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Normal cones and strictly real algebra structure

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Abstract

The notion of an s-normal cone, introduced here, allows different characterizations of strictly real Banach algebras. The normality of H_+, the cone of positive elements, makes of the algebra a function algebra. The nuclearity of H_+ implies finite dimensionality.

Key words: Real Banach algebras, convex cone, s-normal cone, normal cone, nuclear cone

1 Introduction

Characterizations of strictly real algebras have been examined in [5]. The specificity of our approach lies in the use of properties of normal cones. We first show that a real Banach algebra with a convex cone which is s-normal (see Definition 2.1) and contains all squares is necessarily strictly real. As a consequence, a real Banach algebra is strictly real if, and only if, the cone K_0 of finite sums of squares is s-normal (Corollary 2.5). Other characterizations are obtained. The normality is stronger than the s-normality. As a matter of fact a real Banach algebra H is isomorphic to an algebra $\mathcal{C}(X, R)$, of continuous functions on a compact space X if, and only if, its cone H_+ of positive elements is normal (Theorem 3.1). In the commutative case, we have the same result with the cone K_0. The strength of nuclearity of a convex cone is illustrated here by the fact that it implies finite dimensionality (Theorem 4.1). Different consequences are obtained.

Let $(E, \|.\|)$ be a normed space. A subspace K, of E, is said to be a convex cone if $x + y \in K$ and $\alpha x \in K$ for every x, y in K and $\alpha \geq 0$; it is said to be salient if $K \cap (-K) = \{0\}$. A partial order, on E, is associated to K by $x \leq y$ if $y - x \in K$. A convex cone is said to be normal (respectively nuclear) if there is $\alpha > 0$ (respectively a continuous real linear form f on E) such that $\|x\| \leq \alpha \|y\|$, whenever $0 \leq x \leq y$ (respectively $\|x\| \leq f(x)$, for every $x \in K$). Let now $(H, \|.\|)$ be a real Banach algebra and designate by $H_C = H + iH$ the complexification of H. Recall that, by definition, $Sp_H x = Sp_{H_\mathbf{C}} x$, for every $x \in H$, where Spx stands for the spectrum. In the sequel, we denote by ρ the spectral radius given by $\rho(x) = \sup \{|\lambda| : \lambda \in Sp_H x\}$. Also, $\mathcal{C}(X, R)$ stands for the algebra of real continuous functions on the compact space X with the usual operations and the norm defined by $\|f\| = \sup \{|f(t)| : t \in X\}$.

2 Spectrally normal cones and strictly real structure

The convex cone of positive elements, in a strictly real algebra, is not always normal as it is shown by the algebra $\mathcal{C}^1([0,1], R)$, of class C^1 functions, with the usual operations and the norm defined by $\|f\|_1 = \|f\|_\infty + \|f'\|_\infty$. However one has $\rho(f) \leq \rho(g)$ whenever $0 \leq f \leq g$. This fact suggests the folowing definition which allows a characterization of strictly real algebras among real ones.

Definition 2.1. Let A be an algebra and K a convex cone in A. We say that K is spectrally normal (s-normal in short) if there is an $\alpha > 0$ such that $\rho(x) \leq \alpha\rho(y)$, whenever $0 \leq x \leq y$.

Proposition 2.2. Let $(H, \|.\|)$ be a strictly real algebra and K a convex cone in H which is stable by product. If K is normal, then it is s-normal.

Proof. K being normal, let $\alpha > 0$ such that $\|x\| \leq \alpha \|y\|$, if $0 \leq x \leq y$. But K is stable by product, hence it follows, by induction on n, that $0 \leq x^{2^n} \leq y^{2^n}$, for every $n \in N^*$, whenever $0 \leq x \leq y$. So $0 \leq x \leq y$ implies $\left\|x^{2^n}\right\|^{\frac{1}{2^n}} \leq \alpha^{\frac{1}{2^n}} \left\|y^{2^n}\right\|^{\frac{1}{2^n}}$, for every $n \in N^*$. Whence the conclusion by convergence.

Let $(H, \|.\|)$ be a real Banach algebra with a convex cone K. We consider the following conditions which are satisfied by the cone of positive elements in a strictly real algebra.

(P_1) $h^2 \in K$, for every $h \in H$.

(P_2) The cone K is s-normal.

The following result is somehow a converse of this fact.

Theorem 2.3. Let $(H, \|.\|)$ be a unitary real Banach algebra and K a (non void) convex cone in H. If K satisfies (P_1) and (P_2), then $(H, \|.\|)$ is strictly real.

