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About Gevrey-L^2-estimates of pseudo-differential operators associated to the Gevrey symbols

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Abstract

This paper deals with the Gevrey regularity of pseudo-differential operators in C^∞. We prove that a result of Taylor [12], remains true in the Gevrey case.

Key words: L^2-continuity, pseudo-differential operators

1 Introduction

Many authors have been interested in the generalisation of the fundamental theorems on the L^2 and H^s continuity of pseudo-differential operators in C^∞ and analytic classes. We can quote, Beals [1], Calderon and Vaillancourt [4], Coifman and Meyer [5], Hwang [8] and Rodino [11]. Boulkhemair [2] gave a survey of these results and improved several of them. To our knowledge, the Gevrey regularity of these operators is relatively slightly explored. Boutet de Monvel and Krée [3], Hazi [7] and Matsuzawa [10] have tackled it. The starting-point of this study is a result mentionned in Taylor [12]. More precisely, we have reconsidered it in the Gevrey case and see if it remains true. The answer is positive.

In the sequel, we will use the following conventions:

- $\mathbb{R}^n$ is the n-dimensional vector space in which every point x is defined by its n coordinates $x_1, x_2, ..., x_n$.
- Ω denote, unless expressed otherwise, an open set of $\mathbb{R}^n$.
- $x + y$ is the point of coordinates $x_1 + y_1, x_2 + y, ..., x_n + y_n$.
- dx refers to the element of hypercube $dx_1 dx_1 ... dx_n$
- The order of a system of integers $p = \{p_1, p_2, ..., p_n\}$ is $|p| = p_1 + p_2 + ... + p_n$
- $D^\alpha = i^{-|\alpha|} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$.

- $\Delta_x = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + ... + \frac{\partial^2}{\partial x_n^2}$.
- $p! = p_1!p_2!...p_n!$
- $\widehat{u}$ stands for the Fourier transform of u.
- A^* is the adjoint of the operator A.
- $\mathcal{E}(\mathbb{R}^n)$ is the space of indefinitely differentiable functions on $\mathbb{R}^n$.
- $\mathcal{D}(\Omega)$ is the space of indefinitely differentiable functions on $\mathbb{R}^n$, with compact support in Ω.

We set in ([7], [6]):

Definition 1 *Let s a real number greater than or equal to 1. A real function f in $C^\infty(\Omega)$ is said of Gevrey class with order s if, for any compact subset $K \subset \Omega$, there exists a constant $C > 0$ such that* $\forall \alpha \in \mathbb{N}^n \; \|D^\alpha f\| \leq C^{|\alpha|+1} (|\alpha|!)^s$.

Definition 2 *Let $m \in \mathbb{R}$ and ρ, δ two real numbers such that $0 \leq \delta < \rho \leq 1$. We say that a real function $a = a(x,\xi)$ in $C^\infty(\Omega \times \mathbb{R}^n)$, is a Gevrey symbol with order s of type (m, ρ, δ) on Ω if, for any compact subset $K \subset \Omega$, there exist positive constants C_0, C_1, B such that*

$$\sup_{(x,\xi)\in(\mathbb{R}^n\times\mathbb{R}^n)} \left|D_\xi^\alpha D_x^\beta a(x,\xi)\right| \leq C_0 C_1^{|\alpha+\beta|} (|\alpha|!)^s (|\beta|!)^s \left(1 + |\xi|^2\right)^{\frac{1}{2}(m-\rho|\alpha|+\delta|\beta|)} \tag{1}$$

for any $\xi \in \mathbb{R}^n$ with $|\xi| \geq B \; |\alpha|^s$ and any $\alpha, \beta \in \mathbb{N}^n$.

The vector space of such symbols, somtimes called usual or classical symbols, is refered to as ${}_{\rho,\delta}S^m_{(G,s)}(\Omega \times \mathbb{R}^n)$.

We are concerned with the class of symbols $(m, 1, 1)$. Let us to point out here that the function $a(x,\xi) = a$ is taken of Gevrey class with order s in x and ξ whereas, often in the literature (see in particular [3]), it is taken of Gevrey class with order s in x and analytic ($s = 1$) in ξ (which amounts to take $s = 1$ in the factor $(|\alpha|!)^s$).

The following theorem gives the asymptotic extension of a symbol.

