Rev. Real Academia de Ciencias. Zaragoza. **59:** 71–77, (2004).

Unification of a class of trilateral generating functions

Manik Chandra Mukherjee

Netaji Nagar Vidyamandir

Calcutta 700 092. India

Abstract

In this paper we have proved a general theorem in connection with the unification of a class of trilateral generating functions of certain special functions. Some particular cases of interest are also pointed out.

Keywords: Unification, Generating functions, Group-theoretic method.
MSC: 33A65

1 Introduction

Theories in connection with the unification of generating functions are of greater importance in the study of special functions. In this direction attempts have been made by some researchers [1–8]. The aim at presenting this article is to state and prove a theorem on the unification of a class of trilateral generating functions of certain special functions. In fact we have proved the following theorem.

Theorem 1 *For a set of functions $\{S_n(x)|n \geq 0\}$ generated by*

$$\sum_{n=0}^{\infty} A_n(m) S_{n+m}(x) t^n = \frac{f(x,t)}{g(x,t)^m} S_m(h(x,t)) \tag{1}$$

where m is a non-negative integer, $A_n(m)$ are arbitrary constants and f, g, h are arbitrarily chosen functions of x and t, let

$$F(x,y,t) = \sum_{n=0}^{\infty} a_n(m) S_{n+m}(x) g_n(t) t^n,$$

then the following trilateral generating relation holds:

$$\sum_{n=0}^{\infty} S_{n+m}(x) \sigma_n(y,z) t^n = \frac{f(x,t)}{g(x,t)^m} F(h(x,t), y, zt/g(x,t))$$

where

$$\sigma_n(y,z)=\sum_{k=0}^{n} a_k A_{n-k}(m+k) g_k(y) z^k.$$

Proof.

$$\begin{aligned}
&\sum_{n=0}^{\infty} S_{n+m}(x)\sigma_n(y,z)t^n\\
&=\sum_{n=0}^{\infty} S_{n+m}(x)\left(\sum_{k=0}^{n} a_k A_{n-k}(m+k)g_k(y)z^k\right)t^n\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} S_{n+m+k}(x)\sigma_k(x)A_n(m+k)g_k(y)z^k t^{n+k}\\
&=\sum_{k=0}^{\infty} a_k g_k(y)(zt)^k \frac{f(x,t)}{g(x,t)^m} S_{m+k}(h(x,t))\\
&=\frac{f(x,t)}{g(x,t)^m}\sum_{k=0}^{\infty} a_k S_{m+k}(h(x,t)) g_k(y)\left(\frac{zt}{g(x,t)}\right)^k\\
&=\frac{f(x,t)}{g(x,t)^m} F\left(h(x,t),y,\frac{zt}{g(x,t)}\right). \qquad \blacksquare
\end{aligned}$$

The result given in [9] can be obtained directly from the theorem. Indeed, by putting $m=0$ in the above theorem, we get

$$F(x,y,t)=\sum_{n=0}^{\infty} a_n S_n(x) g_n(y) t^n,$$

then,

$$\sum_{n=0}^{\infty} S_n(x)\sigma_n(y,z)t^n = f(x,t)F\left(h(x,t),y,\frac{zt}{g(x,t)}\right),$$

where

$$\sigma_n(y,z)=\sum_{k=0}^{n} a_k A_{n-k}(k) g_k(y) z^k$$

which is found derived in [9].

2 Applications

We now apply our theorem in the derivation of generating functions of various special functions

2.1 On Hermite polynomials

We first consider the following generating relation involving Hermite polynomials [9]

$$\sum_{n=0}^{\infty} H_{n+m}\frac{t^n}{n!} = \exp(2xt-t^2)H_m(x-t). \tag{2}$$

Now it is evident that the relation (2) being of the form (1), one can easily obtain the following result with the help of theorem 1.

Proposition 1 *If there exists a generating relation of the form*

$$F(x,y,t)=\sum_{n=0}^{\infty}a_nH_{n+m}(x)g_n(y)\frac{t^n}{n!},$$

then

$$\sum_{n=0}^{\infty}H_{n+m}(x)\sigma_n(y,z)\frac{t^n}{n!}=\exp(2xt-t^2)F((x-t),y,zt),$$

where

$$\sigma_n(y,z)=\sum_{k=0}^{n}\binom{n}{k}a_kg_k(y)z^k.$$

2.2 On Laguerre polynomials

Next we consider the following generating relation involving Laguerre polynomials [10,11]

$$\sum_{n=0}^{\infty}\binom{m+n}{n}L_{n+m}^{\alpha}(x)t^n=(1-t)^{-1-\alpha-m}\exp\left(\frac{-xt}{1-t}\right)L_n^{\alpha}\left(\frac{x}{1-t}\right). \tag{3}$$

The relation (3) being of the form (1), one can state the following result with the help of theorem 1.

