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Trajectories of zeros of Sobolev-Gegenbauer orthogonal polynomials

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Abstract

In this paper we study the distribution of the zeros of a particular family of Sobolev-Gegenbauer polynomials. We show graphically that their behaviour is much more complicated that in the standard case when the parameter λ of the Gegenbauer family is changed.

1 Introduction

The zeros of orthogonal polynomials play an important role in interpolation theory, quadrature formulas, spectral theory of some lineal operators, digital filter design, ... hence their study has a great interest. In this paper we analyse the distribution of the zeros of a particular family of Sobolev-Gegenbauer polynomials when we change the parameter λ of this family.

First, let us introduce the usual inner product of polynomials. Given a positive and finite Borel measure μ with support S_μ in $\mathbb{R}$, the following formula

$$(p,\, q) = \int_{S_\mu} p\, q\, d\mu, \qquad p,\, q \in \mathcal{P},$$

defines an inner product in the space of the polynomials with real coefficients ($\mathcal{P}$), which has an orthogonal polynomial sequence (unique up to a multiplicative factor) associated.

The zeros of classical orthogonal polynomials have good properties. They are placed in the interior of the convex envelope of S_μ (hence they are reals), they are simple and the zeros of two consecutive polynomials are interlaced [8].

Our purpose is to describe the trajectories of the zeros of Sobolev orthogonal polynomials corresponding to inner products defined by

$$(p,\, q)_S = \int_I p\, q\, d\mu + \sum_{i=0}^{m} M_i\, p^{(i)}(c)\, q^{(i)}(c), \tag{1}$$

where μ is a finite positive Borel measure supported in an interval I of $\mathbb{R}$, $c \notin I^0$ (the interior of I), $m \geq 1$, $M_i \geq 0$ for $i = 0, \ldots, m-1$ and $M_m > 0$. Let us denote by Q_n the orthogonal polynomials with respect to the inner product (1).

Now, let us review several results about the distribution of the zeros of Q_n (for more details see [1, 2, 7]).

Theorem 1 *The polynomials Q_n have at least $n-(m+1)$ changes of sign or zeros of odd multiplicity in I^0 whenever $n \geq m+1$.*

Theorem 2 *The number of zeros of Q_n in I^0 does not depend on the order of the differentials on the product* (1)*, it depends on the terms of the discrete part of the scalar product.*

Now, let us consider the inner product (1), but with $M_i \in \mathbb{R} \setminus \{0\}$. Let $\mathbb{Z}_+$ be the set of the positive integers and let Q_n be the n-th monic polynomial of less degree, not equal to zero, such that $(p,\, Q_n)_S = 0$, $\forall p \in \mathcal{P}_{n-1}$. For the polynomials Q_n, which are orthogonal with respect to the Sobolev inner product considered, we have the following result about the zero location:

Theorem 3 *For each $n \in \mathbb{Z}_+$, Q_n has at least $n - \overline{n}$ sign changes in the interior of the convex envelope of I, where $\overline{n}$ denotes the number of terms in the discrete part of the considered product whose differential order is less than n.*

As consequence of the last theorem and of the Theorem 4 in [7], we have that for n sufficiently big and the measures μ for which exist the called relative asymptotic of the (Q_n) sequence, the orthogonal polynomials Q_n with respect to (1) with $M_i \in \mathbb{R} \setminus \{0\}$, $i = 1, \ldots, m$, have exactly $n-m$ simple zeros in the interior of I and the m remaining zeros are attracted to the c point.

In this paper we study the pattern of the trajectories of the zeros of the orthogonal polynomial of degree $n = 8$ corresponding to the following inner product:

$$(p,\, q) = \int_{-1}^{1} p\, q\, d\mu + p'(2)\, q'(2) \tag{2}$$

where $d\mu = (1-x^2)^{\lambda-\frac{1}{2}}\,dx$. Note that although we have taken $c = 2$ any case $c > 1$ will give similar results and the case $c < -1$ will give symmetric results. Let us denote by $\widetilde{C}_n^{\lambda}$ its corresponding Sobolev orthogonal polynomial of degree n. In addition to this, we compare those trajectories with the trajectories of the zeros of the classical Gegenbauer polynomial C_8^{λ} of 8-degree, that is, the orthogonal polynomial of 8-degree with respect to the inner product:

$$(p,\, q) = \int_{-1}^{1} p(x)\, q(x)\, (1-x^2)^{\lambda-\frac{1}{2}}\, dx. \tag{3}$$