Theorem 1 *Let a_j a symbol of ${}_{\rho,\delta}S^{m_j}_{(G,s)}(\Omega\times\mathbb{R}^n)$, where $(m_j)_j$ is a real sequence decreasing to $-\infty$. Then, there exists a symbol a of ${}_{\rho,\delta}S^{m_0}_{(G,s)}(\Omega \times \mathbb{R}^n)$ such that, for any $N > 0$, there holds*

$$a - \sum_0^{N-1} a_j \in_{\rho,\delta} S^{m_N}_{(G,s)}(\Omega \times \mathbb{R}^n).$$

We also write in this case $a \sim \sum_0^\infty a_j$.

A pseudo-differential operator of class s, $A = a(x, D)$, associated to a symbol a of the space ${}_{\rho,\delta}S^{m}_{(G,s)}(\Omega \times \mathbb{R}^n)$ is defined, relatively to the standard quantization, by the formula

$$a(x, D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} a(x, \xi)\widehat{u}(\xi)d\xi, \qquad u \in \mathcal{D}(\mathbb{R}^n).$$

We write $A = opa$ and say that A belongs to $Op_{\rho,\delta}S^{m}_{(G,s)}(\Omega \times \mathbb{R}^n)$.

The distribution-kernel T of $a(x, D)$ is defined by

$$T(x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\xi} a(x, \xi)d\xi.$$

2 Chronological recall of some results.

Among the considerable results devoted to the L^2 continuity of pseudo-differential operators in the case of $C^\infty-$quantizations, (see [2] in particular), we recall

Theorem 2 *$A = a(x, D)$ sends continuously $L^2(\mathbb{R}^n)$ in itself whenever*

$$\left\| D_x^\beta D_\xi^\alpha a(x, \xi) \right\| \leq C_{\alpha\beta} \tag{2}$$

for all multi-indices α, β such that $|\alpha|$, $|\beta| \leq 3n + 4$ ($C_{\alpha\beta}$ being a positive constant).

In addition, if we set

$$\|A\|_0 = \sup_{|\alpha,\beta| \leq 3n+4} C_{\alpha\beta}$$

where $C_{\alpha\beta}$ are given by (2), then

$$\|a(x, D)u\|_{L^2(\mathbb{R}^n)} \leq C \|A\|_0 \|u\|_{L^2(\mathbb{R}^n)}$$

where C is a positive constant depending only on n.

Theorem 3 *$a(x, D)$ defines a bounded operator on $L^2(\mathbb{R}^n)$ whenever*

$$D_x^\beta D_\xi^\alpha a(x, \xi) \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$$

for all multi-indices α, β such that $|\alpha|, |\beta| \leq \left[\frac{n}{2}\right]$ or $\alpha, \beta \in \{0, 1\}^n$. ($[u]$ denotes the integer part of the real u.)

In 1972, Calderón and Vaillancourt, [4], proved the following result :

Theorem 4 *$a(x, D)$ is bounded in $L^2(\mathbb{R}^n)$ if there exists δ such that $0 \leq \delta < 1$ and*

$$\left| D_x^\beta D_\xi^\alpha a(x, \xi) \right| \leq C |\xi|^{\delta(|\beta|-|\alpha|)}$$

for

$$|\alpha| \leq n + 2 \left[\frac{n}{2}\right] \quad \text{and} \quad |\beta| \leq 2N, \quad \text{with} \quad N \geq \frac{5n}{4(1-\delta)}.$$

In 1978, Coifman and Meyer, [5], improved this result:

Theorem 5 *an opa is bounded in $L^2(\mathbb{R}^n)$ if there exists δ such that $0 \le \delta < 1$ and*

$$\left|D_x^\beta D_\xi^\alpha a(x,\xi)\right| \le C \ \ |\xi|^{\delta(|\beta|-|\alpha|)} \quad \text{for } |\alpha|\,,|\beta| \le \left[\frac{n}{2}\right] + 1.$$

In 1987, Hwang, [8], proved that:

Theorem 6 *an opa is bounded in $L^2(\mathbb{R}^n)$ if there exists δ such that $0 \le \delta < 1$ and*

$$\left|D_x^\beta D_\xi^\alpha a(x,\xi)\right| \le C\,|\xi|^{\delta(|\beta|-|\alpha|)}$$

for $\alpha_j = 0$ or 1 and $\beta_j = 0$ or 1 if $n = 1$ and $\beta_j = 0, 1$ or 2 in general.

