Rev. Real Academia de Ciencias. Zaragoza. **59:** 89–94, (2004).

Commutativity of some p -normed algebras with or without involution

A. El Kinani and A. Ifzarne

École Normale Supérieure,
B.P. 5118-Takaddoum. 10105 Rabat, Marrocco

Abstract

The Le Page inequality in a Banach algebra is $\|xy\| \leq \alpha \|yx\|$, for every $x, y \in A$ and some constant $\alpha > 0$. We examine inequalities of Le Page type in p-Banach algebras with or without involution. As a consequences, some commutativity results are obtained.

Key words: p-Banach algebra, commutativity, generalized involution, hermitian algebra, C^*-algebra.

MSC: 46J40, 46H20.

1 Introduction.

Le Page ([5]) considered in a complex Banach algebra $(A, \|.\|)$ the following condition $(C) : \|xy\| \leq \alpha \|yx\|$, for every $x, y \in A$, and some constant $\alpha > 0$. In the unital case, this condition ensures the commutativity of A. The principal ingredient in the proof is Liouville's theorem for holomorphic functions and the Hahn Banach theorem. This last result is in general not valid in p-normed algebras. Here we consider the support pseudo-norm introduced in [9] to examine inequalities of type (C). We show that the condition $(C_1) : \|xy\|_p \leq \alpha \|yx\|_p$, for every $x, y \in A$ and some constant $\alpha > 0$, implies that $A/RadA$ is commutative. This makes it possible to obtain some commutativity results. In the unital case with a generalized involution $x \longmapsto x^*$, we show that the condition $(C_2) : \|x^*y\|_p \leq \alpha \|yx\|_p$, for every $x, y \in A$ and some constant $\alpha > 0$, forces the algebra to be a commutative C^*-algebra for a norm equivalent to $\|.\|_p$.

A vector involution $x \longmapsto x^*$ ([1]) on a complex algebra A is said to be an involutive anti-morphism if $(xy)^* = x^*y^*$, for every $x, y \in A$. A generalized involution is an algebra involution or an involutive anti-morphism. An element a of A is said to hermitian (resp., normal) if $a = a^*$ (resp., $a^*a = aa^*$). We designate by $H(A)$ (resp., $N(A)$) the set of hermitian (resp., normal) elements of A. In the unital case, we say that a is unitary if $a^*a = aa^* = e$, where e is the unit element. The set of all unitary elements of A will be denoted $U(A)$. A linear p-norm on A, $0 < p \leq 1$, is non-negative function $x \longmapsto \|x\|_p$ such that $\|x\|_p = 0$ if and only if $x = 0$, $\|x+y\|_p \leq \|x\|_p + \|y\|_p$ and $\|\lambda x\|_p = |\lambda|^p \|x\|_p$, for all $x, y \in A$ and $\lambda \in C$. By a p-normed algebra $\left(A, \|.\|_p\right)$, we mean an algebra A endowed with a linear p-norm $\|.\|_p$ such that $\|xy\|_p \leq \|x\|_p \|y\|_p$, for all $x, y \in A$. In this paper, all p-normed algebras are not necessarily assumed to be complete. This is different from [10] and [11]. A complete p-normed algebra is called a p-Banach algebra. A p-Banach algebra $\left(A, \|.\|_p\right)$ with a generalized involution $x \longmapsto x^*$ is said to be hermitian if the spectrum of every hermitian element is real. Throughout the paper, all algebras considered will be associative and complex. We denote Pták's function by $|.|$, that is $|a|^2 = \rho(a^*a)$, for every $a \in A$, where ρ is the spectral radius i.e., $\rho(a) = \sup\{|z| : z \in Spa\}$. The center of A will be denoted $C(A)$. For a, $b \in A$, we designate by $[a,b]$ the commutator $ab - ba$.

2 Condition (C_1)

Our first condition is the analog of Le Page's inequality in p-normed Q-algebras

$$\|xy\|_p \leq \alpha \|yx\|_p, \text{ for every } x, y \in A \text{ and } \alpha > 0, \qquad (C_1)$$

In this case, we can say the following

Theorem 2.1 *Let $(A, \|.\|_p)$, $0 < p \leq 1$, be a p-normed Q-algebra. If A satisfies (C_1), then $A/RadA$ is commutative.*

