The Arzéla-Ascoli theorem for non-locally convex weighted spaces

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Abstract

We deal with Arzéla-Ascoli type theorems in non-locally convex weighted spaces $CV_0(X, E)$ and $CV_p(X, E)$.

Key words: Arzéla Ascoli theorem, Weighted spaces of continuous functions

1 Introduction and Preliminaries

In this paper we characterize the precompact subsets of the weighted spaces $CV_0(X, E)$ and $CV_p(X, E)$ for an arbitrary topological vector space (TVS) $E$. This extends to the non-locally convex setting Arzéla-Ascoli type theorems given in [12]. Whenever the space $E$ happens to be quasicomplete, this turns out to be characterizations of relative compactness. The importance of Arzéla-Ascoli type theorems is evident. Their applications already in the case of scalar-valued functions are numerous, namely in differential equations, in finding extremal curves, in Mazure-Orlicz criterion for the consistency of systems involving certain inequalities etc. For example, by the Arzéla-Ascoli theorem, every bounded equicontinuous sequence in $C(X)$, with $X$ compact, has a uniformly convergent subsequence. This observation is very useful in the existence of solutions of differential equations. In [12], W.M. Ruess and W.H. Summers used effectively the Arzéla-Ascoli theorem for the locally convex weighted spaces to obtain a solution of the Cauchy problem.
concerning the asymptotic almost periodic behavior of motion solutions. Our results may then provide a framework for further applications in the non-locally convex setting.

Throughout this paper, unless stated otherwise, \( X \) will denote a completely regular Hausdorff space and \( E \) a non-trivial Hausdorff topological vector space (TVS) with a base \( W \) of closed balanced neighborhoods of 0. A Nachbin family \( V \) on \( X \) is a set of non-negative upper semicontinuous functions on \( X \), called \textit{weights}, such that given \( u, v \in V \), \( t \geq 0 \) and \( x \in X \), there exists \( w \in V \) with \( tu, tv \leq w \) (pointwise) and \( w(x) > 0 \). Let \( C(X, E) \) be the vector space of all continuous \( E \)-valued functions on \( X \), and \( C_b(X, E) \) (resp. \( C_p(X, E) \), \( C_0(X, E) \)) the subspace of \( C(X, E) \) consisting of those functions which are bounded (resp. have precompact range, vanish at infinity). Further, let

\[
CV_0(X, E) = \{ f \in C(X, E) : vf(X) \text{ is bounded in } E \text{ for all } v \in V \}, \\
CV_p(X, E) = \{ f \in C(X, E) : vf(X) \text{ is precompact in } E \text{ for all } v \in V \}, \\
CV_b(X, E) = \{ f \in C(X, E) : vf \text{ vanishes at infinity on } X \text{ for all } v \in V \}
\]

Clearly, \( CV_0(X, E) \subset CV_p(X, E) \subset CV_b(X, E) \). The first inclusion is due to the upper semicontinuity of the weights. When \( E = \mathbb{K} \), the above spaces are denoted by \( C(X) \), \( C_b(X) \), \( C_p(X) \), \( C_0(X) \), \( CV_b(X) \), and \( CV_0(X) \). If \( \varphi \in C(X) \) and \( a \in E \), then \( \varphi \otimes a \) is the function in \( C(X, E) \) defined by \( (\varphi \otimes a)(x) = \varphi(x)a; x \in X \). The \textit{weighted topology} \( w_V \) \cite{11,4} on \( CV_b(X, E) \) is defined as the linear topology which has a base of neighborhoods of 0 consisting of all sets of the form

\[
N(v, G) = \{ f \in CV_b(X, E) : vf(X) \subset G \}
\]

where \( v \in V \) and \( G \in W \). We mention that, in the non-locally convex setting, the weighted function spaces \( CV_0(X, E) \) and \( CV_b(X, E) \) have been studied by several authors in recent years for a variety of problems; see e.g. \cite{4,5,7,8,10,13}.

The following are some instances of weighted spaces.

1. If \( V = K^+(X) = \{ \lambda \chi_X : \lambda > 0 \} \), the set of all non-negative constant functions on \( X \), then \( CV_b(X, E) = C_b(X, E) \), \( CV_p(X, E) = C_p(X, E) \), \( CV_0(X, E) = C_0(X, E) \), and \( w_V \) is the \textit{uniform topology} \( \sigma \).

