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About Gevrey Hypoellipticity of pseudodifferential operators associated to the class of Hörmander symbols of Gevrey type

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Abstract

Treves, F., [15], proved the hypoellipticity of some regular operators. Later, Hörmander, L.,[11] Durand, M., [7], and Oleinik, O.A. & Radkevic, E. V., [13], considered independently second order operators which are sum of squares of real vector fields; they gave criteria of hypoellipticity. Derridj, M. & Zuily, C., [5], studied their Gevrey regularity. Moreover, Baouendi, M. S.& Goulaouic, C., [1] and then Derridj, M. & Zuily, C., [6], obtained some results of analyticity and Gevrey regularity for some degenerate elliptic operators. Here, we aim to prove Gevrey hypoellipticity of pseudodifferential operators associated to Hörmander symbols of Gevrey type.

Key words: Gevrey class, hypoellipticity, pseudodifferential operators, symbols of Gevrey.

MSC: 47 G 30.

Notations

In the sequel, we will use the following conventions:

- $\mathbb{R}^n$ is the n-dimensional vector space in which every point x is defined by its n coordinates $x_1, x_2, \ldots, x_n$.
- Ω denote, unless specified otherwise, an open subset of $\mathbb{R}^n$.
- s is a real greater than or equal to 1.
- Δ is the diagonal of $\Omega \times \Omega$.
- dx stands for the hypervolume element $dx_1 dx_2 \ldots dx_n$.
- $x + y$ is the element the coordinates of which are $x_1 + y_1, x_2 + y_1, \ldots, x_n + y_n$.

- $x \geq 0$ means $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$.
- $x \geq y$ means $x_1 \geq y_1, x_2 \geq y_2, \ldots, x_n \geq y_n$.
- The order of a system $p = \{p_1, p_2, \ldots ., p_n\}$,of integers is $|p| = p_1 + p_2 + \ldots + p_n$.
- $\mathcal{E}(\mathbb{R}^n)$ is the space of infinitely differentiable functions on $\mathbb{R}^n$; $\mathcal{E}'(\mathbb{R}^n)$ is its topological dual.
- $\mathcal{D}(\Omega)$ is the space of infinitely differentiable functions on $\mathbb{R}^n$, with compact support in Ω; $\mathcal{D}'(\Omega)$ is its topological dual.
- $\widehat{u}$ is the Fourier transform of u.
- $\overline{U}$ is the closure of U.
- $[x]$ is the integer part of the real number x.

1 Pseudodifferential operators of class s

To settle the notion of pseudodifferential operators of Gevrey class, we first recall that of Gevrey function then we restore that of symbol of the same class. To the latter, we associate the introduction of above mentioned operators.

Definition 1 *A real function f in $C^\infty(\Omega)$ is said of Gevrey class with order s if, for any compact subset $K \subset \Omega$*

$$\forall \alpha \in \mathbb{N}^n \, \|D^\alpha f\| \leq C^{|\alpha|+1} \left(|\alpha|!\right)^s . \tag{1}$$

In (1), the choice of the norm does not matter (see [2,9,10]). However, we will use the sup-norm in this paper.

Definition 2 *Let $m \in \mathbb{R}$, ρ and δ two real numbers such that $0 \leq \delta < \rho \leq 1$. We say that a real function $a = a(z, x)$ in $C^\infty(\Omega \times \mathbb{R}^n)$, is a symbol (or amplitude) of Gevrey type with class s and of type(m, ρ, δ) on Ω if and only if: for any compact subset $K \subset \Omega$, there exist C_0, C_1, R positive constants such that*

$$\sup_{x \in K} \left| D_\xi^\alpha D_x^\beta a(x, \xi) \right| \leq C_0 C_1^{|\alpha+\beta|} \left(|\alpha|!\right)^s \left(|\beta|!\right)^s \left(1 + |\xi|^2\right)^{m - \rho|\alpha| + \delta|\beta|} \tag{2}$$

for any $\xi \in \mathbb{R}^n$ with, $|\xi| \geq R, |\alpha|^s$, α and $\beta \in \mathbb{N}^n$.

We denote by ${}_{\rho,\delta}S^m_{G^s}(\Omega \times \mathbb{R}^n)$ the set of such symbols.