Proof. Without loss of generality, we may suppose A complete for the inequality (C_1) extends to the completion $\hat{A}$ and $A \cap Rad\ \hat{A} \subset RadA$ for A is a Q-algebra. Moreover, considering $A/RadA$ instead of A, we may suppose A semi-simple. For any $x \in A$, put

$$\|x\| = \inf \sum_{i=1}^{n} \|x_i\|_p^{\frac{1}{p}},$$

where the infimum is taken over all decompositions of $x = \sum_{i=1}^{n} x_i$, $x_i \in A$. By [9], $\|.\|$ is a submultiplicative semi-norm on A. Moreover, we have

$$\rho(x) \leq \|x\| \leq \|x\|_p^{\frac{1}{p}}, \quad \forall x \in A. \qquad (1)$$

It follows from (C_1), that

$$\|xy\| \leq \alpha^{\frac{1}{p}} \|yx\|_p^{\frac{1}{p}}, \quad \forall x, y \in A. \qquad (2)$$

Now if A is not unital, consider its unitization $A^1 = A \oplus C$. For $a, b, c \in A$, consider the map f defined, on C, by

$$f(\lambda) = (\exp(\lambda a))\, bc \exp(-\lambda a).$$

One cheks that, for any φ in the topological dual of $(A, \|.\|)$, $\varphi \circ f$ is holomorphic. It is also bounded by (2). By Liouville's theorem $\varphi \circ f$ is constant, and so the coefficient of λ in the power series expansion of $\varphi \circ f$ is zero, i.e., $\varphi([a, bc]) = 0$. By Hahn-Banach theorem, we obtain $\|[a, bc]\| = 0$. It follows from (1) that $\rho([a, bc]\, y) = 0$, for every $y \in A$. Whence $[a, bc] \in RadA$ since $RadA = \{x \in A / \rho(xy) = 0, \forall y \in A\}$. This shows that $A^2 = \{xy / x, y \in A\}$ is contained in the centre of A. Thus, for any integer $n > 0$ and all $x, y \in A$, we obtain $(xy)^n = x^n y^n$. Using the fact that $\rho(x)^p = \lim_n \|x^n\|_p^{\frac{1}{n}}$, we deduce that the spectral radius is submultiplicative on A and hence the set of quasi-nilpotent elements of A coincides with the radical of A. Now, for every x and y in A, we have $(xy - yx)^2 = 0$. Whence $xy - yx \in RadA = \{0\}$ and so A is commutative. ■

Let us notice that if $\|x\|_p \leq \alpha \rho(x)^p$, for every $x \in A$, then C_1 is verified. Whence the following classical result.

Corollary 2.1 *Let* $(A, \|.\|_p)$, $0 < p \leq 1$, *be a* p *-normed* Q*-algebra such that* $\|x\|_p \leq \alpha \rho(x)^p$, *for every* $x \in A$ *and some* $\alpha > 0$. *Then* $A/RadA$ *is commutative.*

The analog of a G. Niestegge's result obtained in [7] is the following

Corollary 2.2 *Let* $(A, \|.\|_p)$, $0 < p \leq 1$, *be a* p*-normed* Q*-algebra such that* $\|xy + y\|_p \leq \alpha \|yx + y\|_p$, *for every* $x, y \in A$ *and some* $\alpha > 0$. *Then* $A/RadA$ *is commutative.*

Proof. The inequality in hypotheses is equivalent to $\|xy\|_p \leq \alpha \|yx\|_p$, for every $x \in A^1 = A \oplus C$ and $y \in A$. ■

Let $(A, \|.\|_p)$, $0 < p \leq 1$, be a p-normed Q-algebra satisfying (C_1). Then we have Theorem 2.1 with $Ker \|.\|$ in place of $RadA$, where $Ker \|.\| = \{x \in A : \|x\| = 0\}$. If moreover the topological dual of $(A, \|.\|)$ separates points on A, then $Ker \|.\| = \{0\}$ and therefore (C_1) ensures the commutativity of A.

3 Condition (C_2)

In this section, we look at the condition

$$\|x^* y\|_p \leq \alpha \|yx\|_p, \text{ for every } x, y \in A \text{ and } \alpha > 0 \qquad (C_2)$$

In a unital Banach algebra, we obtained in [2] that the condition (C_2) forces the algebra to be a C^*-algebra. In the non unital case, it implies that the algebra is hermitian. In the

p-Banach algebras with a generalized involution condition (C_2) appears also to be strong as the following result shows.

Theorem 3.1 *Let $(A, \|.\|_p)$, $0 < p \leq 1$, be a unital p-Banach algebra with a generalized involution $x \longmapsto x^*$. If A satisfies (C_2), then*

1) There exists a positive constant M such that

$$\left\|h^2\right\|_p \leq M\rho(h^2)^p, \text{ for every } h \in H(A).$$

2) A is a commutative C^*-algebra for a norm equivalent to $\|.\|_p$.

