Monografías de la Real Academia de Ciencias de Zaragoza **38:** 137–145, (2012)
ISSN 1132-6360

An inverse diffusion problem in a degenerate parabolic equation

Jacques Tort *

This paper is dedicated to Prof. Monique Madaune Tort.

Abstract

In this paper, we prove a Lipschitz stability result for an inverse problem that consists in determining a constant in the diffusion term of some degenerate one-dimensional linear parabolic equation. Our proof is based on the results investigated in [5] and presented during the Eleventh International Conference Zaragoza-Pau.
Keywords: inverse problems, Lipschitz stability, degenerate parabolic equations, weak maximum principle.
AMSCode: 35R30, 35K65, 35B45, 26D10, 35B50

1 Introduction

Inverse problems for nondegenerate parabolic equations have been largely studied in many articles and books. The method for Lipschitz stability results is due to both Puel-Yamamoto in [11] for the wave equation and Imanuvilov-Yamamoto in [10] for parabolic equations. It is based on a global Carleman estimate corresponding to the considered equation. This method has been applied to various equations and inverse problems (see references in [5]). In particular, the determination of diffusion coefficients has been studied in [13]. For non continuous diffusion coefficients, the problem has been solved in [1, 2]. As for degenerate linear equations, very few results about inverse problems are known. To our knowledge, the first papers dealing with Lipschitz stability results for such equations are [5, 12] and concern inverse source problems. We want to apply the results therein to some inverse diffusion problem.

Let us now go deeper into details and describe the model. Let $\alpha \in [0, 2)$ and consider the following initial-boundary value problem:

*Institut de Mathématiques de Toulouse (IMT) - U.M.R. C.N.R.S. 5219 - Université Paul Sabatier Toulouse III, 118 Route de Narbonne, F-31062 Toulouse Cedex 9, France.
email:jacques.tort@math.univ-toulouse.fr

$$
\begin{cases}
u_t - (Kx^\alpha u_x)_x = h & (t,x) \in (0,T) \times (0,1)\,, \\
u\,(t,1) = 0 & t \in (0,T)\,, \\
\text{and} \begin{cases} u\,(t,0) = 0 & \text{for} \quad 0 \le \alpha < 1 \\ (x^\alpha u_x)\,(t,0) = 0 & \text{for} \quad 1 \le \alpha < 2 \end{cases} & t \in (0,T)\,, \\
u\,(0,x) = u_0(x) & x \in (0,1)\,.
\end{cases} \tag{1}
$$

As we recall in section 2, the functional framework in which problem (1) is well posed slightly changes following the fact that $0 \le \alpha < 1$ and $1 \le \alpha < 2$. In particular we do not consider the same boundary condition at $x = 0$.

As a motivation, let us mention that the study of linear equations like (1) is a first step to the study of the non linear one-dimensional Budyko-Sellers climate model. This climate model aims at understanding the effects of many parameters (such as for instance greenhouse gazes, albedo or advection fluxes) on the ice covering on the Earth. Since the nineties, the questions of well-posedness of the model, long-time behaviour of solutions and many others have been largely studied by Diaz, Hetzer and connected researchers (see [8, 9] and references therein). As for controllability issues, which are close to inverse problem ones, very few results are known about the Budyko-Sellers model (see [9]). For the linear equation (1), a recent paper by Cannarsa-Martinez-Vancostenoble [4] gives a positive answer to the question of null-controllability by a locally distributed control. Finally, let us mention that inverse problems for the Sellers model are going to be considered in a forthcoming paper.

Now, let us turn to technical assumptions for our inverse problem. Let $t_0 \in (0,T)$ and set:

$$
T' := \frac{T + t_0}{2}.
$$

The topic of this paper can be stated as follows: is it possible to recover the constant K in (1) from the knowledge of $(x^\alpha u_x)_x\,(T',.)$ and a boundary observation $u_{tx}\,(.,1)\,\big|_{(t_0,T)}$? For this purpose, we assume below that the constant K to be determined belongs to some bounded interval.

Assumption 1. *Set* $0 < K_0 < K_1$ *and assume* $K \in I := [K_0, K_1]$.

Before stating and proving our Lipschitz stability results, we investigate the well-posedness of (1).

