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Dual space of certain locally convex sequence spaces

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Abstract

Let E be a locally convex space and Λ a perfect sequence space. Denote by $\Lambda(E)_r$ the space of all Λ-summable sequences from E which are the limit of their finite sections. In this note we characterize the continuous linear functionals on $\Lambda(E)_r$ in terms of strongly Λ^*-summable sequences in the dual E' of E.

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1 Introduction

For a perfect sequence space Λ and a locally convex space E, A. Pietsch [14] introduced the space $\Lambda[E]$ of all weakly Λ-summable sequences in E and the space $\Lambda(E)$ of all Λ-summable sequences in E. He then characterized the nuclearity of E in terms of the summability of its sequences. Since then, several authors have been interested in vector-valued sequence spaces [1], [3, 4, 5], [9, 10, 11, 12] and [15]. For instance, different aspects of the space $\Lambda[E]$, particularly when $\Lambda = \ell_p$, were studied in [3] and in [5]. Whenever Λ is endowed with the Köthe normal topology, properties of $\Lambda(E)$ were studied in [8], such as the description of the continuous dual in terms of prenuclear sequences in E'. More general cases were considered by M. Florencio and P. J. Paúl in [10] and [11].

In this paper, we consider on Λ an arbitrary locally convex polar topology with respect to the Köthe duality and deal with the following problem: Characterize the E'-valued sequences determining continuous functionals on $\Lambda(E)_r$, the space of Λ-summable sequences which are the limit of their finite sections. To this end, we extend to any perfect Λ the notion of strongly p-summable sequences introduced by H. Apiola [1]. We then make use of such sequences to characterize the continuous dual of $\Lambda(E)_r$.

We refer the reader to Section 30 of [8] and Chapter 2 of [16] for the concepts and Köthe theory of sequence spaces and to [7] for the terminology and notations concerning the general theory of locally convex spaces.

Most of our results remain true when Λ is only normal and E not necessarily sequentially complete. However, for the comfort of the reader, we assume these hypotheses all over the paper unless the contrary is clearly expressed.

2 Preliminaries

Let E be a sequentially complete locally convex space over the field $\mathbb{K}$ of real or complex numbers and E' its continuous dual. Denote by $\mathcal{M}$ the family of all equicontinuous subsets of E' which are absolutely convex and $\sigma(E', E)$-closed. If $M \in \mathcal{M}$, let E'_M denote the linear subspace of E' spanned by M, furnished with the Minkowski functional $\| \ \|_M$ of M as a norm. Consider on E the seminorm P_M defined by

$$P_M(x) = \sup\{|g(x)| : g \in M\}$$

and put $E/M^\perp$ to denote the quotient of E by the kernel $M^\perp$ of P_M. Equip it with the quotient norm induced by P_M. It is easily seen that the Banach space $(E/M^\perp)'$ and E'_M are isometrically isomorphic (compare with Proposition 8.7.7 of [7]).

Now, let Λ be a perfect sequence $\mathbb{K}$-space and Λ^* its Köthe dual. We will write e_n to designate the sequence whose only non zero term is the n^{th} one, which is equal to 1. A sequence $(x_n)_n \subset E$ is said to be Λ-summable if the series $\sum \alpha_n x_n$ converges in E for every $(\alpha_n)_n \in \Lambda^*$. It is weakly Λ-summable if $(a(x_n))_n \in \Lambda$ for every $a \in E'$. We denote by $\Lambda(E)$ the space of all Λ-summable sequences in E and by $\Lambda[E]$ that of weakly Λ-summable ones. We will say that a sequence $(x_n)_n \subset E$ is strongly Λ-summable if, for every $M \in \mathcal{M}$, the series $\sum_n f_n(x_n)$ converges absolutely for all $(f_n)_n \in \Lambda^*[E'_M]$. The space of all such sequences will be denoted by $\Lambda\langle E\rangle$.