To every symbol a of ${}_{\rho,\delta}S^m_{G^s}(\Omega \times \mathbb{R}^n)$, we associate an operator $a(x, D) = A$ in the following manner:

For any $\varphi \in \mathcal{D}(\Omega)$ and any $x \in \Omega$, we set

$$(A\varphi)(x) = a(x, D)\varphi(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} a(x, \xi) \, \widehat{\varphi}(\xi) \, d\xi,$$

or, equivalently, by making explicit the Fourier transform

$$(A\varphi)(x) = a(x,D)\varphi(x) = (2\pi)^{-n} \iint e^{i<x-y,\xi>} a(x,\xi)\ \varphi(y)\ dy\ d\xi.$$

We write $A = Opa$ and say that A belongs to $Op_{\rho,\delta}S^m_{G^s}(\Omega \times \mathbb{R}^n)$.

The distribution-kernel T of $a(x,D)$ is defined by

$$T(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\xi}\ a(x,\xi)\ d\xi.$$

The symbol of the pseudodifferential operator is often given by formal series, called asymptotic expansion (see, for instance, [2], [12]):

$$\sum_{j=0}^{\infty} a_j(x,\xi), \quad a_j(x,\xi) \in C^\infty(\Omega \times \mathbb{R}^n), \quad j = 0,1,\dots \tag{3}$$

In this case, we assume the following hypothesis:

For any compact subset K de Ω, there exist three positive constants C_0, C_1, R such that

$$\sup_{x\in K} \left|\partial_x^\beta \partial_\xi^\alpha a_j(x,\xi)\right| \le C_0 C_1^{|\alpha+\beta|} \left(|\beta|+j\right)!^s \left(1+|\xi|\right)^{m-\delta|\alpha|+\delta j}, \quad |\xi| \ge R\left(j+|\alpha|\right)^s, \tag{4}$$

then, we can construct the genuine Gevrey symbol of the formal series (3) as in the analytical case by F. Treves.

Afterwards, we make use of a sequence of functions $\varphi_j(x,\xi) \in C^\infty(\mathbb{R}^n)$, $j = 0,1,\dots$, satisfying

$$0 \le \varphi_j(\xi) \le 1, \quad \text{for any} \quad \xi \in \mathbb{R}^n,$$

$$\varphi_j(\xi) = \begin{cases} 0, & \text{if} \quad |\xi| < 2L\max(j^s,1), \\ 1, & \text{if} \quad |\xi| > 2L\max(j^s,1), \end{cases}$$

$$|D^\alpha \varphi_j| \le \left(\frac{C}{Lj^{s-1}}\right)^{|\alpha|}, \quad \text{if} \quad |\alpha| \le 2j,$$

where L is a suitable positive constant.

If we take $L > 2^{s-1}R$, in which R is the number given in (4), and set:

$$a(x,\xi) = \sum_{j=0}^{\infty} \varphi_j(\xi)\ a_j(x,\xi),$$

then $a \in {}_{\rho,\delta}S^m_{G^s}(\Omega \times \mathbb{R}^n)$. The proof is similar to that given by Treves in [16].

Definition 3 *A pseudodifferential operator A of $Op_{\rho,\delta}S^m_{G^s}(\Omega\times\mathbb{R}^n)$ is Gevrey-pseudolocal with class s, if for any $u\in\mathcal{E}'(\Omega)$, Au is Gevrey type of class s in any open set where u is so.*

That is:

$$Gevrey-SuppsingAu\subset Gevrey-Suppsingu.$$

In this context, we prove

Theorem 4 *Let a be a symbol of ${}_{\rho,\delta}S^m_{G^s}(\Omega\times\mathbb{R}^n)$, $0<\delta<\rho<1$, $m\leq\rho-\delta$ satisfying the constrains (2) for multi-indices α et $\beta\in\mathbb{N}^n$ such that*

$$|\alpha|,|\beta|\leq[n/2]+1.$$

Then,

1. *The kernel T of $a(x,D)$ is a Gevrey function of class s on $\Omega\times\Omega\backslash\Delta$.*
2. *$a(x,D)$ is Gevrey-pseudolocal of class s.*