2 Well-posedness of (1), regularity of solutions, maximum principle

2.1 Functional framework and regularity of solutions of (1)

The functional framework of such degenerate equations has been largely studied in the case $K \equiv 1$ (see [3] and references in [4, 5]). Obviously, since K is a positive constant, only the definition of the operator changes a little bit. For the reader convenience, we recall the main definitions. Fix $K \in [K_0, K_1]$. For $0 \le \alpha < 2$, let us define $H^1_\alpha\,(0,1)$ as

$$H^1_\alpha(0,1) := \left\{ u \in L^2(0,1) : x^{\frac{\alpha}{2}} u_x \in L^2(0,1) \right\}.$$

Of course, $H^1_\alpha(0,1)$, embedded with the weighted inner product

$$(u,v)_{H^1_\alpha(0,1)} := (u,v)_{L^2(0,1)} + \left(x^{\frac{\alpha}{2}} u_x, x^{\frac{\alpha}{2}} v_x \right)_{L^2(0,1)},$$

is a Hilbert space.

In the $\alpha \in [0,1)$ case, the elements of $H^1_\alpha(0,1)$ have a boundary value at both extremities $x = 0$ and $x = 1$. Therefore, one may define $H^1_{\alpha,0}(0,1)$ as

$$H^1_{\alpha,0}(0,1) := \left\{ u \in H^1_\alpha(0,1) : u(0) = u(1) = 0 \right\}.$$

Now, we define the unbounded operator $A : D(A) \subset L^2(0,1) \longrightarrow L^2(0,1)$ by

$$D(A) := \left\{ u \in H^1_{\alpha,0}(0,1) : x^\alpha u_x \in H^1(0,1) \right\},$$

$$\text{and } \forall u \in D(A), \quad Au := (Kx^\alpha u_x)_x .$$

In the other case ($1 \leq \alpha < 2$), the boundary value at $x = 0$ for an element of $H^1_\alpha(0,1)$ does not exist anymore (see also [5]). That is why we change the definition of $H^1_{\alpha,0}(0,1)$ into

$$H^1_{\alpha,0}(0,1) := \left\{ u \in H^1_\alpha(0,1) : u(1) = 0 \right\}.$$

Then, the unbounded operator $A : D(A) \subset L^2(0,1) \longrightarrow L^2(0,1)$ is defined by

$$D(A) := \left\{ u \in L^2(0,1) : x^\alpha u \in H^1_0(0,1), \, x^\alpha u_x \in H^1(0,1) \text{ and } (x^\alpha u_x)(0) = 0 \right\},$$

$$\text{and } \forall u \in D(A), \quad Au := (Kx^\alpha u_x)_x .$$

Moreover, it can be proved that $D(A)$ may also be characterized by (see [4]; see also [3] for a proof in a similar context):

$$D(A) = \left\{ u \in H^1_{\alpha,0}(0,1) : x^\alpha u_x \in H^1(0,1) \right\}.$$

We chose to use the same notations for $H^1_{\alpha,0}(0,1)$ and $D(A)$ in both $0 \leq \alpha < 1$ and $1 \leq \alpha < 2$ cases because these spaces have nearly the same properties. For instance, in both cases, one has the following result:

Lemma 1. *$(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup of contractions on $L^2(0,1)$. Moreover this semigroup is analytic.*

Proof. Since K is a positive constant, Lemma 1 can be proved as Lemma 2.1 in [5]. □

As a consequence of Lemma 1, the following standard well-posedness result holds:

Theorem 1. *1. For all $u_0 \in D(A)$, for all $h \in H^1(0,T;L^2(0,1))$, problem (1) has a unique solution*

$$u \in C([0,T]\,;D(A)) \cap C^1([0,T]\,;L^2(0,1)).$$

2. For all $u_0 \in L^2(0,1)$, for all $h \in L^2(0,T;L^2(0,1))$, problem (1) has a unique solution u satisfying, for all $\epsilon \in (0,T)$,

$$u \in L^2(\epsilon,T;D(A)) \cap H^1(\epsilon,T;L^2(0,1)).$$

In order to apply the results obtained in [5], we need to know some properties of the time-derivative of the solution of problem (1). As in [5], the following lemma holds:

Lemma 2. *Let $K \in I$, $u_0 \in D(A)$, and $h \in H^1(0,T;L^2(0,1))$. Setting $z := u_t$ where u is the solution of (1) given by Theorem 1, we get, for all $\epsilon > 0$, $z \in L^2(\epsilon,T;D(A)) \cap H^1(\epsilon,T;L^2(0,1))$ and is the weak solution of*

$$\begin{cases} z_t - (Kx^\alpha z_x)_x = h_t & (t,x) \in (0,T)\times(0,1)\,, \\ z(t,1) = 0 & t \in (0,T)\,, \\ \text{and} \begin{cases} z\,(t,0) = 0 & \text{for} \quad 0 \le \alpha < 1 \\ (x^\alpha z_x)\,(t,0) = 0 & \text{for} \quad 1 \le \alpha < 2 \end{cases} & t \in (0,T)\,, \\ z\,(0,x) = u_t(0,x) = (Kx^\alpha u_{0,x})_x(x) + h(0,x) & x \in (0,1)\,. \end{cases} \tag{2}$$

