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An extension of the bilateral generating functions of modified Hypergeometric polynomial

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Abstract

In this note we have obtained an extension of a bilateral generating function of modified Hypergeometric polynomial from the existence of a quasi bilateral generating relation by group-theoretic methods.

1 Introduction

In [1], the "quasi bilinear" generating function for the special function $p_n^{(\alpha)}(x)$ is given by

$$G(x,u,t) = \sum_{n=0}^{\infty} a_n p_n^{(\alpha)}(x) p_m^{(n)}(u) t^n. \tag{1}$$

In this note we proceed to prove the existence of a more general generating relation from the existence of a quasi bilinear generating relation involving certain special function.

In [2], the following theorem on bilateral generating function involving Hypergeometric Polynomial has been obtained by group-theoretic method.

Theorem 1 *If*

$$G(x,w) = \sum_{n=0}^{\infty} a_n \, {}_2F_1(-n,\beta;v;x) w^n,$$

then

$$(1-w)^{(v-1)}(1-xw)^{-\beta} G\left(\frac{x(1-w)}{1-xw},\, wv\right) = \sum_{n=0}^{\infty} w^n \sum_{q=0}^{n} a_q \frac{-v+1}{(n-q)!} \, {}_2F_1(-n,\beta;v-n;x) v^q.$$

Here we have obtained the following extension of the above result from the concept of quasi-bilinear generating function by group-theoretic method.

Theorem 2 *If there exists a quasi bilinear generating relation of the form*

$$G(x,u,w)=\sum_{n=0}^{\infty} a_n\, {}_2F_1(-n,\beta;v;x)\, {}_2F_1(-m,\beta;n;u)w^n,$$

then

$$\begin{aligned}
&(1-w)^{\beta-m}(1+w)^{(v-1)}(1+xw)^{-\beta}G\left(\frac{x(1+w)}{1+xw},\, u+(1-u)w\,,\, \frac{wv}{1-w}\right)\\
&=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!q!} v^q(-1)^q(-v+1)_q \times \qquad (2)\\
&\quad\times \frac{(n+m)_p(n-\beta)_p}{(n)_p}\, {}_2F_1(-(n+q),\beta;v-q;x)\, {}_2F_1(-m,\beta;n+p;u).
\end{aligned}$$

Proof

At first, we consider the following two linear partial differential operators R_1 and R_2, [3, 4]:

$$R_1 = x(1-x)y^{-1}z\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - (x\beta+1)y^{-1}z,$$

$$R_2 = (1-u)t\frac{\partial}{\partial u} + t^2\frac{\partial}{\partial t} + (m-\beta)t,$$

such that

$$R_1({}_2F_1(-n,\beta;v;x)y^v z^n) = (v-1){}_2F_1(-(n+1),\beta;v-1;x)y^{v-1}z^{n+1},$$

$$R_2({}_2F_1(-m,\beta;n;u)t^n) = \frac{(n+m(n-\beta))}{n}{}_2F_1(-m,\beta;n+1;u)t^{n+1},$$

and

$$\exp(wR_1)f(x,y,z) = (1+zw/y)^{-1}(1+zw/y)^{-\beta} f\left(x\frac{(1+zw/y)}{(1+xzw/y)}, y(1+zw/y), z\right),$$
$$\exp(wR_2)f(u,t) = (1-wt)^{\beta-m} f\left(u+(1-utw),\, \frac{t}{(1-tw)}\right).$$

Let

$$G(x,u,w)=\sum_{n=0}^{\infty} a_n\, {}_2F_1(-n,\beta;v;x)\, {}_2F_1(-m,\beta;n;u)w^n. \qquad (3)$$

Now replacing w by $wztv$ and then multiplying both sides of (3) by y^v, we get

$$y^v G(x,u,w)=\sum_{n=0}^{\infty} a_n\, {}_2F_1(-n,\beta;v;x)\, y^v z^n\, {}_2F_1(-m,\beta;n;u)t^n(wv)^n. \qquad (4)$$

On operating both sides of (4) by $\exp(wR_1)\exp(wR_2)$, we get

$$\begin{aligned}&\exp(wR_1)\exp(wR_1)(y^v G(x,u,wztv)) = \\ &= \exp(wR_1)\exp(wR_1)\sum_{n=0}^{\infty} a_n \left({}_2F_1(-n,\beta;v;x)\, y^v z^n\right) {}_2F_1(-m,\beta;n;u)t^n(wv)^n.\end{aligned} \tag{5}$$

The left member of (5) becomes

$$(1+zw/y)^{-1+v}(1+xzw/y)^{-\beta}(1-wt)^{\beta-m}y^v G\left(x\frac{(1+zw/y)}{(1+xzw/y)}, u+(1-u)tw, \frac{wztv}{1-tw}\right). \tag{6}$$

On the other hand, the right member of (5) becomes

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!q!} v^n(-1)^q(-v+1)_q \frac{(n+m)_p(n-\beta)_p}{(n)_p} \\ &\quad \times {}_2F_1(-(n+q),\beta;v-q;x)y^{v-q}z^{n+q}\, {}_2F_1(-m,\beta;n+p;u)t^{n+p}.\end{aligned} \tag{7}$$

Now, equation (6) and (7) then using $y=z=t=1$, we get Equation (2), which is our desired result.

Particular Case: If we put $m=0$, we notice from our theorem that $G(x,u,w)$ becomes $G(x,w)$ for ${}_2F_1(-m,\beta;n;u)$ at $m=0$ becomes 1. Hence, from our theorem, we obtain

$$\begin{aligned}&(1-w)^{\beta}(1+w)^{(v-1)}(1+xw)^{-\beta}G\left(\frac{x(1+w)}{1+xw}, \frac{wv}{1-w}\right) \\ &\sum_{n=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+q}}{q!} v^n(-1)^q(-v+1)_q\, {}_2F_1(-(n+q),\beta;v-q;x)\sum_{p=0}^{\infty}\frac{(n-\beta_n)}{p!}w^p \\ &= \sum_{n=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{(-w)^{n+q}}{q!}(-v+1)_q\, {}_2F_1(-(n+q),\beta;v-q;x)\left(\frac{-v}{1-w}\right)^n(1-w)^{\beta}.\end{aligned}$$

At first we replace $(-v/(1-w))$ by v and then w by $(-w)$, we get

$$\begin{aligned}&(1-w)^{(v-1)}(1-xw)^{-\beta}G\left(\frac{x(1-w)}{1-xw}, wv\right) \\ &= \sum_{n=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{(w)^{n+q}}{q!}(-v+1)_q\, {}_2F_1(-(n+q),\beta;v-q;x)v^n \\ &= \sum_{n=0}^{\infty} w^n \sum_{q=0}^{\infty} a_q \frac{(-v+1)_{n-q}}{(n-q)!}\, {}_2F_1(-n,\beta;v-n+q;x)v^q,\end{aligned}$$

which is Theorem 1.

References

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