In the sequel we will assume that Λ is equipped with a locally convex polar topology defined by a family $\mathcal{S}$ of closed absolutely convex normal and $\sigma(\Lambda^*, \Lambda)$-bounded subsets of Λ^* such that $\mathcal{S}$ is closed under finite unions and scalar multiples and $\mathcal{S}$ covers Λ^*. Such a topology is generated by the seminorms P_S, $S \in \mathcal{S}$, where

$$P_S(\alpha) = \sup\left\{\sum_{n=1}^{\infty} |\alpha_n \beta_n| : (\beta_n)_n \in S\right\}, \alpha = (\alpha_n)_n \in \Lambda.$$

As shown in Proposition 1 of [10], a natural locally convex topology can be defined on $\Lambda(E)$ by means of the $\mathcal{S}$-topology on Λ and the topology of E. This topology is generated by the family $(\varepsilon_{S,M})_{S\in\mathcal{S},\ M\in\mathcal{M}}$ of seminorms, where

$$\varepsilon_{S,M}(x) = \sup\left\{\sum_{n=1}^{\infty} |\alpha_n a(x_n)| : (\alpha_n)_n \in S, a \in M\right\}, x = (x_n)_n \in \Lambda(E).$$

Note that, with a similar proof as in [10], we can regard the $\varepsilon_{S,M}$'s as seminorms on $\Lambda[E]$. In all what follows, $\Lambda(E)$ as well as $\Lambda[E]$ will be equipped with the topology given by these seminorms.

The following proposition will be needed in the sequel. Since we have not been able to locate any reference (which may exist) for it, we include a proof for the sake of completeness.

Proposition 1 *(i) For every $n \in \mathbb{N}$, the projection $I_n : \Lambda[E] \longrightarrow E$, $I_n(x) = x_n$ is a continuous linear map.*
(ii) $\Lambda[E]$ is (sequentially) complete if, and only if, Λ and E are (sequentially) complete.
(iii) $\Lambda(E)$ is a closed subspace of $\Lambda[E]$.
(iv) $\Lambda(E)_r$ is a closed subspace of $\Lambda(E)$.

Proof: (i) is a consequence of the very definition of the $\varepsilon_{S,M}$'s.

(ii): The necessity is also a consequence of the definition. For the sufficiency, let $(x^i)_i$ be a Cauchy net in $\Lambda[E]$. The continuity of I_n implies that $(x^i_n)_i$ is a Cauchy net in E for all n. Hence it must converge to some $x_n \in E$. Let us prove that $x = (x_n)_n \in \Lambda[E]$. If $a \in E'$, then $(a(x^i_n))_n$ is a Cauchy net in Λ. Indeed, for all $S \in \mathcal{S}$ and $M \in \mathcal{M}$ such that $a \in M$, we have

$$P_S((a(x^i_n))_n - (a(x^j_n))_n) \leq \varepsilon_{S,M}(x^i - x^j), \text{ for all } i, j.$$

Let $\alpha = (\alpha_n(a))_n$ be the limit of $(a(x^i_n))_i$. The continuity of I_n gives $a(x_n) = \alpha_n(a)$. Thus $(a(x_n))_n \in \Lambda$ and $x = (x_n)_n \in \Lambda[E]$. Moreover, for $\epsilon > 0$, there exists k such that for all $i, j > k$, $\varepsilon_{S,M}(x^i - x^j) < \epsilon$. This leads to $\varepsilon_{S,M}(x^i - x) \leq \epsilon$ for all $i > k$, which proves the convergence of $(x^i)_i$ to x. The same proof holds for the sequential completeness too.

(iii) Let $x = (x_n)_n \in \overline{\Lambda(E)}$, $\epsilon > 0$, $\alpha \in \Lambda^*$, $M \in \mathcal{M}$ and $S \in \mathcal{S}$ be given with $\alpha \in S$. There exists $y = (y_n)_n \in \Lambda(E)$ such that $\varepsilon_{S,M}(x - y) \leq \epsilon$. Let $k \in \mathbb{N}$ be so that, for all $i \geq j \geq k$, $P_M(\sum_j^i \alpha_n y_n) \leq \epsilon$. Then,

$$P_M(\sum_j^i \alpha_n x_n) \leq P_M(\sum_j^i \alpha_n y_n) + \varepsilon_{S,M}(x - y) \leq 2\epsilon,$$

so that $x \in \Lambda(E)$. Therefore, $\Lambda(E)$ is closed in $\Lambda[E]$.
(iv) Let $x = (x_n)_n \in \overline{\Lambda(E)_r}$, $\epsilon > 0$, $M \in \mathcal{M}$ and $S \in \mathcal{S}$ be given. There exists $y = (y_n)_n \in$