Proposition 2 *If*

$$F(x,y,t)=\sum_{n=0}^{\infty}a_nL_{n+m}^{\alpha}(x)g_n(y)t^n,$$

then

$$\sum_{n=0}^{\infty}L_{n+m}^{\alpha}(x)\sigma_n(y,z)t^n=(1-t)^{-1-\alpha-m}\exp\left(\frac{-xt}{1-t}\right)F\left(\frac{x}{1-t},y,\frac{zt}{1-t}\right),$$

where

$$\sigma_n(y,z)=\sum_{k=0}^{n}\binom{m+n}{m+k}a_kg_k(y)z^k.$$

2.3 On modified Laguerre polynomials

Next we consider the following generating relation involving modified Laguerre polynomials [10]

$$\exp(xt)(1-t)^{-\beta-m}f_m^{\beta}[x(1-t)]=\sum_{n=0}^{\infty}\frac{(m+1)_n}{n!}f_{m+n}^{\beta}(x)t^n. \tag{4}$$

The relation (4) being of the form (1), one can obtain the following result with the help of theorem 1.

Proposition 3 *If there exist a generating relation of the form*

$$F(x,y,t)=\sum_{n=0}^{\infty}a_nf_{m+n}^{\beta}(x)g_n(y)t^n,$$

then

$$\sum_{n=0}^{\infty} f_{n+m}^{\beta}(x)\sigma_n(y,z)t^n = \exp(xt)(1-t)^{-\beta-m} F\left(x(1-t), y, \frac{zt}{1-t}\right),$$

where

$$\sigma_n(y,z) = \sum_{k=0}^{n} \frac{(m+k+1)_{n-k}}{(n-k)!} a_k g_k(y) z^k.$$

2.4 On ultra spherical polynomials

Next we consider the following generating relation involving ultra spherical polynomials [11]

$$\sum_{n=0}^{\infty} \binom{m+n}{n} P_{n+m}^{\lambda}(x)t^n = \rho^{-m-2\lambda} P_m^{\lambda}\left(\frac{x-t}{\rho}\right), \tag{5}$$

where $\rho = (1-2xt+t^2)^{1/2}$.

The relation (5) being of the form (1), one can state the following result with the help of theorem 1.

Proposition 4 *If*

$$F(x,y,t) = \sum_{n=0}^{\infty} a_n P_{n+m}^{\lambda}(x) g_n(y) t^n,$$

then

$$\sum_{n=0}^{\infty} P_{n+m}^{\lambda}(x)\sigma_n(y,z)t^n = \rho^{-2\lambda-m}\, F\left(\frac{x-t}{\rho}, y, \frac{zt}{\rho}\right),$$

where

$$\sigma_n(y,z) = \sum_{k=0}^{n} \binom{m+n}{m+k} a_k\, g_k(y)\, z^k.$$

2.5 On modified Jacobi polynomials

Next we consider the following generating function involving modified Jacobi polynomials [12].

$$\begin{aligned} &\sum_{n=0}^{\infty} \binom{m+n}{n} P_{n+m}^{(\alpha-m-n,\beta-m-n)}(x)t^n \\ &= \left(1+\tfrac{1}{2}(x+1)\right)^{\alpha-m} \left(1+\tfrac{1}{2}(x-1)\right)^{\alpha-n} P_m^{(\alpha-m,\beta-n)}\left(x+\tfrac{1}{2}(x^2-1)t\right). \end{aligned} \tag{6}$$

The relation (6) being of the form (1), one can easily state the following result with the help of theorem 1.

Proposition 5 *If*

$$F(x,y,t) = \sum_{n=0}^{\infty} a_n P_{n+m}^{(\alpha-m-n,\beta-m-n)}\, g_n(y)\, t^n,$$

then

$$\sum_{n=0}^{\infty} P_{n+m}^{(\alpha-m-n,\beta-m-n)}(x)\,\sigma_n(y,z)\,t^n$$
$$= \left(1+\tfrac{1}{2}(x+1)\right)^{\alpha-m}\left(1+\tfrac{1}{2}(x-1)\right)^{\alpha-n}$$
$$\times F\left(1+\tfrac{1}{2}(x^2-1)t,\, y,\, \frac{zt}{\left(1+\frac{1}{2}(x+1)t\right)\left(1+\frac{1}{2}(x-1)t\right)}\right),$$

where

$$\sigma_n(y,z) = \sum_{k=0}^{n} \binom{m+n}{m+k} a_k\, g_k(y)\, z^k.$$

If instead of (6) we consider the following generating realtion [13]