Proof. Such a result can be proved by the differential quotients method (see [5] for references). □

Corollary 1. *Let $K \in I$, $u_0 \in D(A^2)$, and $h \in H^2(0,T;L^2(0,1))$ such that $h(0,.) \in D(A)$. Setting $Z := z_t$ where z is the solution of (2) given by Theorem 1, we get, for all $\epsilon > 0$, $Z \in L^2(\epsilon,T;D(A)) \cap H^1(\epsilon,T;L^2(0,1))$ and is the weak solution of*

$$\begin{cases} Z_t - (Kx^\alpha Z_x)_x = h_{tt} & (t,x) \in (0,T)\times(0,1)\,, \\ Z(t,1) = 0 & t \in (0,T)\,, \\ \text{and} \begin{cases} Z\,(t,0) = 0 & \text{for} \quad 0 \le \alpha < 1 \\ (x^\alpha Z_x)\,(t,0) = 0 & \text{for} \quad 1 \le \alpha < 2 \end{cases} & t \in (0,T)\,, \\ Z\,(0,x) = z_t(0,x) = A^2u_0(x) + Ah(0,.)(x) + h_t(0,x) & x \in (0,1)\,. \end{cases} \tag{3}$$

Proof. Since $h \in H^2(0,T;L^2(0,1))$, $h_t \in H^1(0,T;L^2(0,1))$. Moreover, since $u_0 \in D(A^2)$, and $h(0,.) \in D(A)$, $(Kx^\alpha u_{0,x})_x + h(0,.) \in D(A)$. Therefore the solution z of (2) actually belongs to $C([0,T]\,;D(A)) \cap C^1([0,T];L^2(0,1))$, according to Theorem 1. Now, applying Lemma 2 to equation (2), we complete the proof of Corollary 1. □

2.2 Maximum principle

In order that the solution might satisfy some boundedness and positivity properties, we restrict the spaces of source terms and initial conditions.

Assumption 2. *Let $0 < \eta_0 < \eta_1$ be two real positive numbers. Define the set of source terms $\mathcal{H}$ as*

$$\mathcal{H} := \{h \in H^2(0,T;L^2(0,1)) : h(0,.) \in D(A), (x^\alpha h_x(0,.))_x \in L^\infty(0,1), \\ h_t \in L^\infty((0,T)\times(0,1)), \eta_0 \leq h \leq \eta_1, h_t \leq 0, h_{tt} \leq 0\}.$$

Moreover, define the set of initial data $\mathcal{U}$ as:

$$\mathcal{U} := \left\{u_0 \in D(A^2) : (x^\alpha(x^\alpha u_{0,x})_x)_x \in L^\infty(0,1) \ \ and\ (x^\alpha u_{0,x})_x \leq \frac{-\eta_1}{K_0}\right\}.$$

First, let us prove the following lemma:

Lemma 3. *Let $u \in H^1_{\alpha,0}(0,1)$. Then, for all $M \geq 0$, $(u-M)^+ := \sup(u-M,0) \in H^1_{\alpha,0}(0,1)$ and $(u+M)^- := \sup(-(u+M),0) \in H^1_{\alpha,0}(0,1)$.*

Proof. Let $u \in H^1_{\alpha,0}(0,1)$. Then, for all $\epsilon > 0$, $u \in H^1(\epsilon,1)$. Hence $(u-M)^+ \in H^1(\epsilon,1)$ (see [6], Proposition 6 pp. 934) and, for almost all $x \in (\epsilon,1)$

$$\left((u-M)^+\right)_x(x) = \begin{cases} u_x(x) & \text{if } (u-M)(x) > 0, \\ 0 & \text{if } (u-M)(x) \leq 0. \end{cases}$$