$\Lambda(E)_r$ such that $\varepsilon_{S,M}(x-y) \le \epsilon$. Set $z^{\langle j \rangle} = (0,0,\ldots,0,z_{j+1},z_{j+1},\ldots)$ and let $k \in \mathbb{N}$ be so that $\varepsilon_{S,M}(y^{\langle j \rangle}) \le \varepsilon$, for all $j \ge k$. Then,

$$\begin{aligned} \varepsilon_{S,M}(x^{\langle j \rangle}) &\le \varepsilon_{S,M}(x^{\langle j \rangle} - y^{\langle j \rangle}) + \varepsilon_{S,M}(y^{\langle j \rangle}) \\ &= \varepsilon_{S,M}((x-y)^{\langle j \rangle}) + \varepsilon_{S,M}(y^{\langle j \rangle}) \\ &\le \varepsilon_{S,M}(x-y) + \epsilon \\ &\le 2\epsilon. \qquad Q.E.D. \end{aligned}$$

Remarks: 1. In the statement of (ii), we need not require Λ to be perfect.

2. If Λ is not perfect, $\Lambda(E)$ need not be a subspace of $\Lambda[E]$. This is the case if $E = \ell_1$, $\Lambda = c_0$. Actually, $x = (e_n)_n$ belongs to $\Lambda(E)$ but not to $\Lambda[E]$. However, with a similar proof as above, if Λ happens to be only normal, then $\Lambda(E) = \Lambda^{**}(E)$ and $\Lambda(E)$ is a closed subspace of $\Lambda^{**}[E]$.

3. In general, whenever Λ is perfect, $\Lambda(E)$ is a proper subspace of $\Lambda[E]$. Conditions under which the equalities $\Lambda(E)_r = \Lambda(E)$ or $\Lambda(E) = \Lambda[E]$ hold can be found in [11], [13] and [5].

We also need the following proposition.

Proposition 2 *If E'_β denotes the strong dual of E, the following equality holds*

$$\Lambda[E'_\beta] = \{(a_n)_n \subset E' : (a_n(x))_n \in \Lambda, x \in E\}.$$

Moreover, the topology of $\Lambda[E'_\beta]$ is given by the seminorms

$$\varepsilon_{S,B}(a) = \sup \left\{ \sum_{n=1}^{\infty} |\alpha_n a_n(x)| : (\alpha_n)_n \in S, x \in B \right\},$$

where S runs over $\mathcal{S}$ and B over the set of all absolutely convex closed and bounded subsets of E.

Proof: Let $a = (a_n)_n \in \Lambda[E'_\beta]$. Since E can be seen as a subspace of $(E'_\beta)^{prime}$, $(a_n(x))_n \in \Lambda$ for all $x \in E$. Conversely, assume that for all $x \in E$, $(a_n(x))_n \in \Lambda$ and let $f \in \left(E'_\beta\right)'$ and $(\beta_n)_n \in \Lambda^*$ be given. We must show that the series $\sum |\beta_n f(a_n)|$ is convergent. Choose $(\epsilon_n)_n$ so that $\epsilon_n f(\beta_n a_n) = |f(\beta_n a_n)|$ for all n and set $A = \left\{ \sum_{n=1}^{p} \epsilon_n \beta_n a_n, p \in \mathbb{N} \right\}$. For all $p \in \mathbb{N}$, one has

$$\sum_{n=1}^{p} |\epsilon_n \beta_n a_n(x)| \le \sum_{n=1}^{\infty} |\beta_n a_n(x))|$$

which is finite since $(a_n(x))_n \in \Lambda$. So A is $\sigma(E', E)$-bounded. Since E is sequentially complete, A is bounded in E'_β. Hence we can find $K_f > 0$ such that $\sum_{n=1}^{p} \epsilon_n \beta_n f(a_n) \le K_f$, for all $p \in \mathbb{N}$. This proves that the series $\sum |\beta_n f(a_n)|$ is convergent and that $(f(a_n))_n \in \Lambda$.

Now, if M is an equicontinuous absolutely convex and closed subset of $(E'_\beta)'$, then the polar $U = M^\circ$ of M with respect to the duality $\langle E'_\beta, (E'_\beta)'\rangle$ is a 0-neighbourhood in E'_β. There exists a closed absolutely convex and bounded subset B of E such that $U = B^\circ$ with respect to the duality $\langle E, E'\rangle$. If P_U is the gauge of U, for all $(\alpha_n)_n \in S$ and $p \in \mathbb{N}$, we have:

$$\begin{aligned} P_U(\sum_{n=1}^{p} \alpha_n a_n) &= P_M(\sum_{n=1}^{p} \alpha_n a_n) \\ &= P_B(\sum_{n=1}^{p} \alpha_n a_n) \end{aligned}$$