3 Our problem

In what follows, we will prove

Theorem 7 *Assume $a(x,\xi) \in_{\delta,\delta} S^m_{(G,s)}(\Omega \times \mathbb{R}^n)$, $m \le 0$, $0 \le \delta < 1$, and*

$$\sup_{x\in K}\left|D_x^\beta D_\xi^\alpha a(x,\xi)\right| \le C_0 C_1^{|\alpha+\beta|}\left(|\alpha|!\right)^s\left(|\beta|!\right)^s\left(1+|\xi|^2\right)^{m-\delta(|\alpha|-|\beta|)} \tag{3}$$

for any $\xi \in \mathbb{R}^n$ with $|\xi| \ge B|\alpha|^s$ and $|\alpha|, |\beta| \le 3n+4 = N$.
(B is the constant in the relation (1)).
Then, the operator $A = a(x,D)$ acts continuously from $L^2(\Omega)$ in itself.

Moreover, if

$$|A|_\delta = \sup_{|\alpha,\beta|\le N} C_0 C_1^{|\alpha+\beta|}\left(|\alpha|!\right)^s\left(|\beta|!\right)^s$$

we get

$$\|a(x,D)u\|_{L^2(\mathbb{R}^n)} \le C\,\|A\|_\delta\,\|u\|_{L^2(\mathbb{R}^n)}$$

where C is a positive constant depending only on δ.

Proof. It is sufficient to prove this theorem for $a \in_{\delta,\delta} S^0_{G^s}(\Omega \times \mathbb{R}^n)$. We make use of two results. The first of which is due to M. Cotlar and E. Stein, on sums of almost orthogonal operators.

Definition 3 (Almost orthogonal operators) *We will call a familly of continuous operators $\{A_i : i \in \mathbb{Z}\}$ almost orthogonal, if they satisfy the following conditions:*

$$\|A_i^* A_j\| \le a(i,j), \qquad \left\|A_i A_j^*\right\| \le b(i,j),$$

where $a(i,j)$ and $b(i,j)$ are non negative symmetric functions on $\mathbb{Z}\times\mathbb{Z}$ which satisfy

$$\|a\|_{\infty,1/2}^{1/2} = \sup_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} a^{1/2}(i,j) < \infty, \qquad \|b\|_{\infty,1/2}^{1/2} = \sup_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} b^{1/2}(i,j) < \infty.$$

Lemma 1 *Let $A_1, A_2, \dots, A_N$ be bounded operators from a Hilbert space H_1 to another H_2 such that*

$$\sum_k \sqrt{\|A_j^* A_k\|} \le M, \qquad \sum_k \sqrt{\|A_j A_k^*\|} \le M, \tag{4}$$

where M is a positive constant. Then, there folows that $\|\sum_k A_j\| \le M$.

Proof If $A = \sum A_j$, we have $\|A\|^2 = \|A^*A\|$, and more generaly, by the spectral theorem, $\|P\|^{2m} = \|(A^*A)^m\|$. We expand in a sum and use the fact that

$$\begin{aligned}&\|A_{j_1}^* A_{j_2} ... A_{j_{2m-1}}^* A_{j_{2m}}\| \le \\ &\quad \min\left(\|A_{j_1}^* A_{j_2}\| ... \|A_{j_{2m-1}}^* A_{j_{2m}}\|, \|A_{j_1}^*\| \|A_{j_2} A_{j_3}^*\| ... \|A_{j_{2m-2}} A_{j_{2m-1}}^*\| \|A_{j_{2m}}\|\right)\end{aligned}$$

Taking the geometric mean of the two estimates and noting that $\|A_j\| \le M$. y hypothesis, we obtain

$$\|A\|^{2m} \le M \sum \sqrt{\|A_{j_1}^* A_{j_2}\|} \sqrt{\|A_{j_2} A_{j_3}^*\|} ... \sqrt{\|A_{j_{2m-1}}^* A_{j_{2m}}\|}$$

The sum is taken over j_1, j_2, ..., j_{2m}. If we use (4) to estimate successivley the sum over j_{2m}, j_2, ..., j_2,then only the sum over j_1 is left over and we see that

$$\|A\|^{2m} = NM^{2m}.$$

Taking $2m$-th roots and, letting m tends to ∞, we get $\|A\| \le M$, as expected. □

We also need the following

Lemma 2 *Let X be a measurable space. Assume $K(x, y)$ to be a kernel-distribution satisfying*

$$\int_X \|K(x, y)\| dy \le C_0, \qquad \int_X \|K(x, y)\| dx \le C_1,$$

with C_0 and C_1 being two positive constants.