2. If \( V = S_0^+(X) \), the set of all non-negative upper semi-continuous functions on \( X \) which vanish at infinity, then \( CV_b(X, E) = CV_p(X, E) = CV_0(X, E) = C_b(X, E) \) and \( w_V \) is the \textit{strict topology} \( \beta_0 \).

3. If \( V = K_0^+(X) = \{ \lambda \chi_K : \lambda > 0 \text{ and } K \subset X, \text{ } K \text{ compact} \} \), then \( CV_b(X, E) = CV_p(X, E) = CV_0(X, E) = C(X, E) \) and \( w_V \) is the \textit{compact-open topology} \( k \).

4. If \( V = K_f^+(X) = \{ \lambda \chi_A : \lambda > 0 \text{ and } A \subset X, \text{ } A \text{ finite } \} \), then \( CV_b(X, E) = CV_p(X, E) = CV_0(X, E) = C(X, E) \) and \( w_V \) is the \textit{pointwise topology} \( p \).
Clearly $p \leq k$ on $C(X, E)$ and $k \leq \beta_0 \leq \sigma$ on $C_0(X, E)$. Moreover, the conditions on a Nachbin family $V$ imply that $p \leq w_V$.

Let $B(E, F)$ denote the algebra of all continuous linear mappings $T$ from a TVS $E$ into another $F$. For any collection $\mathcal{A}$ of subsets of $E$, $B_\mathcal{A}(E, F)$ denotes the subspace of $B(E, F)$ consisting of those $T$ which are bounded on the members of $\mathcal{A}$ together with the topology $\tau_\mathcal{A}$ of uniform convergence on the elements of $\mathcal{A}$. This topology has a base of neighborhoods of 0 consisting of all sets of the form

$$M(A, U) := \{T \in B(E, F) : T(A) \subset U\},$$

where $A \in \mathcal{A}$ and $U$ is a neighborhood of 0 in $F$.

Finally, we will say that a net $\{x_\alpha : \alpha \in I\}$ is a $V$-net if it is contained in $S_{v,1} := \{x \in X : v(x) \geq 1\}$ for some $v \in V$. Following Bierstedt [1], $X$ is said to be a $V_{\mathbb{R}}$-space if a function $f : X \to \mathbb{R}$ is continuous whenever, for each $v \in V$, the restriction of $f$ to $S_{v,1}$ is continuous. If $V = K(X)$, then $X$ is a $V_{\mathbb{R}}$-space means that $X$ is a $k_{\mathbb{R}}$-space.

2 Main Results

For any $x \in X$, let $\delta_x : CV_0(X, E) \to E$ denote the evaluation map $\delta_x(f) = f(x)$ at $x$. Clearly, $\delta_x \in B(CV_0(X, E), E)$. Next, define the evaluation map $\Delta : X \to B(CV_0(X, E), E)$ by $\Delta(x) = \delta_x$, $x \in X$. If the subscript $c$ in $B_c(CV_0(X, E), E)$ stands for the topology of uniform convergence on precompact subsets of $CV_0(X, E)$, then one has the following lemma given in [10], see also [1, 12]

**Lemma 2.1** The evaluation map $\Delta : X \to B_c(CV_0(X, E), E)$ is continuous if and only if every precompact subset of $CV_0(X, E)$ is equicontinuous. In particular, if $X$ is $V_{\mathbb{R}}$-space, then every precompact subset of $CV_0(X, E)$ is equicontinuous.

The following theorem extends Theorem 2.1 of [12] to the general setting of TVS’s.

**Theorem 2.2** Let $A$ be a subset of $CV_0(X, E)$. Then $A$ is precompact whenever the following conditions hold.

i. $A$ is equicontinuous.

ii. $A(x) = \{f(x) : f \in A\}$ is precompact in $E$ for each $x \in X$.

iii. $vA$ vanishes at infinity on $X$ for each $v \in V$. This is for each $v \in V$ and $G \in \mathcal{W}$, there exists a compact set $K \subset X$ such that $v(y)f(y) \in G$ for all $f \in A$ and $y \in X \setminus K$. If $X$ happens to be a $V_{\mathbb{R}}$-space, then the converse is also true.