Then $\int_\epsilon^1 x^\alpha \left((u-M)^+\right)_x^2 = \int_{A_\epsilon} x^\alpha u_x^2$, where $A_\epsilon := \{x \in (\epsilon,1) : (u-M)(x) > 0\}$.
Since $\int_{A_\epsilon} x^\alpha u_x^2$ is bounded from above by $\int_0^1 x^\alpha u_x^2$ which does not depend on ϵ, we get (passing to the limit as ϵ converges to 0)

$$\int_0^1 x^\alpha \left((u-M)^+\right)_x^2 \leq \int_0^1 x^\alpha u_x^2 < +\infty.$$

Hence, $(u-M)^+ \in H^1_\alpha(0,1)$. Moreover, $(u-M)(1) = -M$, therefore $(u-M)^+(1) = 0$. The same result holds for the boundary value at 0 in the $\alpha \in [0,1)$ case. Thus, Lemma 3 is proved. □

Theorem 2. *Let $u_0 \in \mathcal{U}$ and $h \in \mathcal{H}$. Then, there exist two positive reals m and M, such that the solution u of (1) satisfies, for almost all $(t,x) \in (0,T)\times(0,1)$*

$$|(x^\alpha u_{tx})_x(t,x)| \leq M, \tag{4}$$

$$|(x^\alpha u_x)_x(t,x)| \geq m. \tag{5}$$

Proof. Let $u_0 \in \mathcal{U}$ and $h \in \mathcal{H}$. Then $u_0 \in D(A^2)$, $h \in H^2(0,T;L^2(0,1))$ and $h(0,.) \in D(A)$. According to Corollary 1, $z := u_t$ is the solution of (2) and $Z = z_t$ is the weak solution of (3). Let us first show that $Z \in L^\infty((0,T)\times(0,1))$. Denote

$$M_1 := K_1^2 \left\|\left(x^\alpha (x^\alpha u_{0,x})_x\right)_x\right\|_{L^\infty(0,1)} + K_1 \|(x^\alpha h_x(0,.))_x\|_{L^\infty(0,1)} + \|h_t(0,.)\|_{L^\infty(0,1)}.$$

First note that, using line 4 of (3), $\|Z(0,.)\|_{L^\infty(0,1)} \leq M_1$. Moreover, $h_{tt} \leq 0$, since $h \in \mathcal{H}$. Then, multiplying the equation satisfied by Z by $(Z - M_1)^+$, we can show, as in [7] (Remark 4 pp. 644), that

$$\|Z\|_{L^\infty((0,T)\times(0,1))} \leq M_1.$$

Yet, using (2), for almost all $(t,x) \in (0,T) \times (0,1)$,

$$(Kx^\alpha u_{tx})_x(t,x) = (Kx^\alpha z_x)_x(t,x) = z_t(t,x) - h_t(t,x).$$

Therefore,

$$\|(Kx^\alpha z_x)_x\|_{L^\infty((0,T)\times(0,1))} \leq \|Z\|_{L^\infty((0,T)\times(0,1))} + \|h_t\|_{L^\infty((0,T)\times(0,1))}.$$

Denote $N := \|h_t\|_{L^\infty((0,T)\times(0,1))}$. Therefore, setting $M = \dfrac{M_1 + N}{K_0}$, we achieve the proof of (4).

Let us now prove (5). Since $u_0 \in \mathcal{U}$, for almost all $x \in (0,1)$, $(x^\alpha u_{0,x})_x(x) \leq -\dfrac{\eta_1}{K_0}$, so that $Au_0(x) + h(0,x) \leq 0$. Moreover, $h_t \leq 0$. We multiply the equation satisfied by z by z^+, and as in Theorem 2 pp. 643 in [7], we prove that, for all $t \in [0,T]$ and for almost all $x \in (0,1)$, $z(t,x) \leq 0$. As a consequence, for almost all $(t,x) \in (0,T) \times (0,1)$,

$$(Kx^\alpha u_x)_x(t,x) = z(t,x) - h(t,x) \leq -\eta_0.$$

Setting $m := \dfrac{\eta_0}{K_1}$, we achieve the proof of (5). □

3 Lipschitz stability in the determination of the diffusion constant

As we said in the introduction, Lipschitz stability results are obtained thanks to global Carleman estimates. For degenerate equations like problem (1), a global Carleman estimate has been proved in [4] and next refined in [5] to treat some inverse source problem. Yet, we do not need to apply this estimate for solving the present inverse problem. Indeed, we can transform the determination of the diffusion constant into the determination of a source term in another degenerate equation like (1). As a consequence, we will use the stability estimate given in [5] in Theorem 3.1. In this step, the only change relies in the fact that [5] deals with the case of a diffusion constant K equal to 1, whereas we need the estimate for any diffusion constant belonging to I. Therefore, we first remind the context in [5] and explain why the case $K \neq 1$ is not more difficult providing the constant can be estimated by some uniform bounds.