When we want to evaluate a family of monic classical orthogonal polynomials we use the third-order recurrence relation that they verify [8]:

$$\begin{aligned} &p_0(x) = 1, \qquad p_1(x) = x - \beta_0, \\ &p_n(x) + (\beta_{n-1} - x)\, p_{n-1}(x) + \gamma_{n-1}\, p_{n-2}(x) = 0, \qquad n \geq 2, \end{aligned} \tag{4}$$

where $\beta_i, \gamma_i \in \mathbb{R}$. In the particular case of the Gegenbauer polynomials (with $\lambda \neq 0$) the coefficients are given by $\beta_n = 0$ and

$$\gamma_n = \frac{n\,(n+2\lambda-1)}{(2n+2\lambda)\,(2n+2\lambda-2)}, \qquad n \geq 1.$$

If we attempt to compute Sobolev orthogonal polynomials (for $\lambda > -1/2$) with the classical Gram-Schmidt algorithm then we will only be able to obtain the orthogonal polynomials of low degree (till 3 or 4) due to the great difficulty in performing in an accurate way the quadratures of the inner product. Instead of, Barrio *et al.* proposed in [3] an alternative way that do not involve the use of such a quadrature:

Proposition 4 *Let* $\{q_0(x),\, q_1(x), \ldots\}$ *be the sequence of monic Sobolev-Gegenbauer orthogonal polynomials with respect to the scalar product*

$$\langle\, p,\, q\,\rangle_W = \int_I p(x)\, q(x)\, (1-x^2)^{\lambda-\frac{1}{2}}\, \mathrm{d}x + p'(c)\, q'(c), \qquad \lambda > -\frac{1}{2},\ \lambda \neq 0$$

and $\{p_0(x),\, p_1(x), \ldots\}$ *the classical monic Gegenbauer orthogonal polynomials (with the same parameter* λ*), then*

$$q_l(x) = p_l(x) - \sum_{s=0}^{l-1} A_{ls}\, q_s(x), \tag{5}$$

where

$$A_{ls} = \frac{p_l'(c)\, q_s'(c)}{\prod_{i=0}^{s} \gamma_i + p_s'(c)\, q_s'(c)} \tag{6}$$

being $\gamma_0 = \sqrt{\pi}\,\Gamma(\frac{1}{2}+\lambda)/\Gamma(1+\lambda)$ *and* $\gamma_n = n\,(n+2\lambda-1)/\left((2n+2\lambda)\,(2n+2\lambda-2)\right)$.

Note that the above result holds for $\lambda > -1/2$ and $\lambda \neq 0$ but formulas (5) and (6) may also be used, although in this case the inner product is not well defined, in the case

$\lambda < -1/2$ with $\lambda \neq -n/2$, $(n \in \mathbb{N})$ if we use in the generation of the classical Gegenbauer polynomials the third-order recurrence relation (4). Therefore, we use the proposition above in order to extend the Gegenbauer-Sobolev orthogonal polynomials to the case $\lambda < -1/2$.

2 Zeros of the polynomials $\widetilde{C}_8^\lambda$

It is known [8] that, for $\lambda > -1/2$, all the zeros of C_n^λ lie on $(-1,\ 1)$. In a series of papers [4, 5], Driver and Duren show that for $\lambda < 1-n$ all the zeros of C_n^λ lie on the imaginary axis. Moreover, for arbitrary real λ the zeros of C_n^λ are symmetric with respect to both the real and imaginary axis. Besides, in [6] they analyse with detail the trajectories of zeros of C_8^λ when $\lambda \in (1-n,\ -1/2)$.

In this section we study and compare numerically the trajectories, depending on the value of the parameter λ, of the zeros of the orthogonal polynomials C_8^λ and $\widetilde{C}_8^\lambda$ corresponding to the inner products (3) and (2), respectively, when $\lambda > -1/2$ and using the recurrence relations (4) and (5) for $\lambda < -1/2$. Besides, we compare the numerical results with the theoretical theorems presented in the previous section. You can see the evolution of the roots of the polynomials C_8^λ and $\widetilde{C}_8^\lambda$ in the Figures 1, 2 and 3. See [6] for more information about the zeros of C_8^λ.