Therefore,

$$\sup\{|\sum_{n=1}^{p} \alpha_n f(a_n)| : (\alpha_n)_n \in S, f \in M\} = \sup\{|\sum_{n=1}^{p} \alpha_n a_n(t)| : (\alpha_n)_n \in S, t \in B\}.$$

Since S is normal,

$$\sup\{\sum_{n=1}^{p} |\alpha_n f(a_n)| : (\alpha_n)_n \in S, f \in M\} = \sup\{\sum_{n=1}^{p} |\alpha_n a_n(t)| : (\alpha_n)_n \in S, t \in B\}.$$

Thus $\varepsilon_{S,M}(a) = \varepsilon_{S,B}(a)$. Q.E.D.

Remark: Proposition 2 remains true whenever the weakly bounded sets of E' are strongly bounded. This holds in particular if E is locally barrelled.

3 Dual space of $\Lambda(E)$

For $S \in \mathcal{S}$, let Λ^*_S denote the vector space spanned by S and $(\Lambda^*_S)^*$ its Köthe dual. We start with the following lemma:

Lemma 3 *1. Let $S \in \mathcal{S}$ and $\alpha \in (\Lambda^*_S)^*$. Then*

$$P_S(\alpha) := \sup\left\{\sum_{n=1}^{\infty} |\alpha_n \beta_n| : (\beta_n)_n \in S\right\} < \infty.$$

Moreover, P_S defines a seminorm on $(\Lambda^*_S)^*$.
2. The space

$$\lambda := \{\alpha \in (\Lambda^*_S)^* : \alpha_n = 0 \text{ whenever } e_n \notin \Lambda^*_S\}$$

is a normal sequence space and (λ, P_S) is a Banach space.

Proof: 1. We only have to prove that $\sup\left\{\sum_{n=1}^{\infty} |\alpha_n \beta_n| : (\beta_n)_n \in S\right\}$ is finite. Since S is absolutely convex, closed and $\sigma(\Lambda^*, \Lambda)$-bounded, it is also so with respect to the Köthe normal topology. The latter being complete on Λ^*, Λ^*_S is a Banach space when it is equipped with the gauge of S as a norm. For $(\alpha_n)_n \in (\Lambda^*_S)^*$, consider the linear mapping

$T_\alpha: \Lambda_S^* \to \ell_1$ defined by $T_\alpha((\beta_n)_n) = (\alpha_n\beta_n)_n$. Then T_α is continuous by the closed graph theorem. Hence it is bounded on S.

2. In order to prove that (λ, P_S) is a Banach space, it is enough to show that it is isometrically isomorphic to a closed subspace of the dual space $(\Lambda_S^*)'$ of Λ_S^*. Actually, if μ is the functional defined on ℓ_1 by $\mu((\beta_n)_n) = \sum_n \beta_n$, then the linear map $\varphi : \alpha \mapsto \mu \circ T_\alpha$ maps isometrically λ into $(\Lambda_S^*)'$. Moreover, for $T \in \overline{\varphi(\lambda)}$ and $\epsilon > 0$, there exists $\alpha \in \lambda$ such that $\|T - T_\alpha\| < \epsilon$. Put $\delta_n = T(e_n)$ if $e_n \in \Lambda_S^*$ and $\delta_n = 0$ otherwise. Then $\delta = (\delta_n)_n$ belongs to λ. Indeed, for every $\beta \in S$ and $k \in \mathbb{N}$, if $\beta^{(k)} = \sum_1^k \beta_n e_n$, then $T(\beta^{(k)}) = T_\delta(\beta^{(k)})$ and

$$\begin{aligned}
\left|T(\beta) - T_\delta(\beta^{(k)})\right| &\leq |T(\beta) - T_\alpha(\beta)| + \left|T_\alpha(\beta) - T_\alpha(\beta^{(k)})\right| + \left|T_\alpha(\beta^{(k)}) - T(\beta^{(k)})\right| \\
&\leq 2\|T - T_\alpha\| + \left|T_\alpha(\beta) - T_\alpha(\beta^{(k)})\right| \\
&\leq 2\varepsilon + \left|\sum_{n=k+1}^{\infty} \alpha_n\beta_n\right|.
\end{aligned}$$

Since $\left|\sum_{n=k+1}^{\infty} \alpha_n\beta_n\right|$ tends to 0,

$$T(\beta) = \lim_{k\to\infty} T(\beta^{(k)}) = \lim_{k\to\infty} \sum_{n=1}^{k} \delta_n\beta_n = \sum_{n=1}^{\infty} \delta_n\beta_n.$$

It follows at once that δ belongs to $(\Lambda_S^*)^*$, then also to λ, and that $T = T_\delta$. Consequently, $\varphi(\lambda)$ is closed. Q.E.D.