3.1 A Lipschitz stability estimate for source terms in linear degenerate equations like (1)

Take $K \in I$ and consider the following initial-boundary value problem:

$$\begin{cases} u_t - (Kx^\alpha u_x)_x = g & (t,x) \in (0,T) \times (0,1), \\ u(t,1) = 0 & t \in (0,T), \\ \text{and } \begin{cases} u(t,0) = 0 & \text{for } 0 \leq \alpha < 1 \\ (x^\alpha u_x)(t,0) = 0 & \text{for } 1 \leq \alpha < 2 \end{cases} & t \in (0,T), \\ u(0,x) = u_0(x) & x \in (0,1). \end{cases} \tag{6}$$

The inverse problem that consists in determining g from standard measurements of u cannot be solved in the whole space $L^2(0,T;L^2(0,1))$. Therefore, a set of admissible source terms is considered in [5]. For the reader convenience, we remind its definition below.

Let $C_0 > 0$. We introduce the following condition on source terms:

$$\left|\frac{\partial g}{\partial t}(t,x)\right| \leq C_0 \left|g(T',x)\right| \text{ for almost all } (t,x) \in (t_0,T)\times(0,1). \tag{7}$$

Then, define $\mathcal{G}(C_0) := \left\{ g \in H^1(0,T;L^2(0,1)) \,|\, g \text{ satisfies } (7) \right\}$. One has the following stability estimate:

Theorem 3. *Let $\alpha \in [0,2)$ and $u_0 \in L^2(0,1)$. Assume $K \in I$. There exists $C = C(T, t_0, \alpha, C_0, K_0, K_1) > 0$ such that for all $g \in \mathcal{G}(C_0)$, the solution u of (1) satisfies:*

$$\|g\|^2_{L^2((t_0,T)\times(0,1))} \leq C\left(\left\|(x^\alpha u_x)_x(T',.)\right\|^2_{L^2(0,1)} + \|u_{tx}(.,1)\|^2_{L^2(t_0,T)}\right). \tag{8}$$

Proof. The proof can be carried out the same way as the proof of Theorem 3.1 in [5]. The fact that I is a bounded interval enables to have bounds and constants in estimates (for instance in the Carleman estimate associated with the operator $-K(x^\alpha u_x)_x$) that do not depend on K. This is essential for solving our inverse problem. □

3.2 A uniqueness and Lipschitz stability result

Now we are able to state our Lipschitz stability result.

Theorem 4. *Let $\alpha \in [0,2)$, $u_0 \in \mathcal{U}$ and $h \in \mathcal{H}$. There exists $C = C(T, t_0, \alpha, K_0, K_1, \eta_0, \eta_1, \|h_t\|_{L^\infty((0,T)\times(0,1))}, \|(x^\alpha h_x(0,.))_x\|_{L^\infty(0,1)})$ such that, for all $\lambda \in I$, for all $\mu \in I$, the corresponding solutions u_λ and u_μ of problem (1) satisfy*

$$|\lambda-\mu|^2 \leq C\left(\left\|\left(x^\alpha\left(u_\lambda-u_\mu\right)_x\right)_x(T',.)\right\|^2_{L^2(0,1)} + \left\|\left(u_\lambda-u_\mu\right)_{tx}(.,1)\right\|^2_{L^2(t_0,T)}\right).$$

Proof. Let $u_\lambda \in C([0,T];D(A)) \cap C^1([0,T];L^2(0,1))$ be the solution of

$$\begin{cases} u_{\lambda,t} - \left(\lambda x^\alpha u_{\lambda,x}\right)_x = h & (t,x) \in (0,T)\times(0,1), \\ u_\lambda(t,1) = 0 & t \in (0,T), \\ \text{and } \begin{cases} u_\lambda(t,0) = 0 & \text{for } 0 \leq \alpha < 1 \\ (x^\alpha u_{\lambda,x})(t,0) = 0 & \text{for } 1 \leq \alpha < 2 \end{cases} & t \in (0,T), \\ u_\lambda(0,x) = u_0(x) & x \in (0,1), \end{cases}$$

and $u_\mu \in C([0,T];D(A)) \cap C^1([0,T];L^2(0,1))$ be the solution of

$$\begin{cases} u_{\mu,t} - \left(\mu x^\alpha u_{\mu,x}\right)_x = h & (t,x) \in (0,T)\times(0,1), \\ u_\mu(t,1) = 0 & t \in (0,T), \\ \text{and } \begin{cases} u_\mu(t,0) = 0 & \text{for } 0 \leq \alpha < 1 \\ (x^\alpha u_{\mu,x})(t,0) = 0 & \text{for } 1 \leq \alpha < 2 \end{cases} & t \in (0,T), \\ u_\mu(0,x) = u_0(x) & x \in (0,1). \end{cases}$$

