Rev. Real Academia de Ciencias. Zaragoza. **59:** 95–100, (2004).

Symbols of pseudodifferential operators associated to Gevrey kernel's type

Mohammed Hazi

École Normale Supérieure, 16050 - Kouba, Algiers, Algeria
e-mail: mohamedhazi@hotmail.com

Abstract

In this article, we aim at proving the correctness of the inverse theorem (1) mentioned in [5]. More precisely, we associated symbols of Gevrey type to pseudodifferential operators when the latter are given by their kernels.

Key words: symbols of Gevrey, kernels, pseudodifferential operators

A.M.S classification subject: 47 G 30.

1 Introduction

In [4], we gave a description of pseudodifferential operators when defined by their Hörmander's symbols or their kernels (cfr. [2] and [3]); and this in the C^∞-case. It appeared that the two approches are equivalent. In [1], it is shown that this equivalence remains also true in the analytic case. This paper is a continuation of [5], where we obtained that the definition by symbols implies the one by kernels. Here, we deal with the converse. And this, in the more fine context of Gevrey classes.

We recall three definitions as they were given in [5].

Definition 1.1. Let n be a non zero positive integer, Ω an open subset of $\mathbb{R}^n$ and s any real number larger or equal to 1. A real function φ in $C^\infty(\Omega)$ is said of Gevrey class with order s if, for any compact subset $K \subset \Omega$, there exists constant a $C > 0$ such that

$$\forall\ \alpha \in \mathbb{N}^n\ \ \|D^\alpha f\| \leq C^{|\alpha|+1} \left(|\alpha|!\right)^s . \tag{1}$$

Definition 1.2. Given $n \in \mathbb{N}^*$, $m \in \mathbb{R}$, $s \geq 1$ and Ω an open subset of $\mathbb{R}^n$. We say that a real function $a = a(z, x)$ in $C^\infty(\Omega \times \mathbb{R}^n)$, is a symbol (or amplitude) of Gevrey type with class s on Ω if, and only if, it satisfies that for any compact subset $K \subset \Omega$, there exist C_0, C_1, R positive constants such that

$$\left|D_\xi^\alpha D_z^\beta a(z,\xi)\right| \leq C_0 C_1^{|\alpha+\beta|} \left(|\alpha|!\right)^s \left(|\beta|!\right)^s \left(1+|\xi|^2\right)^{m-|\alpha|} \tag{2}$$

for any $z \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$ with , $|\xi| \geq R$, α and $\beta \in \mathbb{N}^n$.

We denote by $\mathcal{S}_{G^s}^m(\Omega \times \mathbb{R}^n)$ the set of such symbols.

Notice that, in general, a is supposed analytic in z. (This is done by taking $s = 1$ in the factor $(|\beta|!)^s$corresponding to the variable z.)

Definition 1.3. We keep the notation of definition (2). Moreover, let U be an open neighborhood of o in $\mathbb{R}^n$and m a positive real number. We say that a distribution $T = T(z, x)$, on $\Omega \times U$, is a Gevrey kernel of order m if, and only if, the following assertions are satisfied:

a) The restriction f of T to $(U\setminus\{0\})$ is Gevrey of order s such that, for every compact K of Ω, there is an open neighborhood V of $U\setminus\{0\}$and a scalar $C > 0$ such that

$$\left|D_x^\alpha D_z^\beta T(z,x)\right| \leq C^{|\alpha+\beta|+1} \left(|\alpha|!\right)^s \left(|\beta|!\right)^s \left(1+|x|^2\right)^{-m-n-|\alpha|} \tag{3}$$

for every (z, x) in $K \times V$.

b) The distribution T is of the form

$$T(z,.) = P_{f_\theta}(z,.) + \sum_{|\alpha|\leq m} C_\alpha(z)\delta^{(\alpha)}, \tag{4}$$

where $(C_\alpha)_\alpha$ is a family of Gevrey functions of order s in Ω and θ is a map of $C_0^\infty(U)$, verifying $\theta \equiv 1$ in a neighborhood of zero, while P_{f_θ} is a distribution given for any ψ in $C_0^\infty(U)$ by

$$< P_{f_\theta}, \psi >= \int f(x) \left(\psi(x) - \sum_{|\alpha|\leq m} \frac{D^\alpha \psi(0)}{\alpha!} x^\alpha \theta(x)\right) dx. \tag{5}$$

c) If m is a positive integer, then, for any compact K in Ω, there is a scalar $C > 0$ such that, for every α in $\mathbb{N}^n$ with $|\alpha| = m$ and any $\varepsilon > 0$

$$\left|\int_{|x|\geq\varepsilon, x\in U} x^\alpha f(z,x)dx\right| \leq C, \tag{6}$$

and this, for every z in K.

