\|_k + 1)^{-1}H. \tag{8}$$

By (6), there exists $j \in \{1, ..., n\}$ such that

$$v(x)h_i(x) - v(x_i)h_i(x_{ij}) \in H. \tag{9}$$

By (7), (8) and (9)

$$\begin{aligned} v(y)f(y) - v(x_{ij})h_i(x_{ij}) &= (v(y)f(y) - v(x)f(x)) + v(x)(f(x) - h(x)) \\ &\quad + (v(x)h_i(x) - v(x_{ij})h_i(x_{ij})) \\ &\in H + \frac{v(x)}{\|v\|_k + 1}H + H \subset G. \end{aligned} \tag{10}$$

i.e. $vA(X) \subset \bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} (v(x_{ij})h_i(x_{ij}) + G)$ and $vA(X)$ is precompact in E.

For $b.$ (ii), let $v \in V$ and $G \in \mathcal{W}$, and suppose these be same as in the proof of $b.$ (i). Also, let h_i and x_{ij} be as above. Set $F = \cup_{i=1}^{n}\{x_{ij} : j = 1, ..., n_i\}$. Then, for any fixed $y \in X$ and $f \in A$, (9) and (10) give

$$\begin{aligned} v(y)f(y) - v(x_{ij})f(x_{ij}) &= (v(y)f(y) - v(x_{ij})h_i(x_{ij})) \\ &\quad + (v(x_{ij})h_i(x_{ij}) - v(x_{ij})f(x_{ij})) \\ &\in (H + H + H) + H \subset G; \end{aligned}$$

that is, $y \in T_x(A, v, G)$ with $x \in F$. Hence $\{T_x(A, v, G) : x \in F\}$ covers X.

The implication $b. \Longrightarrow c.$ is trivial.

$c. \Longrightarrow d.$: Suppose that $c.$ holds. Fix any $v \in V$ and $G \in \mathcal{W}$ and choose a balanced $H \in \mathcal{W}$ with $H + H + H \subset G$. By $c.$ (ii) there exists a finite set $F \subset X$ such that $\{T_x(A, v, H) : x \in F\}$ covers X. By $c.$ (i), A is p-precompact, and so there exists $h_i, ..., h_n \in A$ such that

$$A \subset \bigcup_{i=1}^{n} (h_i + N(\chi_F, (\|v\|_F + 1)^{-1}H). \tag{11}$$

We claim that $A \subset \bigcup_{i=1}^{n}(h_i + N(v, G))$. Fix any $f \in A$. By (11), there exists $i \in \{1, ..., n\}$ such that

$$(f - h_i)(F) \subset (\|v\|_F + 1)^{-1}H).$$

Then, for any $y \in X$, $y \in T_x(A, v, H)$ for some $x \in F$ and so

$$\begin{aligned} v(y)f(y) - v(y)h_i(y) &= (v(y)f(y) - v(x)) + v(x)(f(x) - h_i(x)) \\ &\quad + (v(x)h_i(x) - v(y)h_i(y)) \\ &\in H + \frac{v(x)}{\|v\|_F + 1}H - H \\ &\subset H + H + H \subset G. \end{aligned}$$

This proves our claim; i.e., $d.$ holds.

Now, assume that X is a $V_{I\!R}$-space and let us show that $d. \Longrightarrow a.$ Since A is a precompact subset of $CV_p(X, E)$, just as in the proof of (2.2), $a.(i)$ and $a.(ii)$ follow. To prove $a.(iii)$, let $v \in V$ and $G \in \mathcal{W}$. Choose a balanced $H \in \mathcal{W}$ such that $H + H + H \subset G$. The precompactness of A gives $h_1, ..., h_n \in A$ such that

$$A \subset \bigcup_{i=1}^{n} (h_i + N(v, H)). \tag{12}$$

Now, consider the function $h : X \to E^n$ defined by $h(x) = (h_1(x), h_2(x), \ldots, h_n(x))$. This is a continuous function such that $(vh)(X)$ is precompact, for it is contained in the product $\prod_{i=1}^{n}(vh_i)(X)$ which is precompact. Hence for the neighborhood H^n, there exists a finite subset F of X such that

$$(vh)(X) = \bigcup_{x \in F} ((vh)(x) + H^n). \tag{13}$$

This gives

$$X \subset \bigcup_{x \in F} T_x(\{h_1, \ldots, h_n\}, v, H). \tag{14}$$

We now show that $\{T_x(A, v, G) : x \in F\}$ covers X. Fix $y \in X$. By (14), $y \in T_x(\{h_i\}_{i=1}^{n}, v, H)$ for some $x \in F$ and so

$$v(y)h_i(y) - v(x)h_i(x) \in H \ \ \forall\, i = 1, ..., n. \tag{15}$$

Given $f \in A$. By (12), there exists $i \in \{1, ..., n\}$ such that $f - h_i \in N(v, H)$; this is

$$v(z)(f(z) - h_i(z)) \in H \ \forall\, z \in X. \tag{16}$$

So, by (15) and (16),

$$\begin{aligned} v(y)f(y) - v(x)f(x) &= (v(y)f(y) - v(y)h_i(y)) \\ &\quad +(v(y)h_i(y) - v(x)h_i(x)) \\ &\quad +(v(x)h_i(x) - v(x)h(x)) \\ &\in H + H + H \subset G. \end{aligned}$$

Hence $y \in T_x(A, v, G)$; i.e., *a.iii)* holds.