The following theorem extends to the general case of locally convex spaces a result of Q. Bu and J. Diestel given in [2] for $\Lambda = \ell^p$, $1 < p < \infty$ and E a Banach space.

Theorem 4 *Let F be a continuous linear functional on $\Lambda(E)$ and, for every $n \in \mathbb{N}$ and $t \in E$, $a_n(t) = F(te_n)$. Then there exists $M \in \mathcal{M}$ and $S \in \mathcal{S}$ such that the sequence $(a_n)_n$ is strongly Λ_S^*-summable in E'_M.*

Proof: Since F is continuous, there exist $S \in \mathcal{S}$ and $M \in \mathcal{M}$ such that

$$|F(x)| \leq \varepsilon_{S,M}(x),\ \forall x = (x_n)_n \in \Lambda(E).$$

Fix $n \in \mathbb{N}$ and $t \in E$. We have

$$|a_n(t)| = |F(te_n)| \leq \varepsilon_{S,M}(te_n) = P_S(e_n)P_M(t),$$

which shows that $(a_n)_n \subset E'_M$.

It remains to show that $(a_n)_n \in \Lambda_S^*\langle E'_M\rangle$. To this end, let $(f_n)_n \in (\Lambda_S^*)^*[(E'_M)']$, $k \in \mathbb{N}$ and $\delta > 0$ be given. Consider the completion $\widetilde{E/M^\perp}$ of $E/M^\perp$. It is known that $(\widetilde{E/M^\perp})' = (E/M^\perp)'$ is isometrically isomorphic to E'_M. Then, due to the the principle of local reflexivity (cf. [6]), there exists a continuous operator $u_k\colon \mathrm{span}\{f_1, f_2, \ldots, f_k\} \to \widetilde{E/M^\perp}$ such that $\|u_k\| \le 1+\delta$ and $a_n(u_k f_n) = f_n(a_n)$ for all $n \in \{1, 2, \ldots, k\}$. Since every a_n is continuous and $E/M^\perp$ is dense in $\widetilde{E/M^\perp}$, there exist $0 < \delta_n \le \dfrac{\delta}{k(1+p_S(e_n))}$ and $x_n \in E$ such that $\|\widehat{x_n} - u_k f_n\| \le \delta_n$ and $|a_n(\widehat{x_n} - u_k f_n)| \le \dfrac{\delta}{k}$, $\widehat{x_n}$ being $x_n + M^\perp$. Now,

$$\begin{aligned}
\left|\sum_{n=1}^{k} f_n(a_n)\right| &= \left|\sum_{n=1}^{k} a_n(u_k f_n)\right| \\
&\le \left|\sum_{n=1}^{k} a_n(\widehat{x_n} - u_k f_n)\right| + \left|\sum_{n=1}^{k} a_n(x_n)\right| \\
&\le \sum_{n=1}^{k} |a_n(\widehat{x_n} - u_k f_n)| + \left|F(\sum_{n=1}^{k} x_n e_n)\right| \\
&\le \delta + \varepsilon_{S,M}(\sum_{n=1}^{k} x_n e_n) \\
&= \delta + \sup\left\{\left|\sum_{n=1}^{k} \alpha_n a(x_n)\right| : (\alpha_n)_n \in S, a \in M\right\}.
\end{aligned}$$

But, for $(\alpha_n)_n \in S$ and $a \in M$,

$$\begin{aligned}
\left|\sum_{n=1}^{k} \alpha_n a(x_n)\right| &\le \left|\sum_{n=1}^{k} \alpha_n a(\widehat{x_n} - u_k f_n)\right| + \left|\sum_{n=1}^{k} \alpha_n a(u_k f_n)\right| \\
&\le \sum_{n=1}^{k} |\alpha_n|\, |a(\widehat{x_n} - u_k f_n)| + \left|\langle a \circ u_k, \sum_{n=1}^{k} \alpha_n f_n\rangle\right| \\
\text{(since } ||a||_M \le 1) \quad &\le \sum_{n=1}^{k} |\alpha_n|\, \delta_n + (1+\delta)\left\|\sum_{n=1}^{k} \alpha_n f_n\right\| \\
&\le \sum_{n=1}^{k} P_S(e_n)\delta_n + (1+\delta)\sup\left\{\left|\sum_{n=1}^{k} \beta_n f_n(x')\right| : (\beta_n)_n \in S, x' \in M\right\} \\
&\le \delta + (1+\delta)\varepsilon_{S,M}((f_n)_n).
\end{aligned}$$