The set of Gevrey kernels, thus defined, is designated by $\mathcal{K}_{G^s}^m(\Omega \times \mathbb{R}^n)$.

2 The main result

We prove the following result:

Theorem 1 *If T is in $\mathcal{K}^m_{G^s}(\Omega \times \mathbb{R}^n)$, then there is a symbol f in $\mathcal{S}^m_{G^s}(\Omega \times \mathbb{R}^n)$ such that $T = \mathcal{F}^{-1} f$ is of Gevrey type of order s on $\Omega \times U'$, where U' is an open 0 neighborhood in $\mathbb{R}^n$.*

Proof. Recall first that $\mathcal{F}^{-1} f$ is the image of f by the inverse of Fourier transform.

We will need the following lemma which is an analogous of the one in [1].

Lemma 1 *Let m be a positive real number, U and U' open neighborhoods of zero, in $\mathbb{R}^n$, such that $U' \subset U$.*

i) Let T be a distribution on U, the restriction g of which, to $U \backslash \{0\}$, is C^∞ and satisfies

$$\left| D^\beta_x g(x) \right| \leq C \left| x \right|^{m-n-|\beta|}, \quad |\beta| \leq m. \tag{7}$$

ii) We suppose $x^\alpha T$ that is an integrable on U, for $|\alpha| < m$.

iii) If m is a positive integer, we suppose the existence of a map φ in $C^\infty_0(U)$, identically equal to 1 on U' and such that, for $|\alpha| = m$,

$$\sup_{0<\varepsilon<1} |\langle x^\alpha T_\varepsilon(x), \varphi_\varepsilon(x)\rangle| \leq C, \tag{8}$$

with $\varphi_\varepsilon(x) = \varphi\left(\frac{x}{\varepsilon}\right)$.

Then, for every ψ in $C^\infty_0(U)$, there exists a scalar $M > 0$ such that

$$\left| D^\alpha_\xi (\psi T)(\xi) \right| \leq CM(1 + |\xi|)^{m-|\alpha|}, \tag{9}$$

and this for every α in $\mathbb{N}^n$ and every ξ in $\mathbb{R}^n$.

Proof of lemma 1.– Since α is in $\mathbb{N}^n$, we have, for $|\xi| \geq 1$,

$$\begin{aligned} D^\alpha_\xi (\widehat{\psi T})(\xi) &= (-1)^{|\alpha|} (\widehat{x^\alpha \psi T})(\xi) = (-1)^{|\alpha|} \langle x^\alpha \psi T, e^{ix\xi} \rangle \\ &= (-1)^{|\alpha|} \left(I_1 + I_2 + I_3 \right) \end{aligned}$$

where

$$\begin{aligned} &I_1 = \langle x^\alpha T, \varphi_\varepsilon \psi \rangle, \\ &I_2 = \langle x^\alpha T, \varphi_\varepsilon \psi \left(e^{-ix\xi} - 1 \right) \rangle, \\ &I_3 = \langle x^\alpha T, (1 - \varphi_\varepsilon)\, \psi e^{-ix\xi} \rangle = x^\alpha T\, \widehat{(1 - \varphi_\varepsilon)}\, \psi. \end{aligned}$$

Let us examine these expressions one by one. First, fix

$$\varepsilon = \frac{1}{|\xi|}. \tag{10}$$

Using (7) and (8) and (10), we get

$$|I_1| \le C\ M_1\ |\xi|^{m-|\alpha|}$$

with a constant $M_1 > 0$.

Concerning the estimation of I_2, the definition of φ assures that

$$\left|e^{-ix\xi} - 1\right| \le |\xi|\,|x|\,.$$

This inequality and (7) yield to

$$|I_2| \le C\ M_2\ |\xi|^{m-|\alpha|}\,,$$

with a constant $M_2 > 0$.

With respect to I_3, we can write

$$\xi_j I_3 = -1\langle D_{x_j}\left(x^\alpha\left(1-\varphi_\varepsilon\right)\psi T, e^{-ix\xi}\right)\rangle, \qquad \forall j = 1, \dots, n.$$

Since the support of $1-\varphi_\varepsilon$ is contained in $U\backslash U'$, we can find a scalar $\mu > 0$ such that

$$0 < \mu \le |\xi|\ \ |x|\,,$$

which permits to have

$$|\xi_j I_3| \le C\ M_3\ |\xi|^{m-|\alpha|+1}\,,$$

where M_3 is a strictly positive scalar.

With these three estimations we obtain (9) in the case $|\xi| \ge 1$ considered.

As (9) obviously remains true when $|\xi| < 1$, the proof is finished. ■

In order to finish the proof of theorem, we will also need the following proposition.