Set $w := u_\lambda - u_\mu$. Then w, which belongs to $C([0,T]\,;D(A))\cap C^1([0,T]\,;L^2(0,1))$, is the solution of

$$\begin{cases} w_t - (\lambda x^\alpha w_x)_x = (\lambda-\mu)(x^\alpha u_{\mu,x})_x & (t,x)\in(0,T)\times(0,1)\,,\\ w\,(t,1)=0 & t\in(0,T)\,,\\ \text{and} \begin{cases} w\,(t,0)=0 & \text{for } 0\le\alpha<1\\ (x^\alpha w_x)\,(t,0)=0 & \text{for } 1\le\alpha<2\end{cases} & t\in(0,T)\,,\\ w\,(0,x)=0 & x\in(0,1)\,.\end{cases} \tag{9}$$

Let us set $g := (\lambda-\mu)(x^\alpha u_{\mu,x})_x$. Our goal is to show that g belongs to $\mathcal{G}(C_0)$ for some $C_0>0$ that is to be determined. Set $z := u_{\mu,t}$. According to Lemma 2, for almost all $(t,x)\in(0,T)\times(0,1)$, one has

$$\left|\frac{\partial}{\partial t}\left((\lambda-\mu)\,(x^\alpha u_{\mu,x})_x\right)(t,x)\right| = |\lambda-\mu|\,|(x^\alpha z_x)_x\,(t,x)|\,.$$

Yet, according to (4) and (5) in Theorem 2, one has, for almost all $(t,x)\in(0,T)\times(0,1)$,

$$|(x^\alpha z_x)_x\,(t,x)| \le \frac{M}{m}\left|(x^\alpha u_{\mu,x})_x(T',x)\right|,$$

so that we deduce immediately that $g\in\mathcal{G}(C_0)$ for $C_0 = \dfrac{M}{m}$.

We now apply the stability estimate in Theorem 3 to the solution w of (9). There exists $C = C(T,t_0,\alpha,K_0,K_1,\eta_0,\eta_1,\|h_t\|_{L^\infty((0,T)\times(0,1))}\,,\|(x^\alpha h_x\,(0,.))_x\|_{L^\infty(0,1)})$ such that

$$\|g\|^2_{L^2((t_0,T)\times(0,1))} \le C\left(\left\|(x^\alpha w_x)_x\,(T',.)\right\|^2_{L^2(0,1)} + \|w_{tx}\,(.,1)\|^2_{L^2(t_0,T)}\right). \tag{10}$$

Moreover, thanks to Theorem 2,

$$\|g\|^2_{L^2((t_0,T)\times(0,1))} = \int_{t_0}^T\int_0^1 |\lambda-\mu|^2\,(x^\alpha u_{\mu,x})_x^2\,(t,x)dxdt \ge (T-t_0)|\lambda-\mu|^2 m^2.$$

Replacing w by $u_\lambda - u_\mu$ in (10), we complete the proof of Theorem 4.

□

Using the stability estimate for source terms in [12], we can prove the same way the following Lipschitz stability result with a locally distributed observation.

Theorem 5. *Let $\alpha\in[0,2)$, $u_0\in\mathcal{U}$, $h\in\mathcal{H}$ and $\omega := (a,b)$ with $0<a<b<1$. There exists $C = C(T,t_0,\alpha,\omega,K_0,K_1,\eta_0,\eta_1,\|h_t\|_{L^\infty((0,T)\times(0,1))}\,,\|(x^\alpha h_x\,(0,.))_x\|_{L^\infty(0,1)})$ such that, for all $\lambda\in I$, for all $\mu\in I$, the corresponding solutions u_λ and u_μ of problem (1) satisfy*

$$|\lambda-\mu|^2 \le C\left(\left\|\left(x^\alpha\,(u_\lambda-u_\mu)_x\right)_x\,(T',.)\right\|^2_{L^2(0,1)} + \left\|(u_\lambda-u_\mu)_t\right\|_{L^2(\omega_T^{t_0})}\right),$$

where $\omega_T^{t_0} := (t_0,T)\times\omega$.

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