Hence

$$\left|\sum_{n=1}^{k} f_n(a_n)\right| \le 2\delta + (1+\delta)\varepsilon_{S,M}((f_n)_n),\ (f_n)_n \in \Lambda[(E'_M)'] \text{ and } k \in \mathbb{N}.$$

Further, let $(\epsilon_n)_n$ be such that $|f_n(a_n)| = \epsilon_n f_n(a_n)$, $n \in \mathbb{N}$. Then $(\epsilon_n f_n)_n \in (\Lambda_S^*)^*[(E'_M)']$ and

$$\sum_{n=1}^{k} |f_n(a_n)| = \sum_{n=1}^{k} \epsilon_n f_n(a_n) \le 2\delta + (1+\delta)\varepsilon_{S,M}((\epsilon_n f_n)_n).$$

It follows that the series $\sum f_n(a_n)$ converges absolutely, for $\varepsilon_{S,M}$ is finite on $(\Lambda_S^*)^*[(E'_M)']$. Q.E.D.

Remark: From the preceding proof, since δ is arbitrary, one has

$$\sum_{n=1}^{\infty} |f_n(a_n)| \le \varepsilon_{S,M}((f_n)_n),\ (f_n)_n \in (\Lambda_S^*)^*[(E'_M)'].$$

To establish the converse of Theorem 4, we need the following lemma:

Lemma 5 *For all* $(a_n)_n \in \Lambda_S^*\langle E_M'\rangle$, $(\|a_n\|_M)_n \in \Lambda_S^*$.

Proof: Given $(\alpha_n)_n \in \Lambda$ and $\varepsilon > 0$. By the definition of $\|\ \|_M$, for every $n \in \mathbb{N}$, there exists $t_n \in M^\circ \subset E$, such that

$$\|\alpha_n a_n\|_M \leq \frac{\varepsilon}{2^n} + |\alpha_n a_n(t_n)|. \tag{1}$$

For every $a \in E_M'$ and $n \in \mathbb{N}$, put $f_n(a) = \alpha_n a(t_n)$. Then $|f_n(a)| \leq \|a\|_M |\alpha_n|$ and the normality of Λ gives $(f_n(a))_n \in \Lambda \subset (\Lambda_S^*)^*$. By Proposition 2, $(f_n)_n \in (\Lambda_S^*)^*[(E_M')']$. Thus the series $\sum \alpha_n a_n(t_n) = \sum f_n(a_n)$ converges absolutely. Consequently, $\sum \|\alpha_n a_n\|_M$ converges by (1), whereby $(\|a_n\|_M)_n \in \Lambda^*$. Now, if $(\alpha_n)_n \in S^\circ \subset \Lambda$, by the preceding remark, we get

$$\sum_{n=1}^{\infty} |\alpha_n a_n(t_n)| = \sum_{n=1}^{\infty} |f_n(a_n)| \leq \varepsilon_{S,M}\left((f_n)_n\right) \leq 1.$$

Hence $\sum_{n=1}^{\infty} \|\alpha_n a_n\|_M \leq \varepsilon + 1$. This is $(\|a_n\|_M)_n \in (1+\varepsilon)S^{\circ\circ} = (1+\varepsilon)S$ which shows that $(\|a_n\|_M)_n$ belongs to Λ_S^*. Q.E.D.

Proposition 6 *For every* $S \in \mathcal{S}$, $M \in \mathcal{M}$ *and* $a = (a_n)_n \in \cup_{S,M}\Lambda_S^*\langle E_M'\rangle$, *the mapping*

$$f_a : x \mapsto \sum_{n=1}^{\infty} a_n(x_n)$$

defines a continuous linear functional on $\Lambda(E)$.