Proposition 1 *Let U be an open 0-neighborhood in $\mathbb{R}^n$, a given α in $\mathbb{N}^n$ and T an element of $\mathcal{K}^m_{G^s}(\Omega \times \mathbb{R}^n)$ such that its restriction f to $U\backslash\{0\}$, satisfies*

$$|\partial^\alpha f(x)| \le C^{|\alpha|+1}(|\alpha|!)\,|x|^{-m-n-|\alpha|}\,. \tag{11}$$

For any map φ in $C_0^\infty(U)$ which is identically equal to 1 on an open 0-neighborhood $U' \subset U$, we get

$$\left|D^\alpha(\widehat{\varphi T})(\xi)\right| \le C^{|\alpha|+1}(|\alpha|!)^s\,(1+|\xi|)^{m-|\alpha|}\,. \tag{12}$$

Proof of Proposition 1.– We define a function ψ on $C_0^\infty(U)$ as follows

$$\psi(x) = \begin{cases} 1 & \text{for} \quad |x| \leq 1, \\ 0 & \text{for} \quad |x| \geq 3. \end{cases} \tag{13}$$

For any β in $\mathbb{N}^n$ such that $|\beta| = |\alpha|$ we get

$$\left|\xi^\alpha D^\alpha(\widehat{\varphi T})(\xi)\right| \leq |I_1| + |I_2|,$$

where

$$|I_1| = \left|\int D_x^\alpha (x^\alpha \varphi T)\left[1 - \psi(|\xi|\, x)\right] e^{-ix\xi} dx\right|,$$

$$|I_2| = \left|\int D_x^\alpha (x^\alpha \varphi T)\, \psi(|\xi|\, x)\, e^{-ix\xi} dx\right|.$$

Let us estimate these expressions. We have

$$|I_2| \leq \int_{|x|<\frac{3}{|\xi|}} |D_x^\alpha (x^\alpha \varphi T)|\, dx. \tag{14}$$

The Leibniz formula and (11) permits to get

$$|I_2| \leq \int_{|x|<\frac{3}{|\xi|}} \left|\sum_{\upsilon\leq\beta} ()\beta\upsilon \partial^{\beta-\upsilon} T \partial^\upsilon (x^\alpha \varphi)\right| dx$$

$$\leq \int_{|x|<\frac{3}{|\xi|}} \sum_{\upsilon\leq\beta} ()\beta\upsilon \left|\partial^\upsilon (x^\alpha \varphi)\right| C^{|\beta-\upsilon|+1} (|\beta-\upsilon|!)^s\, |x|^{-n-m-|\beta-\nu|}\, dx.$$

Whence

$$|I_2| \leq \int_{|x|<\frac{3}{|\xi|}} A_2 C^{|\beta|+1} (|\beta|!)^s\, |x|^{-n-m}\, dx \leq B_2 C^{|\beta|+1} (|\beta|!)^s\, (1 + |\xi|)^m. \tag{15}$$

By proceeding in the same way for $|I_1|$, we obtain

$$|I_1| \leq \int_{\frac{1}{|\xi|}<|x|<\frac{3}{|\xi|}} A_1 C^{|\beta|+1} (|\beta|!)^s\, |x|^{-n-m}\, dx \leq B_1 C^{|\beta|+1} (|\beta|!)^s\, (1 + |\xi|)^m. \tag{16}$$

Notice that the scalars A_1 and A_2 depend on β while B_1 and B_2 depend on β, m and n. The constant C does not depend on β, nor n, nor m.

On the other hand, using (14) and (15), we easily obtain

$$\left|D^\alpha(\widehat{\varphi T})(\xi)\right| \leq A C^{|\alpha|+1} (|\alpha|!)^s\, (1 + |\xi|)^{m-|\alpha|}. \tag{17}$$

This inequality complets the proof of the relation (12) and thus, the theorem is proved.■

References

[1] Baouendi, M.S. and Goulaouic, C. and Métivier, G.: "Kernels and symbols of analytic pseudodifferential operators", *J. Differ. Equations*, **48** (1983), pp. 227–240.

[2] Boutet De Monvel, L. and Krée, P.: "Pseudodifferential operators and Gevrey classes", *Ann. Inst. de Fourier, Grenoble*, **17**, (1967), p. 295–323.

[3] Boutet De Monvel, L.: "Opérateurs pseudodifférentiels analytiques et opérateurs d'ordre infini", *Ann. Inst. de Fourier, Grenoble*, **22**, (1972), p. 229–268.

[4] Hazi, M.: "Noyaux et symboles des opérateurs pseudodifférentiels en C^∞" *Mémoire de D.E.A*, École Polytechnique, Paris. (1982)

[5] Hazi, M.: "Kernels of pseudodifferential operators associated to Hörmander symbols of Gevrey type", *Arab Gulf Journal of Scientific Research*, **19**, (2001), Manama, pp. 52–58.