$$\sum_{n=0}^{\infty} \frac{(r+1)_n}{n!} P_{n+r}^{(k-r-n,\beta)}(x)\,t^n$$
$$= (1+t)^{k-r}\left(1+\tfrac{1}{2}(1-x)t\right)^{-1-\beta-k} P_r^{(k-r,\beta)}\left(\frac{x-\frac{1}{2}(1-x)t}{1+\frac{1}{2}(1-x)t}\right), \quad (7)$$

then we get the following result

Proposition 6 *If*

$$F(x,y,t) = \sum_{n=0}^{\infty} a_n P_{n+r}^{(k-r-n,\beta)}\, g_n(y)\, t^n,$$

then

$$\sum_{n=0}^{\infty} P_{n+r}^{(k-r-n,\beta)}(x)\,\sigma_n(y,z)\,t^n$$
$$= (1+t)^{k-r}\left(1+\tfrac{1}{2}(1-x)t\right)^{-1-\beta-k} \times F\left(\frac{x-\frac{1}{2}(1-x)t}{1+\frac{1}{2}(1-x)t}, y, \frac{zt}{1+t}\right),$$

where

$$\sigma_n(y,z) = \sum_{k=0}^{n} \frac{(r+k+1)_{n-k}}{(n-k)!} a_k\, g_k(y)\, z^k.$$

2.6 On Bessel functions

Next we consider the following generating relation involving Bessel functions [14].

$$\sum_{n=0}^{\infty} J_{n+m}(x)\frac{t^n}{n!} = \left(1-\frac{2t}{x}\right)^{-m/2} J_m(\sqrt{x^2-2xt}). \quad (8)$$

The relation (8) being of the form (1), one can easily state the following result with the help of theorem 1.

Proposition 7 *If*

$$F(x,y,t) = \sum_{n=0}^{\infty} a_n J_{n+m}(x)\, g_n(y)\frac{t^n}{n!},$$

then

$$\sum_{n=0}^{\infty} J_{n+m}(x)\,\sigma_n(y,z)\frac{t^n}{n!} = \left(1-\frac{2t}{x}\right)^{-m/2} F\left(\sqrt{x^2-2xt}, y, \frac{xtz}{\sqrt{x^2-2xt}}\right),$$

where

$$\sigma_n(y,z) = \sum_{k=0}^{n} \binom{n}{k} a_k\, g_k(y)\, z^k.$$

2.7 On Bessel polynomials

Next we consider the following generating relation involving Bessel polynomials [10].

$$\sum_{n=0}^{\infty} Y_{n+m}(x)\frac{t^n}{n!} = (1-2xt)^{-(m+1)/2} \exp\left(\frac{1-\sqrt{1-2xt}}{x}\right) Y_m\left(\frac{x}{\sqrt{1-2xt}}\right). \tag{9}$$

The relation (9) being of the form (1), one can easily state the following result with the help of theorem 1.

Proposition 8 *If*

$$F(x,y,t) = \sum_{n=0}^{\infty} a_n Y_{n+m}(x)\, g_n(y)\frac{t^n}{n!},$$

then

$$\sum_{n=0}^{\infty} J_{n+m}(x)\,\sigma_n(y,z)\frac{t^n}{n!}$$

$$= (1-2xt)^{-(m+1)/2} \exp\left(\frac{1-\sqrt{1-2xt}}{x}\right) F\left(\frac{x}{\sqrt{1-2xt}}, y, \left(\frac{zt}{\sqrt{1-2xt}}\right)\right),$$

where

$$\sigma_n(y,z) = \sum_{k=0}^{n} \binom{n}{k} a_k\, g_k(y)\, z^k.$$

2.8 On modified generalised Bessel polynomiasl

Next we consider the following generating relation involving modified generalised Bessel polynomiasl [15,16].

$$\sum_{n=0}^{\infty} \frac{\beta^n}{n!} Y_{n+m}^{\alpha-m-n}(x) t^n = (1-xt)^{1-\alpha} \exp(\beta t) Y_m^{\alpha-m}\left(\frac{x}{1-xt}\right). \tag{10}$$

The relation (10) being of the form (1), one can easily state the following result with the help of theorem 1.

Proposition 9 *If*

$$F(x,y,t) = \sum_{n=0}^{\infty} a_n Y_{n+m}^{\alpha-m-n}(x)\, g_n(y)\, t^n,$$

then

$$\sum_{n=0}^{\infty} Y_{n+m}^{\alpha-m-n}(x)\,\sigma_n(y,z)\, t^n = (1-xt)^{1-\alpha} \exp(\beta t) F\left(\frac{x}{1-xt}, y, zt\right),$$

where

$$\sigma_n(y,z) = \sum_{k=0}^{n} \frac{\beta^{n-k}}{(n-k)!} a_k\, g_k(y)\, z^k.$$

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