We mention that in the particular case of $V = S_0^+(X)$, (2.2) and (2.3) reduce to Theorem 3.6 of [3]. Further, the Corollaries 2.5.1(a), 2.5.2(a), 2.5.3, 2.5.4(a) of [12] remain valid in the above general setting and are stated as follows:

Corollary 2.4 *([2], p. 81) Let X be a $k_{\mathbb{R}}$-space and E a quasicomplete TVS. A subset A of $(C(X, E), k)$ is relatively compact if and only if the following conditions hold.*

(i) A is equicontinuous on each compact subset of X,

(ii) $A(x)$ is relatively compact in E for each $x \in X$.

Corollary 2.5 *Let X be a locally compact space and E a quasicomplete TVS. A subset A of $(C_0(X,E),u)$ is relatively compact if and only if the following conditions hold.*

(i) A is equicontinuous,

(ii) $A(x)$ is relatively compact in E for each $x \in X$,

(iii) A uniformly vanishes at infinity on X (i.e., for any $G \in \mathcal{W}$, there exists a compact set $K \subset X$ such that $f(y) \in G$ for all $f \in A$ and $y \in X \backslash K$).

Corollary 2.6 *([3], Theorem 3.6) Let X be a $k_{\mathbb{R}}$-space and E a quasicomplete TVS. A subset A of $(C_b(X,E),\beta_0)$ is relatively compact if and only if the following conditions hold.*

(i) A is equicontinuous on each compact subset of X

(ii) $A(x)$ is relatively compact in E for every $x \in X$,

(iii) A is uniformly bounded (i.e., $A(X)$ is bounded in E).

Corollary 2.7 *Let E be a quasicomplete TVS. A subset A of $(C_p(X,E),u)$ is relatively compact if and only if the following conditions hold.*

(i) $A(X)$ is relatively compact in E,

(ii) given $G \in \mathcal{W}$, there exists a finite open cover $\{K_i : i = 1,...,n\}$ of X such that, for any $i \in \{1,...,n\}$ and $x, y \in K_i$,

$$f(x) - f(y) \in G, \quad \forall f \in A.$$

Acknowledgement

We wish to thank Professor W. Ruess for his suggestions used in the proof of Theorem 2.3.

References

[1] K.D. Bierstedt: Tensor product of weighted spaces, Bonner Math. Schriften 81 (1975), 25-58.

[2] J. L. Kelley, I. Namioka and co-authors: *Linear Topological Spaces* (Van Nostrand, 1963).

[3] L.A. Khan: The strict topology on a space of vector-valued functions, Proc. Edinburgh Math. Soc. 22 (1979), 35-4.

[4] L.A. Khan : Weighted topology in the non-locally convex setting. Mat. Vesnik. 37 (1985), 189-195.

[5] L.A. Khan and A.B. Thaheem : Multiplication operators on weighted spaces in the non-locally convex framework, Internat. J. Math. & Math. Sci., 20 (1997), 75-80.

[6] L.A. Khan and A.B. Thaheem : Operator-valued multiplication operators on weighted function spaces, Demonst. Math. 25 (2002), 599-605.

[7] K. Kour, B. Singh : Weighted composition operators on non-locally convex weighted spaces of continuous functions. J. Indian Math. Soc. 66 (1999), 1-4.

[8] J.S. Manhas, R. K. Singh : Weighted composition operators on nonlocally convex weighted spaces of continuous functions, Analysis Math. 24 (1998), 275-292.

[9] L. Oubbi: Multiplication operators on weighted spaces on continuous functions, Portugalia Math. (N.S.) 59 (2002), no.1, 111-124; Erratum, Portugalia Math. (N.S.) 60 (2003), no. 1, i.

[10] L. Oubbi : Weighted composition operators on non-locally convex weighted spaces, Rocky Mountain J. Math. (to appear).

[11] J.B. Prolla : Weighted spaces of vector-valued continuous functions. Ann. Mat. Pura Appl. 89 (1971), 145-158.

[12] W.M. Ruess and W.H. Summers : Compactness in spaces of vector-valued continuous functions and assymptotic almost periodicity, Math. Nachr. 135 (1988), 7-33.

[13] B. Singh and K. Kour : On weighted composition operators on non-locally convex function spaces. Indian J. Appl. Math. 28 (11) (1997), 1505-1512.

[14] A. Wilansky, *Topology for Analysis* (Ginn, Waltham 1970).