**Proof.** Suppose i. – iii. hold. Since $CV_0(X, E) \subset C(X, E)$, by i., $A$ is an equicontinuous subset of $C(X, E)$. Further, since, by ii., $A$ is $p$-precompact, it follows from ([14], 109,
p. 289) that $A$ is a precompact subset of $(C(X, E), k)$. To show that $A$ is precompact in $CV_0(X, E)$, let $v \in V$ and $G \in W$. Choose a balanced $H \in W$ such that $H + H \subset G$. By $iii.$, there exists a compact $K \subset X$ such that
\[ v(y)f(y) \in H \text{ for all } f \in A \text{ and } y \in X \setminus K. \quad (1) \]

Since $v$ is upper-semicontinuous, $\|v\|_K = \sup\{v(y) : y \in K\} < \infty$. But $A$ is precompact in $(C(X, E), k)$. Then there exist $h_1, \ldots, h_n \in A$ such that
\[ A \subset \bigcup_{i=1}^n (h_i + N(\chi_K, (\|v\|_k + 1)^{-1}H)). \quad (2) \]

We claim that
\[ A \subset \bigcup_{i=1}^n \{h_i + N(v, G)\}. \]

Let $f \in A$ and $y \in X$. If $y \in K$, by (2), there exists $j \in \{1, \ldots, n\}$ such that
\[ v(y)(f(y) - h_j(y)) \in v(y)(\|v\|_k + 1)^{-1}H \subset H. \]

If $y \notin X \setminus K$, then, for any $i \in \{1, \ldots, n\}$, (1) gives
\[ v(y)(f(y) - h_i(y)) = v(y)f(y) - v(y)h_i(y) \in H - H \subset G. \]

This establishes our claim, and so $A$ is precompact in $CV_0(X, E)$. Now, Suppose $X$ is a $V_{\mathcal{K}}$-space and that $A$ is a precompact subset of $CV_0(X, E)$. Let us verify $i. - iii.$ Since $X$ is a $V_{\mathcal{K}}$-space, by (2.1), $A$ is equicontinuous whereby $i$. On the other hand, since $p \leq w_V$, $H$ is $p$-precompact. Hence, for each $x \in X$, $A(x)$ is precompact in $E$. Whence $ii.$ Finally, let $v \in V$ and $G \in W$ and choose a balanced $H \in W$ with $H + H \subset G$. Since $A$ is precompact, there exist $h_1, \ldots, h_n \in A$ such that
\[ A \subset \bigcup_{i=1}^n (h_i + N(v, H)). \quad (3) \]

Put $K = \bigcup_{i=1}^n \{y \in X : v(y)h_i(y) \notin H\}$. Since $h_i \in CV_0(X, E)$ for each $i$, $K$ is compact. Now, let $f \in A$ and $y \in X \setminus K$. By (3), there exists $i \in \{1, \ldots, n\}$ such that $f \in h_i + N(v, H)$. Hence
\[ v(y)f(y) = v(y)(f(y) - h_i(y)) + v(y)h_i(y) \in H + H \subset G. \]

Thus $vA$ vanishes at infinity on $X$.

The following result is an extension of Theorem 2.2 of [12]. Following W.M. Ruess and W.H. Summers, we shall set, for any $A \subset CV_p(X, E)$, $v \in V$ and $G \in W$,
\[ T_x(A, v, G) = \{y \in X : v(y)f(y) - v(x)f(x) \in G \text{ for all } f \in A\}, \quad x \in X. \]
Theorem 2.3 Consider the following assertions:

a. (i) $A$ is equicontinuous;
   (ii) $A(x)$ is precompact in $E$ for each $x \in X$,
   (iii) given $v \in V$ and $G \in W$, there exists a compact set $K \subset X$ such that $\{T_x(A, v, G) : x \in K\}$ covers $X$.

b. (i) $vA(X) = \{v(x)f(x) : x \in X, f \in A\}$ is precompact in $E$ for each $v \in V$;
   (ii) given $v \in V$ and $G \in W$, there exists a finite set $F \subset X$ such that $\{T_x(A, v, G) : x \in F\}$ covers $X$.

c. (i) $A(x)$ is precompact in $E$ for each $x \in X$;
   (ii) given $v \in V$ and $G \in W$, there exists a finite set $F \subset X$ such that $\{T_x(A, v, G) : x \in F\}$ covers $X$.

d. $A$ is precompact.

Then $a. \implies b. \implies c. \implies d.$ If $X$ is in addition a $V_{\text{IR}}$-space, then also $c. \implies d.$

Proof. $a. \implies b.$ Suppose $a.$ holds. We first note that (i) and (ii) together imply, as in the proof of (2.2), that $A$ is a precompact subset of $(C(X, E), k)$. We now verify $b.$ (i) and (ii).