Proof: Choose $M \in \mathcal{M}$, $S \in \mathcal{S}$ and $a = (a_n)_n \in \Lambda_S^*\langle E_M'\rangle$. For every $x = (x_n)_n \in \Lambda(E)$, we have $(\delta_n)_n \subset (E_M')'$, where δ_n is the evaluation $u \mapsto u(x_n)$ at x_n. Thanks to Proposition 1, $(\delta_n(u))_n \in \Lambda \subset (\Lambda_S^*)^*$. By Proposition 2, $(\delta_n)_n \in (\Lambda_S^*)^*[(E_M')']$. Hence $\sum |\delta_n(a_n)|$ converges and that f_a is well defined.

Next, consider the map φ_a defined from $\lambda[(E_M')']$ into ℓ_1 by $\varphi_a((f_n)_n) = (f_n(a_n))_n$, where λ is the sequence space defined in Lemma 3. Then φ_a is well defined. Moreover, suppose that $(f^i)_{i\in\mathbb{N}} \in \lambda[(E_M')']$ converges to $f := (f_n)_n$ and $(\varphi_a(f^i))_i$ converges in ℓ_1 to $(\alpha_n)_n$. By the continuity of the projections, $(f_n^i)_{i\in\mathbb{N}}$ converges to f_n for every $n \in \mathbb{N}$ and then $(f_n^i(a_n))_{i\in\mathbb{N}}$ converges to $f_n(a_n)$ as well. It follows that $(f_n(a_n))_n = (\alpha_n)_n$ showing that the graph of φ_a is closed and then that φ_a is continuous, since $\lambda[(E_M')']$ is a Banach space (Proposition 1). Let $K > 0$ be so that

$$\sum_{n=1}^{\infty} |f_n a_n| \leq K\varepsilon_{S,M}((f_n)_n),\ (f_n)_n \in \lambda[(E_M')']$$

and define $g_n = \delta_n$ if $e_n \in \Lambda_S^*$ and $g_n = 0$ otherwise. Then $(g_n)_n \in \lambda[(E_M')']$. In view of Lemma 5, $a_n = 0$ whenever $e_n \notin \Lambda_S^*$. So we have

$$|f_a(x)| = \left|\sum_{n=1}^{\infty} a_n(x_n)\right| = \left|\sum_{e_n\in\Lambda_S^*} a_n(x_n)\right| \le \sum_{n=1}^{\infty} |g_n(a_n)| \le K\varepsilon_{S,M}((g_n)_n) \le K\varepsilon_{S,M}(x).$$

This shows that f_a is continuous on $\Lambda(E)$. Q.E.D.

We now obtain the promised characterization of continuous linear forms on $\Lambda(E)_r$.

Theorem 7 *The following equality holds algebraically*

$$(\Lambda(E)_r)' = \bigcup\{\Lambda_S^*\langle E_M'\rangle, S \in \mathcal{S}, \mathcal{M} \in \mathcal{M}\}.$$

Proof: By (6), the map $a \mapsto f_a$ from $\bigcup\{\Lambda_S^*\langle E_M'\rangle, S \in \mathcal{S}, \mathcal{M} \in \mathcal{M}\}$ into $(\Lambda(E)_r)'$ is well defined, linear and one to one. It is onto by (6) and the definition of $\Lambda(E)_r$. Q.E.D.

The following result describes a fundamental base of equicontinuous subsets of $(\Lambda(E)_r)'$. In order to establish it, let us introduce the following notation. For $S \in \mathcal{S}$ and $M \in \mathcal{M}$, we set

$$K_{S,M} := \{(f_n)_n \in \Lambda[(E_M')'], \forall a \in M, (f_n(a_n))_n \in S^\circ\}.$$

Note that $K_{S,M}$ is nothing but the closed unit ball of $\lambda[(E_M')']$. We then have:

Theorem 8 *The sets of the form*

$$S\langle M\rangle = \left\{(a_n)_n \in \Lambda_S^*\langle E_M'\rangle : \forall (f_n)_n \in K_{S,M}, \sum_{n=1}^{\infty} |f_n(a_n)| \le 1\right\}$$

yield a fundamental system of equicontinuous subsets of $(\Lambda(E)_r)'$*,* S *running over* $\mathcal{S}$ *and* M *over* $\mathcal{M}$*.*