Proof. We have

$$T(x,y)=(2\pi)^{-n}\int_{\mathbb{R}^n}e^{i(x-y)\xi}\ a(x,\xi)\ d\xi.$$

To elucidate (1), one must prove that $(x-y)^\alpha T=F$ is a Gevrey function of class s on $\Omega\times\Omega$. First, let us observe that

$$F(x,y)=\int e^{i(x-y)\xi}\left(-D_\xi\right)^\alpha a(x,\xi)\ d\xi.$$

Then

$$D_x^\beta F(x,y)=\int D_x^\beta\left(e^{i(x-y)\xi}\left(-D_\xi\right)^\alpha a(x,\xi)\right)\ d\xi.$$

Since

$$\left|D_x^\beta\left(e^{i(x-y)\xi}\left(-D_\xi\right)^\alpha a(x,\xi)\right)\right|\leq CC_0C_1^{|\alpha+\beta|}\left(|\alpha|!\right)^s\left(1+|\xi|^2\right)^{m-\rho|\alpha|+2\delta|\beta|},$$

where C_0 and C_1 are the constants involved in (2) whereas C is positive constant not depending on α and β, it is allowable to get

$$\left|D_x^\beta F(x,y)\right|=C'C_0C_1^{|\alpha+\beta|}\left(|\alpha|!\right)^s,$$

which means that F satisfies (1). Then, it is of Gevrey type of class s with respect to x. Of course, the same argument shows that F is of Gevrey type of class s with respect to y; the first claim of the theorem follows.

Regarding the second one, we prove that

$$\forall u \in \mathcal{E}'(\Omega), Gevrey - Suppsing\ Tu \subset Gevrey - Suppsing\ u.$$

To this end, consider an open set $U \subset \Omega$ and $\varphi \in \mathcal{D}'(\Omega)$ such that $\varphi \equiv 1$ on a neighbourhood of $\overline{U}$. Let $u \in \mathcal{E}'(\Omega)$ of Gevrey type of class s on U. We have:

$$Tu = T(\varphi u) + T((1-\varphi)u).$$

Clearly, φu is Gevrey type of class s on U. Since T is separately regular, it follows that $T(\varphi u) \in \mathcal{E}(\Omega)$.

Moreover, it holds

$$T((1-\varphi)u)(x) = \int T(x,y)(1-\varphi(y))\ u(y)\ dy$$

For $x \in U$ and $y \in Supp(1-\varphi)$ (the complementary of some neighbourhood of the diagonal Δ where T is a Gevrey function of class s). Therefore, differentiating and making use of the Lebesgue dominated convergence theorem, we find that $T(1-\varphi)u)$ is of Gevrey type of class s in V, whence the claim.

2 Gevrey-Hypoelliptic operators

The operators, such as elliptic ones, are said to be Gevrey-hypoelliptic. More explicitly (see [12]):

Definition 5 *A linear operator $A : \mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega)$ is said Gevrey-hypoelliptic if, for any distribution u in $\mathcal{E}'(\Omega)$ and any open set $U \subset \Omega$, the restriction u/U of u to U is of Gevrey type of class s whenever the restriction Au/U of Au to U is.*

In other words

Definition 6 *An operator A is said Gevrey-hypoelliptic if it satisfies*

$$Gevrey - Suppsing\ u = Gevrey - Suppsing\ Au. \qquad (5)$$

One of the most essential consequences of the construction of a parametrization is described by this

Theorem 7 *If $A \in Op_{\rho,\delta}S^m_{G^s}(\Omega \times \mathbb{R}^n)$ is an eigen-support, elliptic with $1 > \rho > \delta > 0$, $m \leq \rho - \delta - n(1+\delta/2)$, then it is Gevrey-hypoelliptic.*

Proof. Let B a two sided parametrization of A in $Op_{\rho,\delta}S^{-m}_{G^s}(\Omega \times \mathbb{R}^n)$. Then $u = B(Au)$ (mod. $S^{-\infty}_{G^s}(\Omega \times \mathbb{R}^n)$). The pseudo-local Gevrey property of B yields

$$Gevrey - Suppsing\ B(Au) \subset Gevrey - Suppsing\ Au,$$

that is,

$$Gevrey - Suppsing\ u \subset Gevrey - Suppsing\ Au.$$

The converse inclusion follows from the Gevrey-pseudolocal property of A; whence (5).