Then $Pu(x) = \int K(x, y)u(y)dy$ defines a continuous operator on $L^2(X)$; moreover

$$\|P\| \le \sqrt{C_0 C_1}$$

Proof We have

$$\begin{aligned}\|\langle Pu, v\rangle\| &\le \int \|K(x, y)\| \|u(x)\| \|v(y)\| dxdy \\ &= \int \left(\sqrt{\|K(x, y)\|} \|u(x)\|\right) \left(\sqrt{\|K(x, y)\|} \|v(y)\|\right) dxdy \\ &\le \sqrt{\int \|K(x, y)\| \|u(x)\|^2 dydx} \sqrt{\int \|K(x, y)\| \|v(y)\|^2 dxdy} \\ &\le \sqrt{C_0} \|u\|_{L^2} \sqrt{C1} \|v\|_{L^2},\end{aligned}$$

and the claim follows. □

Let us turn back to the proof of our main theorem. We shall put the operator A under the form of a sum of quasi-orthogonal operators $A = \sum A_j$. To this end, we shoose a partition of the unity φ_j on $[0,\infty[$, $-1,0,1,2,\ldots$,such that φ_{-1} has support in $[0,1[$, φ_j has support in $]2^{j-1},2^{j+1}[$, $j\geq 0$, and that

$$\varphi_j(t)=1, \quad \text{if} \quad \left|t-2^j\right|\leq \frac{1}{4}2^j; \quad j\geq 0, \qquad \text{and} \qquad \varphi_j^{(k)}(t)\leq C_k 2^{jk}, \quad j\geq 0.$$

Such partition exists. Set

$$a_j(x,\xi)=\varphi_j\left(C\left(1+|\xi|^2\right)^{\frac{\delta}{2}}\right)a(x,\xi),$$

for a certain constant $C>0$.

We aim to apply lemma 1. Firstly, we estimate the norm of the operator $a_j(x,D)$. On the support of $a_j(x,\xi)$, we have

$$2^{j-1}\leq C\left(1+|\xi|^2\right)^{\frac{\delta}{2}}\leq 2^j.$$

As a consequence, Eq. (3) yields

$$\sup_{x\in K}\|D_\xi^\alpha D_x^\beta a_j(x,\xi)\|\leq C_0\tilde{C}_1^{\|\alpha+\beta\|}\left(\|\alpha\|!\right)^s\left(\|\beta\|!\right)^s\left(1+\|\xi\|^2\right)^{j(\|\beta\|-\|\alpha\|)}, \tag{5}$$

where $\tilde{C}$ is a positive constant depending on j. Now, let us consider U_j the unit operator on $L^2(\mathbb{R}^n)$, defined by

$$U_j\psi(x)=2^{\frac{nj}{2}}\psi(2^jx).$$

There follows that $B_j=U_j^*A_jU_j$ is a pseudo-differential operator of Gevrey symbol type $b_j(x,\xi)=a_j(2^{-j}x,2^j\xi)$, of class s, and (2.3) implies

$$\sup_{x\in K}\|D_\xi^\alpha D_x^\beta b_j(x,\xi)\|\leq C_0\tilde{C}_1^{\|\alpha+\beta\|}\left(\|\alpha\|!\right)^s\left(\|\beta\|!\right)^s \tag{6}$$

Theorem 2 yields $\|A_j\|\leq CH$, where

$$H=\sup_{|\alpha|,|\beta|\leq N} C_0\tilde{C}_1^{|\alpha+\beta|}\left(|\alpha|!\right)^s\left(|\beta|!\right)^s.$$

Now, we give estimates of the norms of the operators $A_k^*A_j$ and $A_jA_k^*$, with $\|k-j\|\geq 4$. In each case, the symbols A_j and A_k have disjoint supports, and $A_k^*A_j$ and $A_jA_k^*$ admit regular kernels. Hence, we may expect to obtain convenient bounds for their norms by elementary tools.