For $b.$ (i), let $v \in V, G \in W$ and choose a balanced $H \in W$ such that $H + H + H + H \subset G$. By $a.$ (iii), there exists a compact $K \subset X$ such that

$$X = \bigcup \{T_x(A, v, H) : x \in K\}. \quad (4)$$

Since $A$ is precompact in $(C(X, E), k)$, there exist $h_1, \ldots, h_n \in A$ such that

$$A \subset \bigcup_{i=1}^n (h_i + N(\chi_k, (\|v\|_k + 1)^{-1}H)). \quad (5)$$

Moreover, since each $vh_i(K)$ is precompact in $E$, there exist $\{x_{ij}\}_{j=1}^{n_i} \subset K$ such that

$$vh_i(K) \subset \bigcup_{j=1}^{n_i} (v(x_{ij})h(x_{ij}) + H). \quad (6)$$

Now, fix any $y \in X$ and $f \in A$. By (4), $y \in T_x(A, v, H)$ for some $x \in K$ and so

$$v(y)f(y) - v(x)f(x) \in H \text{ for all } f \in A. \quad (7)$$

By (5), there exists $i \in \{1, \ldots, n\}$ such that

$$(f - h_i)(K) \subset (\|v\|_k + 1)^{-1}H. \quad (8)$$

By (6), there exists $j \in \{1, \ldots, n\}$ such that

$$v(x)h_i(x) - v(x_i)h_i(x_{ij}) \in H. \quad (9)$$
By (7), (8) and (9)

\[ v(y)f(y) - v(x_{ij})h_i(x_{ij}) = (v(y)f(y) - v(x)f(x)) + v(x)(f(x) - h(x)) + (v(x)h_i(x) - v(x_{ij})h_i(x_{ij})) \]

\[ \in H + \frac{v(x)}{\|v\|_F + 1}H + H \subset G. \quad (10) \]

i.e. \( vA(X) \subset \bigcup_{i=1}^{n} \bigcup_{j=1}^{n} (v(x_{ij})h_i(x_{ij}) + G) \) and \( vA(X) \) is precompact in \( E \).

For \( b. (ii) \), let \( v \in V \) and \( G \in \mathcal{W} \), and suppose these be same as in the proof of \( b. (i) \).

Also, let \( h_i \) and \( x_{ij} \) be as above. Set \( F = \bigcup_{i=1}^{n} \{ x_{ij} : j = 1, \ldots, n_i \} \). Then, for any fixed \( y \in X \) and \( f \in A, (9) \) and (10) give

\[ v(y)f(y) - v(x_{ij})f(x_{ij}) = (v(y)f(y) - v(x_{ij})h_i(x_{ij})) + (v(x_{ij})h_i(x_{ij}) - v(x_{ij})f(x_{ij})) \]

\[ \in (H + H + H) + H \subset G; \]

that is, \( y \in T_x(A, v, G) \) with \( x \in F \). Hence \( \{ T_x(A, v, G) : x \in F \} \) covers \( X \).

The implication \( b. \implies c. \) is trivial.

\( c. \implies d. \) : Suppose that \( c. \) holds. Fix any \( v \in V \) and \( G \in \mathcal{W} \) and choose a balanced \( H \in \mathcal{W} \) with \( H + H + H \subset G \). By \( c. (ii) \) there exists a finite set \( F \subset X \) such that \( \{ T_x(A, v, H) : x \in F \} \) covers \( X \). By \( c. (i) \), \( A \) is \( p \)-precompact, and so there exists \( h_i, \ldots, h_n \in A \) such that

\[ A \subset \bigcup_{i=1}^{n} (h_i + N(\chi_F, (\|v\|_F + 1)^{-1}H)). \quad (11) \]

We claim that \( A \subset \bigcup_{i=1}^{n} (h_i + N(v, G)) \). Fix any \( f \in A \). By (11), there exists \( i \in \{1, \ldots, n\} \) such that

\( (f - h_i)(F) \subset (\|v\|_F + 1)^{-1}H) \).

Then, for any \( y \in X \), \( y \in T_x(A, v, H) \) for some \( x \in F \) and so

\[ v(y)f(y) - v(y)h_i(y) = (v(y)f(y) - v(x_{ij}h_i(y))) + (v(x_{ij}h_i(x_{ij}) - v(y)h_i(y)) \]

\[ \in H + \frac{v(x)}{\|v\|_F + 1}H - H \]

\[ \subset H + H + H \subset G. \]

This proves our claim; i.e., \( d. \) holds.