Proof: Let us prove that $S\langle M\rangle$ is equicontinuous. If $x = (x_n)_n \in \Lambda(E)$ satisfies $\varepsilon_{S,M}(x) \le 1$, as shown in the proof of (6), $(\delta_n)_n \in K_{S,M}$. Moreover, if $a = (a_n)_n \in S\langle M\rangle$, then $\sum_{n=1}^{\infty} |\delta_n(a_n)| = \sum_{n=1}^{\infty} |a_n(x_n)| \le 1$. So $S\langle M\rangle$ is equicontinuous.
Conversely, if $H \subset (\Lambda(E)_r)'$ is equicontinuous, there exists $S \in \mathcal{S}$ and $M \in \mathcal{M}$, such that

$$\forall x = (x_n)_n \in \Lambda(E)_r, \forall a = (a_n)_n \in H, |\langle a, x\rangle| = \left|\sum_{n=1}^{\infty} a_n(x_n)\right| \le \varepsilon_{S,M}(x).$$

Let $(f_n)_n \in K_{S,M}$, then $\varepsilon_{S,M}((f_n)_n) \le 1$. By the remark following the proof of Proposition 2, $\sum_{n=1}^{\infty} |f_n(a_n)| \le \varepsilon_{S,M}((f_n)_n) \le 1$. Hence, $H \subset S\langle M\rangle$. Q.E.D.

Combining Proposition 2 of [11], Theorem 7 and Theorem 8, we get the following result. At this point, let us recall that Λ is said to verify the AK property if every $\alpha \in \Lambda$ is the limit of its finite sections.

Corollary 9 *If* E *is a complete locally convex space and* Λ *satisfies the* AK *property, then the dual space of* $\Lambda\tilde{\otimes}_\varepsilon E$ *is given by* $\bigcup\{\Lambda_S^*\langle E_M'\rangle, S \in \mathcal{S}, \mathcal{M} \in \mathcal{M}\}$*. Moreover, the sets of the form* $S\langle M\rangle$ *yield a basis of equicontinuous subsets of* $(\Lambda\tilde{\otimes}_\varepsilon E)'$*.*

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References

[1] H. Apiola: *Duality between spaces of p-summing operators and characterization of nuclearity.* Math. Ann. **219**, (1974), 53-64.

[2] Q. Bu, J. Diestel: *Observations about the projective tensor product of Banach space* $\ell^p\widehat{\otimes}X$, $1 < p < \infty$. Quaestiones Mathematicae **24** (2001), 519-533.

[3] W. Congxin, Q. Bu: *Köthe dual of Banach spaces* $\ell_p[E]$ $(1 \leq p < \infty)$ *and Grothendieck space.* Comment. Math. Univ. Carolinae **34**, (2) (1993), 265-273.

[4] N. De Grande-De Kimpe: *Generalized Sequence spaces.* Bull. Soc. Math. Belgique, **23** (1971), 123-166.

[5] M. Gupta, Q. Bu: *On Banach-valued sequence spaces* $\ell_p[X]$. J. Anal. **2** (1994), 103-113.

[6] D. W. Dean: *The equation* $L(E, X^{**}) = L(E, X)^{**}$ *and the principle of the local reflexivity.* Proc. Amer. Math. Soc., **40** (1973), 146-148.

[7] H. Jarchow: *Locally convex spaces.* B. G. Teubner Stuttgart(1981).

[8] G. Köthe: *Topological Vector Spaces I and II.* Springer-Verlag, Berlin, Heidelberg, New York.

[9] M. A. Ould Sidaty: *Reflexivity and AK-property of certain vector sequence spaces.* Bull. Belg. Math. Soc., Simon Stevin 10 (4) (2003), 579-783.

[10] M. Florencio, P. J. Paúl: *Una representación de ciertos* ε*-productos tensoriales.* Actas de las Jornadas Matematicas Hispano Lusas. Murcia (1985), 191-203.

[11] M. Florencio, P. J. Paúl: *La propiedad AK en ciertos espacios de suecsiones vectoriales.* Proc. Eleventh Spanish-Portuguese Conference on Mathematics. **1**, 197-203, Dep. Mat. Univ. Extremadura, **18**, (1987).

[12] M. Florencio, P. J. Paúl: *Barrelledness conditions on vector valued sequence spaces* Arch. Math. (Basel) Vol. **48** (1987), 153-164.

[13] M. Florencio, P. J. Paúl: *A note on* λ*-multiplier convergente series.* Casopis Pest. Mat., **113** (1988), 421-428.

[14] A. Pietsch: *Nuclear locally convex spaces.* Springer-Verlag, Berlin, Heidelberg, New York (1972).

[15] R. C. Rosier: Dual spaces of certain vector sequence spaces. Pacific J. Math. **46**, 487-501.

[16] M. Valdivia Ureña: *Topics in Locally Convex Spaces.* Amsterdam-New York-Oxford (1982).