For $\lambda > -1/2$ each zero of C_8^λ is real and each one belongs to the interval $(-1, 1)$ as we can see in the graphic corresponding to Gegenbauer for $-1/2 < \lambda < 17/2$. On the other hand, for $\lambda > -1/2$, each zero of the polynomial $\widetilde{C}_8^\lambda$ is real and all of them belong to the interval $(-1, 1)$ except one that is placed to the right of that interval and is attracted by the mass point $c = 2$. This situation can be seen in the graphic corresponding to Sobolev-Gegenbauer for $-1/2 < \lambda < 17/2$ of Figure 1. Note that this zero distribution is deduced theoretically from Theorems 1, 2 and 3. Besides, when the degree n increases, the zero placed to the right of the interval $(-1, 1)$ will tend to the point $c = 2$.

As soon as λ starts to decrease from $-1/2$ the zeros of C_8^λ and $\widetilde{C}_8^\lambda$ start to spread from the interval $(-1, 1)$ onto the real axis. When λ decreases from $-1/2$ to $-3/2$ for both C_8^λ and $\widetilde{C}_8^\lambda$, two zeros emerge from the interval $(-1, 1)$, one from each opposite end of the interval, but keeping in both cases (C_8^λ and $\widetilde{C}_8^\lambda$) the two zeros on the real axis (see Gegenbauer and Sobolev-Gegenbauer for $-3/2 < \lambda < -1/2$ in Figure 1). Hence, in the Gegenbauer case C_8^λ, for $-3/2 < \lambda - 1/2$, we have that its eight zeros are on the real axis, six of them belonging to the interval $(-1, 1)$ and the other two depart from each one of the opposite ends of that interval. In the Sobolev case, $\widetilde{C}_8^\lambda$, for $-3/2 < \lambda < -1/2$ we also have that its eight roots are on the real axis, five of them are placed on the interval $(-1, 1)$, another one is localised on the right of the mass point $c = 2$ and the two remaining zeros emerge from each opposite end of the interval.

When λ decreases from $-3/2$ to $-5/2$, in both cases, C_8^λ and $\widetilde{C}_8^\lambda$, two zeros emerge from each opposite end of the interval $(-1, 1)$, into the upper and lower vertical directions $\pm i$, in paths symmetric with respect to the real axis, and in the case of C_8^λ the paths are also symmetric with respect to the imaginary axis. In the case of $\widetilde{C}_8^\lambda$ the symmetry with respect to the imaginary axis is broken due to the non-symmetric discrete part of its corresponding inner product. The remaining four zeros of C_8^λ are on the real axis, specifically they are placed on the interval $(-1, 1)$, while the remaining four zeros of $\widetilde{C}_8^\lambda$ are on the real axis, three of them are in the interval $(-1, 1)$ and the remaining zero is localised on the right of the mass point $c = 2$. This phenomena can be seen in the graphics corresponding to Gegenbauer and Sobolev-Gegenbauer for $-5/2 < \lambda < -3/2$ of Figure 1. Note that we also present a magnification of a small interval around -1 in order to see with more detail the bifurcation of the complex roots (that we have called "the ear bifurcation").

When $-7/2 < \lambda < -5/2$ the trajectories of the zeros of C_8^λ and $\widetilde{C}_8^\lambda$ change. In the case of C_8^λ, when λ decreases from $-5/2$ to $-7/2$, two zeros depart from each opposite end of the interval $(-1, 1)$ into the vertical directions $\pm i$ and as soon as λ tend to $-7/2$ that zeros return to the same endpoints performing loops. The four remaining zeros of C_8^λ are on the real axis, two of them are placed in the interval $(-1, 1)$ while the other two zeros emerge onto the real axis, one from each end of the interval $(-1, 1)$. That evolution of the trajectories of the zeros of C_8^λ can be seen in the Gegenbauer graphic for $-7/2 < \lambda < -5/2$ of Figure 1. In the case of $\widetilde{C}_8^\lambda$, when λ start decreasing from $-5/2$, two zeros depart from each end of the interval $(-1, 1)$ into the vertical directions $\pm i$ but, in contrast to C_8^λ, these zeros do not return to the same endpoints when λ tends to $-7/2$. The four remaining zeros are on the real axis, one of them belongs to the interval $(-1, 1)$, another one is localised to the right of the mass point $c = 2$ and the other two zeros emerge from the opposite ends of that interval onto the real axis. While λ decreases from $-5/2$ to a certain value $\lambda_0 \in (-7/2, -5/2)$ the zero that departs from the interval $(-1, 1)$ by its right end tends to 2, and the zero that is placed on the right of that interval decreases to 2. And eventually, when λ_0 both zeros coincide. When λ decreases from λ_0 to $-7/2$ both zeros depart from the real axis (in $x = 2$) in opposite vertical directions $\pm i$. That evolution of the zeros of $\widetilde{C}_8^\lambda$ can be seen in the Sobolev-Gegenbauer graphics for $\lambda_0 < \lambda < -5/2$ and $-7/2 < \lambda < \lambda_0$ of Figure 1.