The following theorem extends this property to pseudodifferential operators associated to amplitudes $a \in_{\rho,\delta,\theta} S^m_{G^s}(\Omega \times \mathbb{R}^n)$, characterised by this

Definition 8 *Let ρ, δ and θ be three real such $0 \leq \delta, \theta < \rho \leq 1$. With the notations of definition (1.2), we say that a function $a \in_{\rho,\delta,\theta} S^m_{G^s}(\Omega \times \mathbb{R}^n)$ if, for any compact subset K in $\Omega \times \Omega$, we have*

$$\left| D^\alpha_\xi D^\beta_\xi D^\gamma_y a(x,\xi) \right| \leq C_0 C_1^{|\alpha+\beta+\theta|} (|\alpha|!)^s (|\beta|!)^s (|\gamma|!)^s \left(1+|\xi|^2\right)^{m-\rho|\alpha|+\delta|\beta|+\theta|\gamma|}. \tag{6}$$

Such operators A are defined by

$$Au(x) = (2\pi)^{-n} \iint e^{i<x-y,\xi>} a(x,y,\xi) u(y) dy d\xi. \tag{7}$$

We assume they have eigen supports, which allow them, as in the C^∞ case, to act continuously from $\mathcal{E}(\Omega)$ into itself (see, for instance, [3], [14]). Let us state

Theorem 9 *Let A be a pseudodifferential operator as in (7) associated to the symbol $D^\beta_y a(x,\xi)$ with*

$$1 > \rho > \max(\delta,\theta) > 0, m \leq \rho - \max(\delta,\theta) - n(1+\max(\delta,\theta)/2). \tag{8}$$

Then its kernel T is a function with class s on $\Omega \times \Omega \backslash \Delta$. Moreover, for any pair of disjoint compact subsets K, L of Ω and any integer r, there exists an integer k and some positive constants B and C such that, for every multi-index α, we have

$$|D^\alpha_x T, \varphi \otimes \psi| \leq C B^{|\alpha+\beta+\theta|} (\alpha!)^s (\beta!)^s \sup_{|\beta|\leq k} \sqrt{\int |D^\beta \psi(y)|^2 dy} \sqrt{\int |\varphi(x)|^2 dx}. \tag{9}$$

Proof. It is based upon two lemmas. First, we recall Frieberg's lemma [8]:

Lemma 10 *Let U a bounded open set in $\mathbb{R}^n$ and $N = [n/2] + 1$, then, for any compact subset K of U and any number $h(0 < h < 1)$, such that $K_h \subset U$, there exists a positive constant C such that,*

$$h^N \sup_{x \in K} |u(x)| \leq C \sum_{|\alpha|} h^{|\alpha|} \left(\int_{K_h} |D^\alpha u(x)|^2 \, dx \right)^{1/2},$$

where $K_h = \{x \in \mathbb{R}^n / dist(x, K) \leq h\}$ (the set of elements of $\mathbb{R}^n$ the distance of which to K is less than or equal to h).

The second lemma follows from lemma 2.2 by L. Hörmander [11]:

Lemma 11 *Let K be a compact subset of $\mathbb{R}^n$ and h a positive real number, there exists a sequence of functions $(\psi_j)_j$ in $C_0^\infty(K_h)$ such that $\psi_j = 1$ on K and ,*

$$|D^\alpha \psi_j(x)| \leq \left(\frac{A}{h} \right)^{js} (\alpha!)^s, \quad |\alpha| \leq j.$$

where C and A are positive constants independent of j, a and h (in case h is bounded).

Let us resume the proof of our theorem. First, we point out that condition (9) implies that T is of Gevrey class with order s in x on $\Omega \times \Omega \backslash \Delta$. By transposition and using (9) for the kernel $T_{^tA}$ of the transpose ^tA, we deduce that T is of Gevrey class with order s in y on $\Omega \times \Omega \backslash \Delta$. Since, thanks to theorem (8) in [4], T is C^∞ on $\Omega \times \Omega \backslash \Delta$, we find that T is Gevrey class with order s in (x, y) on $\Omega \times \Omega \backslash \Delta$. So, this amounts to prove (9). The same argument, up to some constants, as under taken in [3], elucidates this point.

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