For $k-j\geq 4$, if $a_k(x,\eta)\ a_j(y,\xi)\neq 0$, then

$$\left(1+\|\eta\|^2\right)^{\frac{\delta}{2}}\sim 2^k \quad \text{and} \quad \left(1+\|\xi\|^2\right)^{\frac{\delta}{2}}\sim 2^j$$

and simultaneously, this implies

$$\|\xi-\eta\|\geq C(2^j+2^k)^{1+\gamma}\left(1+\|\xi-\eta\|^2\right)^{\frac{\gamma}{2}}, \quad \text{with} \quad \gamma=\frac{1-\delta}{\delta(1+\delta)} \tag{7}$$

Now, $A_k^* A_j u(x) = \int F(x,y)u(y)dy$, where

$$F(x,y) = \int \overline{a_k(x,\xi)} a_j(z,\eta) e^{i(x\xi - z\xi + z\eta - y\eta)} dz d\xi d\eta.$$

An integration by parts gives

$$F(x,y) = \int b_L(x,y,z,\xi,\eta) e^{i(x\xi - z\xi + z\eta - y\eta)} dz d\xi d\eta, \tag{8}$$

with

$$b_L(x,y,z,\xi,\eta) = \left(1 + \|x-z\|^2\right)^{-L} \left(1 + \|z-y\|^2\right)^{-L}$$
$$(1-\Delta_\xi)^N (1-\Delta_\eta)^L \|\xi - \eta\|^{2L} (-\Delta_z)^L \overline{a_k(z,\xi)} a_j(z,\eta).$$

Then

$$\|b_L(x,y,z,\xi,\eta)\| \le C \left[\left(1+\|x-z\|^2\right)^{-\frac{L}{2}} \left(1+\|z-y\|^2\right)^{-\frac{L}{2}} \|\xi-\eta\|^{-1}(2^j + 2^k) \right]^{2L}. \tag{9}$$

Hence, in Supp (b_L), if $\|k-j\| \ge 4$, the relation (7) is plausible. Substituting into (9), yields

$$\|b_L(x,y,z,\xi,\eta)\| \le C \left[\left(1+\|x-z\|^2\right)^{-\frac{L}{2}} \left(1+\|z-y\|^2\right)^{-\frac{L}{2}} \left(1+\|\xi-\eta\|^2\right)^{-\frac{\gamma}{2}} (2^j+2^k)^{-\gamma} \right]^{2L}. \tag{10}$$

If

$$L > \max\left(\frac{n}{2}, \frac{3}{2\gamma}, \frac{n}{2\gamma}\right),$$

we may make an integration in (10), and with (8), we deduce

$$\|F(x,y)\| \le C \left(1+\|x-y\|^2\right)^{-L-\frac{\gamma}{2}} (2^j+2^k)^{-\gamma},$$

from which there folows

$$\|A_k^* A_j\| \le C(2^j+2^k)^{-\gamma}$$

provided $\|k-j\| \ge 4$. Now, for $\|k-j\| \le 4$, we have

$$\|A_k^* A_j\| \le \|A_k^*\| \, \|A_j\| \le H^2.$$

Then, in all cases, we obtain

$$\|A_k^* A_j\| \le C 2^{-\gamma\|j-k\|}. \tag{11}$$

Estimating $A_j A_k^*$ is easier. Indeed, $\widehat{A_j A_k^*} u(\xi) = \int \chi(\xi,\eta)\widehat{u}(\eta)d\eta$, where

$$\chi(x,\eta) = \int a_j(x,\zeta)\overline{a_k(y,\zeta)} e^{i(-x\xi + x\zeta - y\zeta + y\eta)} dx d\zeta dy.$$

Now, $\|k-j\| \ge 4$ implies $\chi(x,\eta) = 0$, then $A_j A_k^* = 0$ for $\|k-j\| \ge 4$. When $\|k-j\| \le 4$, we make use the inequality

$$\|A_k^* A_j\| \le \|A_k^*\| \, \|A_j\| \le H^2$$

to get

$$\|A_j A_k^*\| \leq C 2^{-\gamma\|j-k\|}. \tag{12}$$

Combining (11) and (12) together with Cotlar-Knapp-Stein lemma, we deduce that the operator $A = a(x, D) = \sum A_j$ is bounded in $L^2(\mathbb{R}^n)$.

The second statement of the theorem is straightforward. □

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