As we can see in Figure 2 the trajectories of the zeros of $\widetilde{C}_8^\lambda$ when $-8 < \lambda < -7/2$ have a highly complicated behaviour in contrast with the non-Sobolev case (on the left). In

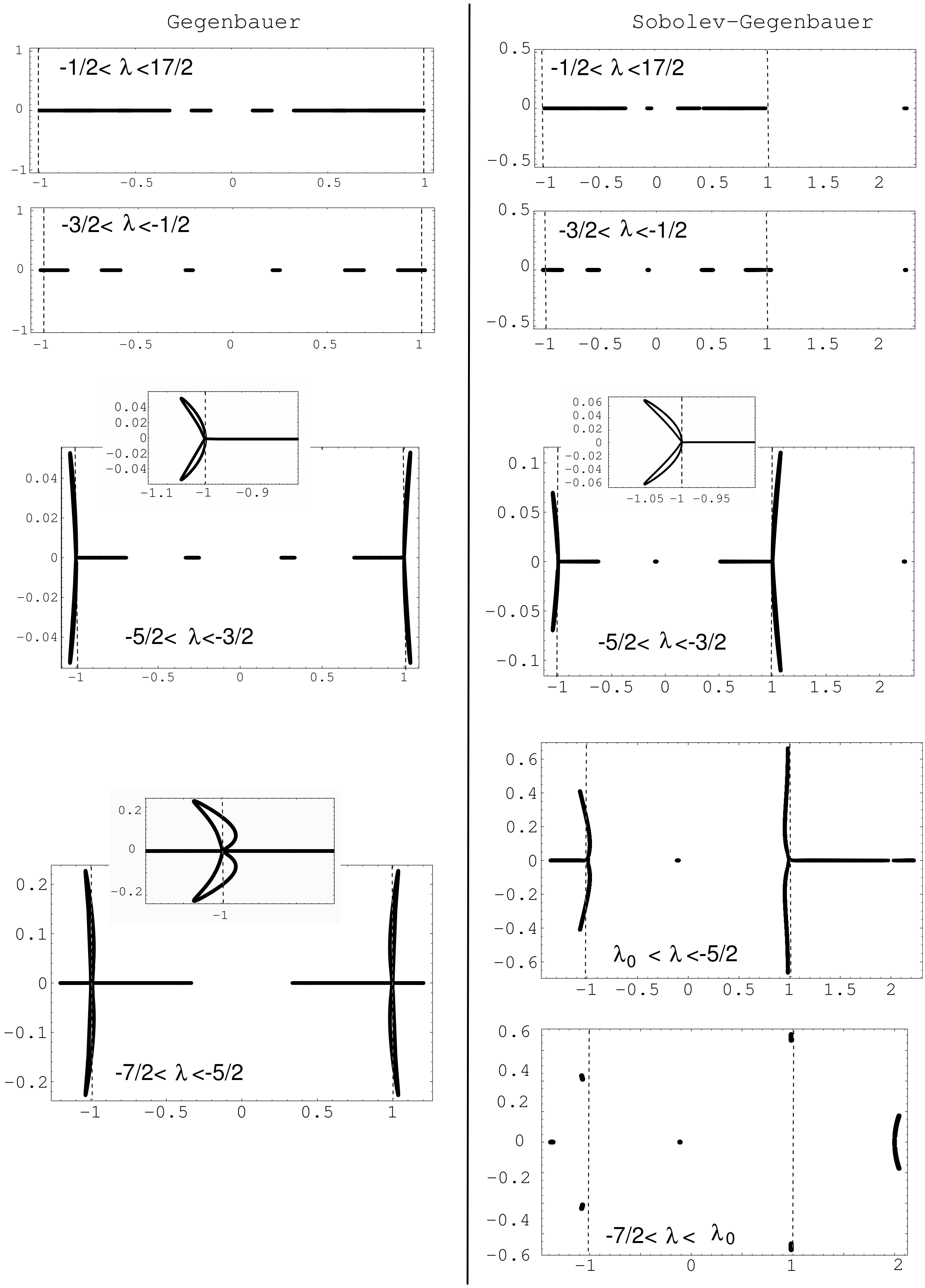

Figure 1: Behaviour of the zeros of classical Gegenbauer and Sobolev-Gegenbauer for $\lambda \in (-7/2,\, 17/2)$.