Now, assume that \( X \) is a \( V_{FR} \)-space and let us show that \( d. \implies a. \) Since \( A \) is a precompact subset of \( CV_p(X, E) \), just as in the proof of (2.2), \( a.(i) \) and \( a.(ii) \) follow. To prove \( a.(iii) \),
let $v \in V$ and $G \in \mathcal{W}$. Choose a balanced $H \in \mathcal{W}$ such that $H + H + H \subset G$. The precompactness of $A$ gives $h_1, \ldots, h_n \in A$ such that

$$A \subset \bigcup_{i=1}^{n} (h_i + N(v, H)). \quad (12)$$

Now, consider the function $h : X \to E^n$ defined by $h(x) = (h_1(x), h_2(x), \ldots, h_n(x))$. This is a continuous function such that $(vh)(X)$ is precompact, for it is contained in the product $\prod_{i=1}^{n} (vh_i)(X)$ which is precompact. Hence for the neighborhood $H^n$, there exists a finite subset $F$ of $X$ such that

$$(vh)(X) = \bigcup_{x \in F} ((vh)(x) + H^n). \quad (13)$$

This gives

$$X \subset \bigcup_{x \in F} T_x(\{h_1, \ldots, h_n\}, v, H). \quad (14)$$

We now show that $\{T_x(A, v, G) : x \in F\}$ covers $X$. Fix $y \in X$. By (14), $y \in T_x(\{h_i\}_{i=1}^{n}, v, H)$ for some $x \in F$ and so

$$v(y)h_i(y) - v(x)h_i(x) \in H \quad \forall \quad i = 1, \ldots, n. \quad (15)$$

Given $f \in A$. By (12), there exists $i \in \{1, \ldots, n\}$ such that $f - h_i \in N(v, H)$; this is

$$v(z)(f(z) - h_i(z)) \in H \quad \forall \quad z \in X. \quad (16)$$

So, by (15) and (16),

$$v(y)f(y) - v(x)f(x) = (v(y)f(y) - v(y)h_i(y)) + (v(y)h_i(y) - v(x)h_i(x)) + (v(x)h_i(x) - v(x)h(x))$$

$$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \in \quad H + H + H \subset G.$$

Hence $y \in T_x(A, v, G)$; i.e., $a.iii$) holds.

We mention that in the particular case of $V = S_0^+(X)$, (2.2) and (2.3) reduce to Theorem 3.6 of [3]. Further, the Corollaries 2.5.1(a), 2.5.2(a), 2.5.3, 2.5.4(a) of [12] remain valid in the above general setting and are stated as follows:

**Corollary 2.4 ([2], p. 81)** Let $X$ be a $k_{IR}$-space and $E$ a quasicomplete TVS. A subset $A$ of $(C(X, E), k)$ is relatively compact if and only if the following conditions hold.

(i) $A$ is equicontinuous on each compact subset of $X$,

(ii) $A(x)$ is relatively compact in $E$ for each $x \in X$. 

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Corollary 2.5 Let $X$ be a locally compact space and $E$ a quasicomplete TVS. A subset $A$ of $(C_0(X,E),u)$ is relatively compact if and only if the following conditions hold.

(i) $A$ is equicontinuous,

(ii) $A(x)$ is relatively compact in $E$ for each $x \in X$,

(iii) $A$ uniformly vanishes at infinity on $X$ (i.e., for any $G \in \mathcal{W}$, there exists a compact set $K \subset X$ such that $f(y) \in G$ for all $f \in A$ and $y \in X\setminus K$).

Corollary 2.6 ([3], Theorem 3.6) Let $X$ be a $k_{\mathbb{R}}$-space and $E$ a quasicomplete TVS. A subset $A$ of $(C_b(X,E),\beta)$ is relatively compact if and only if the following conditions hold.

(i) $A$ is equicontinuous on each compact subset of $X$

(ii) $A(x)$ is relatively compact in $E$ for every $x \in X$,

(iii) $A$ is uniformly bounded (i.e., $A(X)$ is bounded in $E$).

Corollary 2.7 Let $E$ be a quasicomplete TVS. A subset $A$ of $(C_p(X,E),u)$ is relatively compact if and only if the following conditions hold.

(i) $A(X)$ is relatively compact in $E$,

(ii) given $G \in \mathcal{W}$, there exists a finite open cover $\{K_i : i = 1, \ldots, n\}$ of $X$ such that, for any $i \in \{1, \ldots, n\}$ and $x, y \in K_i$,

$$f(x) - f(y) \in G, \quad \forall f \in A.$$

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References