Proof. Let H_C, the complexification of H, be endowed with the involutive anti-morphism $* : h+ik \longmapsto h-ik$, for h, k in H. By (P_2), there is $\alpha > 0$ such that $\rho(u+v) \geq \alpha\rho(u)$, for every u and v in K. If $x = h + ik \in H_C$, with $hk = kh$, then $xx^* = h^2 + k^2$ with h^2, k^2 in K. One, then, shows that there is $\beta > 0$ such that $\rho(xx^*) \geq \beta\rho^2(x)$. Writing this for x^n and using the normality of x, we obtain $\rho^2(x) \geq \rho(xx^*) \geq \beta^{\frac{1}{n}}\rho^2(x)$. Whence $\rho(xx^*) = \rho^2(x)$, for every normal element x in H_C. Now let $h \in H$. If $\alpha + i\beta \in Sp_H h$ with α and β real, put $a = h + it$ with t a real number. Then a is a normal element, of H_C, such that $\alpha + i(\beta + t) \in Sp_{H_C} a$. Since $aa^* = h^2 + t^2$, it follows that

$$\alpha^2 + (\beta + t)^2 \leq \rho(a)^2 = \rho(aa^*) \leq \rho(h)^2 + t^2.$$

Whence $\alpha^2 + \beta^2 + 2\beta t \leq \rho(h)^2$, for each $t \in R$. This implies that $\beta = 0$.

Remark 2.4. None of conditions (P_1) or (P_2) alone is sufficient. Indeed, in any real algebra, the cone $K_0 = \left\{\sum_{\text{finite}} h^2,\ h \in H\right\}$ satisfies (P_1); and in any commutative real algebra the cone $H_+ = \{x \in H : Sp_H x \subset R^+\}$ satisfies (P_2). In fact we have the following.

Corollary 2.5. The algebra $(H, \|.\|)$ is strictly real if, and only if, the cone K_0 is s-normal.

Corollary 2.6. The algebra $(H, \|.\|)$ is strictly real if, and only if, $\overline{K_0} = H_+$; where $\overline{K_0}$ is the closure of K_0.

Now, it is worth to establish a link between s-normality and real strictness for a convex cone, in general.

Proposition 2.7. Let $(H, \|.\|)$ be a unitary real Banach algebra and K a closed convex cone, in H, satisfying (P_1). The following assertions are equivalent.

i) K is s-normal and stable by product.

ii) $\rho(y) \geq \rho(x)$, whenever $y \geq x \geq 0$.

iii) H is strictly real and $K = H_+$.

Proof. i)$\Longrightarrow$ii) Since K is stable by product, one has $y^{2^n} \geq x^{2^n} \geq 0$, for every $n \in N^*$, if $y \geq x \geq 0$. But, K being s-normal, there is $\alpha > 0$ such that $\rho(x) \geq \alpha^{\frac{1}{2^n}}\rho(y)$, for every n in N^*, if $y \geq x \geq 0$. Whence ii) by convergence.

ii)$\Longrightarrow$iii) By theorem 2.3, the algebra H is strictly real. By (P_1), we obtain $H_+ \subset K$. Now, if $x \in K$ and $r > 0$ are such that $-r \in Sp_H x$, then $\rho(x) \geq \rho(x) + r$ which is absurd; the inequality follows from ii) for $\rho(x)$, $\rho(x) - x$ and $\rho(x) - (\rho(x) - x)$ are in K.

iii)$\Longrightarrow$i) By theorem 4.8 of [7].

Combining (P_1) et (P_2), one obtains characterizations of real strictness.

Theorem 2.8. Let $(H, \|.\|)$ be a unitary real Banach algebra. The following assertions are equivalent.

i) $(H, \|.\|)$ is strictly real.

ii) $\rho(h^2 + k^2) \geq \rho(h^2)$; $h, k \in H$.

iii) There is $\alpha > 0$ such that $\rho(h^2 + k^2) \geq \alpha\rho(h^2)$; $h, k \in H$.

iv) There is $\alpha > 0$ such that $\rho(h^2 + k^2) \geq \alpha\rho(h^2)$; $h, k \in H$ with $hk = kh$.

In [5], L. Ingelstam defines, in non unitary Banach algebras, a modified exponential function by $e^x = -\sum_{n\geq 1} \frac{x^n}{n!}$; and gives a sufficient, but not necessary, condition for a real Banach algebra to be strictly real. Here are some conditions which are necessary and sufficient.

Proposition 2.9. Let $(H, \|.\|)$ be real Banach algebra. The following assertions are equivalent.

i) H is strictly real.

ii) $\rho(e^{-\alpha h^2}) \leq 1$; $h \in H$, $\alpha > 0$.

iii) $(\forall h \in H)$ $(\exists \beta > 0) : \rho(e^{-\alpha h^2}) \leq \beta$; $\alpha > 0$.

In the unitary case, i), ii) and iii) are also equivalent to

iv) $\rho(h^2 - k^2) \leq \max(\rho(h^2), \rho(k^2))$; $h, k \in H$.

v) $\exists \alpha > 0 : \rho(h^2 - k^2) \leq \alpha \max(\rho(h^2), \rho(k^2))$; $h, k \in H$.

3 Normal cones and function algebra structure

The cone of positive elements H_+ in a strictly real algebra H is not normal in general (cf section 2). It turns out that this condition is a strong one as the following result shows.