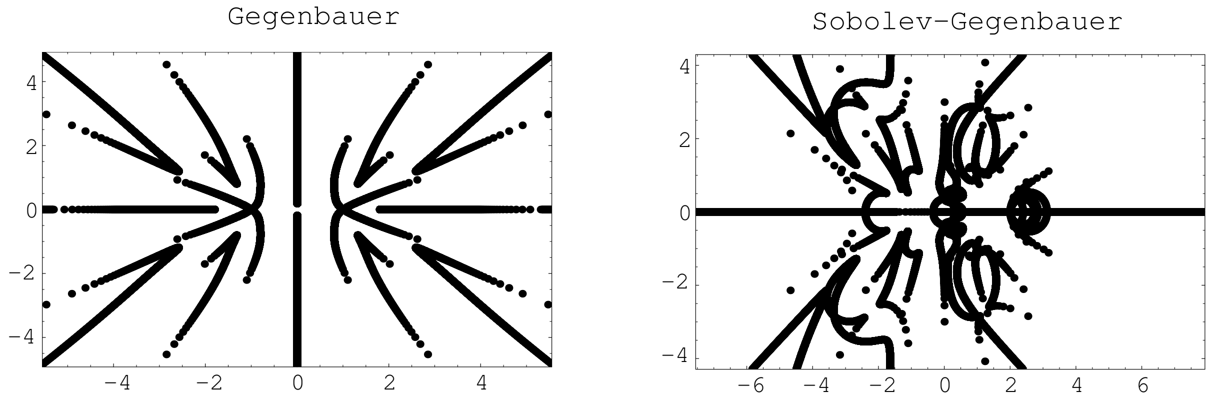

Figure 2: Behaviour of the zeros of classical Gegenbauer and Sobolev-Gegenbauer for $\lambda \in (-8, -7/2)$.

the classical Gegenbauer polynomials we can observe several patterns on the trajectories of the zeros, making possible a detailed analysis of them (see [6]). Therefore, we note that the discrete part of the inner product in the Sobolev case makes the behaviour of the zeros of the orthogonal polynomials much more complicated.

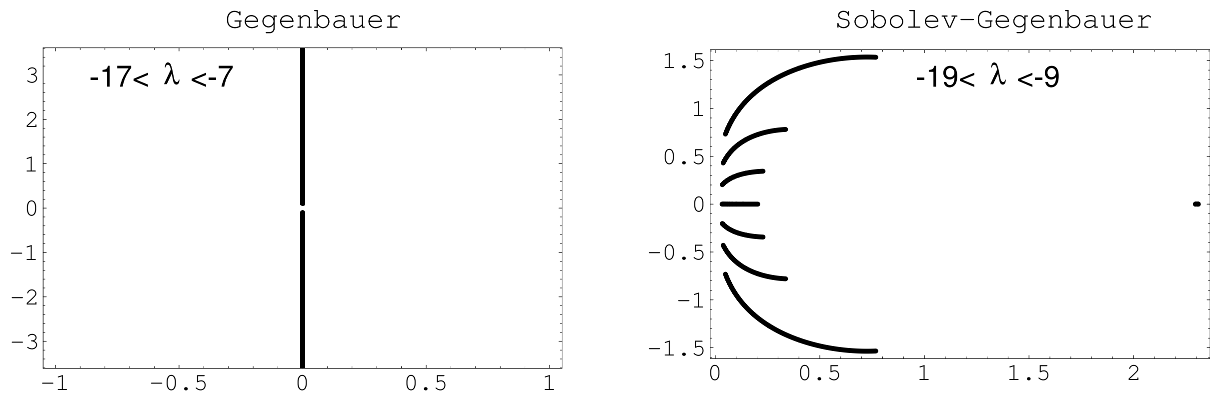

Figure 3: Asymptotic behaviour of classical Gegenbauer and Sobolev-Gegenbauer orthogonal polynomials.

However, this unstable behaviour is not indefinitely hold, when λ continues decreasing that behaviour changes into a stable one, again analogously to the case of the Gegenbauer polynomial C_8^λ when λ decreases from -7 to $-\infty$. In the case of the polynomial $\widetilde{C}_8^\lambda$ two zeros are real and the rest of zeros are complex, one of the real zeros is placed on the right the mass point $c = 2$ while the another real zero tends to 0 when λ tends to $-\infty$, and the six complex zeros also tend to 0 when λ tends to $-\infty$. In the case of C_8^λ its eight zeros lie on the imaginary axis tending to 0 when λ tends to $-\infty$. Those behaviours can be seen in Figure 3.

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