Proof. 1) Let $h \in H(A)$ such that $\rho(h) < 1$. Then $\rho(h^2) < 1$. By an analog result of Ford's square root lemma, there is $k \in H(A)$ such that $k^2 = e - h^2$ and $hk = kh$, where e is the unit element. Put $u = h + ik$. It is easy to show that $u \in U(A)$ and $u^2 = e + 2iku$. We have

$$\|ku\|_p = \frac{1}{2^p}\left\|u^2 - e\right\|_p \leq \frac{\alpha+1}{2^p}\|e\|_p$$

Therefore

$$\left\|h^2\right\|_p = \left\|e - k^2\right\|_p \leq \|e\|_p + \|(ku)(u^*k)\|_p \leq \|e\|_p + \alpha\|ku\|_p^2 \leq M$$

where

$$M = \|e\|_p\left[1 + \frac{\alpha(\alpha+1)^2}{2^{2p}}\|e\|_p\right].$$

Hence for an arbitrary $h \in H(A)$, we have $\|h^2\|_p \leq M\rho(h^2)^p$.

2) It follows from condition (C_2) that $\|u^2\|_p \leq \alpha\|e\|_p$, for every $u \in U(A)$. We obtain

$$\rho(u) = \sqrt{\rho(u^2)} \leq \left\|u^2\right\|_p^{\frac{1}{2p}} \leq \left(\alpha\|e\|_p\right)^{\frac{1}{2p}}.$$

By standard arguments ([8]), one shows that the algebra is hermitian. Let $h \in H(A)$ such that $\rho(h) < 1$. There is $k \in H(A)$ such that $k^2 = e - h$ and $hk = kh$. Then, by 1) there exists a constant $M > 0$ such that $\|k^2\|_p \leq M\rho(k^2)^p$. Whence

$$\|h\|_p = \left\|e - k^2\right\|_p \leq \|e\|_p + M\rho(e-h)^p \leq \|e\|_p + 2^pM$$

Now for an arbitrary $h \in H(A)$, we have $\|h\|_p \leq c\rho(h)^p$, where $c = \|e\|_p + 2^pM$. Given $x \in RadA$, we have $h = \frac{1}{2}(x + x^*) \in RadA$ and $k = \frac{1}{2i}(x - x^*) \in RadA$. This implies that A is semi-simple. We consider first the case where $x \longmapsto x^*$ is an algebra involution. Since A is hermitian and semi-simple, we show as in the Banach case ([8]) that Pták's function $|.|$ is an algebra norm. The inequality $\|h\|_p \leq c\rho(h)^p$, for every $h \in H(A)$, implies that, for $x = h + ik \in A$, where $h, k \in H(A)$, we have

$$\|x\|_p \leq c(|h|^p + |k|^p) \leq 2c|x|^p.$$

Moreover

$$|x|^{2p} \leq \|x^*x\|_p \leq \alpha^2 \|x\|_p^2 .$$

Hence $|.|$ is equivalent to $\|.\|_p$ and $(A, |.|)$ is a C^*-algebra. To see that A is commutative, observe that $|a|^2 \leq \mu |a^2|$, for some $\mu > 0$ and every $a \in A$. Then, by induction $\left|a^{2^n}\right| \geq \left(\frac{1}{\mu}\right)^{2^n-1} |a|^{2^n}$. Whence $\rho(a) \geq \frac{1}{\mu}|a|$ which implies commutativity by Corollary 2.2. Suppose now that $x \longmapsto x^*$ is an involutive anti-morphism.We will show that the algebra A is commutative. In this case, $H(A)$ is a real p-Banach algebra. Moreover $Rad\,(H(A)) = \{0\}$, since $\|h\|_p \leq c\rho(h)^p$, for every $h \in H(A)$. Hence $H(A)$ is a real semi-simple p-Banach algebra in which every square is quasi-invertible for A is hermitian. By an analog result, in p-Banach algebra, of theorem 4.8 of Kaplansky ([4]), the algebra $H(A)$ is commutative. This completes the proof. ■

The trivial case of any Banach space with the product zero and any generalized involution shows that the existence of a unit is essential in thorem 3.1 ([2]). In the case A is not unital, we have the following

Theorem 3.2 *Let $(A, \|.\|_p)$, $0 < p \leq 1$, be a non unital p-Banach algebra with a generalized involution $x \longmapsto x^*$ satisfying (C_2). Then*

1) A is hermitian.