Theorem 3.1. A unitary and strictly real Banach algebra $(H, \|.\|)$ is isomorphic to $\mathcal{C}(X, R)$ if, and only if, H_+ is normal.

Proof. Only sufficiency has to be shown. Since H is strictly real, one has $0 \leq h(\rho(h) + \frac{1}{n})^{-1} + e \leq 2e$, for any $h \in H$ and any $n \in N^*$. So, H_+ being normal, there is $\alpha > 0$ such that $\rho(h) \geq \alpha \|h\|$, for every $h \in H$; hence H is semi-simple. But then, it is commutative by theorem 4.8 of [7]. And so, its complexification $H_{\mathbf{C}}$, endowed with the involution $* : h + ik \longmapsto h - ik$, is a hermitian Banach algebra such that $\rho(h) \geq \alpha \|h\|$, for every $h \in H$. Hence, H_C is a C^*-algebra for an equivalent norm, by theorem 8.4 of [12]. We conclude by the well known Gelfand-Naïmark theorem.

Proposition 3.2. Let $(H, \|.\|)$ be a unitary real Banach algebra and K a convex cone, in H, closed and stable by product. If K is normal and satisfies (P_1), then H is isomorphic to an algebra $\mathcal{C}(X, R)$.

Proof. cf. propositions 2.2 and 2.7.

Corollary 3.3. Let H be a unitary and commutative real Banach algebra. It is isomorphic to an algebra $\mathcal{C}(X, R)$ if, and only if, the cone K_0 is normal.

Proof. It is sufficient to notice that the closure of K_0 is also normal and apply proposition 3.2.

Remark 3.4. In the three previous results, one can not expect the isomorphism between H and $\mathcal{C}(X, R)$ to be an isometry. Indeed, the normality of a convex cone does not depend on the norm defining the topology.

Now, we reconsider theorem 2.4 of [8]; it appears, in particular, that commutativity is implicitely contained in hypotheses.

Proposition 3.5. Let $(H, \|.\|)$ be a unitary real Banach algebra. The following assertions are equivalent.

i) H is isomorphic to an algebra $\mathcal{C}(X, R)$.

ii) $(\exists \alpha > 0), (\exists \beta > 0) : \|h^2\| \geq \alpha \|h\|^2$ and $\|h^2 + k^2\| \geq \beta \|h^2\|$; $h, k \in H$.

iii) H is strictly real and H_+ is normal.

Proof. All what we have to show is ii)$\Longrightarrow$iii). Since $\|h^2\| \geq \alpha \|h\|^2$, for every h in H, one obtains, by iteration, and convergence that $\rho(h) \geq \alpha \|h\|$, for every h in H. Now $\rho(h^2 + k^2) \geq \alpha \|h^2 + k^2\| \geq \alpha\beta\rho(h^2)$, for every h and k in H. Hence H is strictly real, by theorem 2.9. On the other hand, for every $u \in H_+$, there is $v \in H_+$ such that $u = v^2$ ([9], theorem 2.2). Hence the relation $\|h^2 + k^2\| \geq \beta \|h^2\|$ is nothing else than the normality of H_+.

Corollary 3.6. Let $(H, \|.\|)$ be a unitary real Banach algebra. If $\|h^2 + k^2\| \geq \|h^2\|$, for every h and k in H. Then H is isometrically isomorphic to $\mathcal{C}(X, R)$.

4. Nuclear cones and finite dimensionality. The nuclearity of a convex cone, introduced in [6], is stronger than its normality. This fact is illustrated here by the following result.

Theorem 4.1. Let $(H, \|.\|)$ be a unitary strictly real algebra. If the cone H_+ is nuclear, then H is of finite dimension.

Proof. The cone H_+, being nuclear, is normal and hence H is isomorphic to an algebra $\mathcal{C}(X, R)$. Now, by theorem 3.4.3 of [11], the algebra H is nuclear; and hence finite dimensional for it is a normed space.

Corollary 4.2. Let H be a unitary strictly real algebra. If there is, on H, a real scalar product $\langle ., . \rangle$ such that $\|x\| \leq \langle x, e \rangle$, for every $x \in H_+$, then H is of finite dimension.

Finally, here, are some particular cases of normality and nuclearity which characterize the algebra R of real numbers.

Proposition 4.3. Let $(H, \|.\|)$ be a unitary strictly real algebra of unit e such that $\|e\| = 1$. The following assertions are equivalent.

i) H is isometrically isomorphic to R.

ii) There is a real scalar product $\langle ., . \rangle$, on H, such that $\langle e, e \rangle = 1$ and $\|u\| \leq \langle u, e \rangle$, for every u in H_+.

iii) There is a real linear form f, on H, such that $f(e) = 1$ and $\|u\| \leq f(u)$, for every u in H_+.

iv) There is a real linear form f, on H, such that $f(e) = 1$ and $\|u\| = f(u)$, for every u in H_+.

v) $\|v\| > \|u\|$, whenever $v \geq u \geq 0$ and $u \neq v$.

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