2) If $x \longmapsto x^*$ is an algebra involution, then $A/RadA$ is commutative.

Proof. 1) For every normal element $a \in A$, condition (C_2) implies $\rho(a)^{2p} \leq \alpha^{\frac{1}{2^n}} \left(\|aa^*\|_p\right)^{\frac{1}{2^n-1}}$. We obtain $\rho(a) \leq |a|$. As in [8], one shows that A is hermitian. To see 2), observe that (C_2) implies $\|xy\|_p \leq M \|yx\|_p$, for a $M > 0$. Whence the conclusion by Theorem 2.1. ■

4 Condition (C_3)

Our last condition is

$$\|x^*y^*\|_p \leq \alpha \|yx\|_p , \text{ for every } x, y \in A \text{ and } \alpha > 0. \qquad (C_3)$$

If $x \longmapsto x^*$ is a continuous algebra involutive, then the condition (C_3) is satisfied. If $x \longmapsto x^*$ is a continuous involutive anti-morphism, (C_3) implies Le Page condition $\|xy\|_p \leq \alpha^2 \|yx\|_p$. If $x \longmapsto x^*$ is only a linear involution, we have the following

Theorem 4.1 *Let $(A, \|.\|_p)$, $0 < p \leq 1$, be a unital p-Banach algebra and $x \longmapsto x^*$ be a continuous linear involution such that $e^* = e$. If A satisfies (C_3) ,then $(ab)^* = b^*a^*$ in $A/RadA$.*

Proof. For $a, b \in A$ and $c \in A$, consider the map f defined, on C and with values in A, by

$$f(\lambda) = ((\exp(\lambda a))\, c)^* \, (b \exp(-\lambda a))^* .$$

For any φ in the topological dual of $(A, \|.\|)$, $\varphi \circ f$ is harmonic. It is also bounded by (C_3). By Liouville's theorem $\varphi \circ f$ is constant. Differentiating relative to the real part (or the imaginary part) of λ, we have $\varphi\left((ac)^* b^* - c^*(ba)^*\right) = 0$. Whence $(ac)^* b^* - c^*(ba)^* \in RadA$ by Hahn-Banach theorem and (1), which proves the result. ■

Remark Under the condition $\|x^* y^*\|_p \leq \alpha \|xy\|_p$, for every $x, y \in A$ and some constant $\alpha > 0$, we obtain $(ca)^* b^* - c^*(ab)^* \in RadA$, for every $a, b, c \in A$. Then $(ab)^* = a^* b^*$ in $A/RadA$.

Indeed, for $a, b, c \in A$, consider the map g defined, on C, by
$g(\lambda) = (c \exp(\lambda a))^* \, ((\exp(-\lambda a))\, b)^*$ and proceed as for theorem 4.1.

References

[1] Bonsall, F. F. and Duncan, J., Complete normed algebras, Ergebrise der Mathematik, Band 20, Springer Verlag (1973).

[2] El Kinani, A., Oudadess, M., Criteria of Le Page type in involutive Banach algebras, Tr. J. of Mathemaztics, 23(1999), 315-321.

[3] Johnson, B. E., The uniqueness of the complete norm topology. Bull. Amer. Math. Soc. 42 (1967), 407-409.

[4] Kaplansky. I, Normed algebras, Duke Math. J. 16 (1949), 399-418.

[5] Le Page, C., Sur quelques conditions impliquant la commutativité dans les algèbres de Banach. C.R.A.S. Paris, Ser. A-B; 265, (1967), A 235-A 237.

[6] Mallios, A, Topological algebras. Selected topics. North-Holland. 1986.

[7] Niestegge, G., A note on criterion of Le Page and Hirschfeld- Zelazko for the commutativity of Banach algebras. Studia . Math. T. LXXIX, (1984), 87-90.

[8] Ptàk. V, Banach algebras with involution, Manuscripta Math. 6 (1972), 245-290.

[9] Xia Dao-Xing (Hsia.Tao-Hsing), On locally bounded topological algebras. Acta Math. Sinica 14 N^0 2 (1964). Engl. Übers. Chinese-Math. 5 N^0 2, 261-276. (1964).

[10] Zelazko, W., Selected Topics in Topological Algebras, Aarhus University Lecture Notes Ser. 31, 1971.

[11] Zelazko, W., On the locally bounded and m-convex topological algebras. Studia Math. t. XIX, 333-355.