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Localized sequences of approximate critical points

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This paper is dedicated to Prof. Monique Madaune Tort.

Abstract

This paper deals with the existence of special sequences of approximate critical points of a C^1-functional on a real Banach space. It extends to a more natural level of generality the approach which was presented recently in the context of what is usually referred to as the mountain pass geometry. In addition to treating more general minimax principles we also use a broader class of weights to formulate properties of the sequence.

1 Introduction

Henceforth $(X, \|\cdot\|)$ is a real Banach space and $(X^*, \|\cdot\|_*)$ is its dual space. For $A \subset X$ and $u \in X$, $d(u, A) = \inf\{\|u - v\| : v \in A\}$ and we recall that $d(u, A) = 0 \Leftrightarrow u \in \overline{A}$, the closure of A. For $\Phi \in C^1(X, \mathbb{R})$ and $d \in \mathbb{R}$, we set $\Phi_d = \{u \in X : \Phi(u) \geq d\}$.

By a sequence of approximate critical points of Φ we mean a sequence $\{u_n\} \subset X$ such that $\|\Phi'(u_n)\|_* \to 0$. Establishing the existence of such a sequence is a important step towards proving the existence of critical points and minimax principles are a useful way of identifying levels c for which such sequences exist with $\Phi(u_n) \to c$. Additional properties of the sequence can also help to prove its relative compactness, either by sharpening the convergence of the derivatives as in Cerami's condition [3, 4, 5] or through some information about its location as in the work of Ghoussoub and Preiss [9]. By introducing a Finsler structure on X, Ekeland [7] and Ghoussoub [11, 10] where able to formulate results of this kind establishing the existence of sequences of approximate critical points with additional properties, see (1.3) below, in which the information about the location of the sequence is given in terms of the Finsler metric. Recently [15], in the context of the mountain pass theorem [1, 13], we obtained conclusions of a similar nature but expressed in terms of the norm on X (see (1.4) below), without any reference to a Finsler structure, by using an appropriate deformation lemma. Subsequently, Rabier [12] has

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shown how to derive these conclusions from (1.3). Here we extend the direct approach of [15] to more general minimax principles and at the same time we replace the weights $(1+\|\cdot\|)^k$ for $k\in[0,1]$ used in [15], by a broader class suggested by Rabier's results.

In what follows an admissible weight is a continuous function $\omega:[0,\infty)\to(0,\infty)$ which is non-decreasing and concave with $\omega'(0+)<\infty$. Some properties of admissible weights are set out in Section 2.

To formulate a general minimax principle we adapt the setting used by Willem in Theorems 2.8 and 2.20 of [16], but one could just as easily use other formulations, for example the one based on homotopy-stable families with extended boundary preferred by Ghoussoub [11]. The proof of our main result, Theorem 1.1, uses only the elementary deformation lemma proved in Section 3 and some simple properties of admissible weights.

Theorem 1.1 *Consider $\Phi\in C^1(X,\mathbb{R})$ and an admissible weight ω. For a metric space (M,ρ) and a non-empty, closed subset M_0 of M, let Γ_0 be a non-empty subset of $C(M_0,X)$ such that $\Gamma_0(M_0)=\cup_{p\in\Gamma_0}p(M_0)$ is a compact subset of X. Set $\Gamma=\{p\in C(M,X):p|_{M_0}\in\Gamma_0\}$ and then*

$$c=\inf_{p\in\Gamma}\sup_{t\in M}\Phi(p(t)). \tag{1.1}$$

Note that $c\geq\inf_{u\in\Gamma_0(M_0)}\Phi(u)>-\infty$.

Suppose that $c<\infty$ and let $\{p_n\}\subset\Gamma$ be an optimal sequence of paths such that $m_n=\sup_{t\in M}\Phi(p_n(t))\to c$ as $n\to\infty$.

Let W be a closed subset of Φ_c such that

$$(I)\,\Gamma_0(M_0)\cap W=\emptyset \text{ and } (II) \text{ for all } p\in\Gamma, p(M)\cap W\neq\emptyset. \tag{1.2}$$

Then there exists a sequence $\{u_n\}\subset X$ such that

$$\Phi(u_n)\to c,\ \omega(\|u_n\|)\|\Phi'(u_n)\|_*\to 0,\ \frac{d(u_n,W)}{\omega(\|u_n\|)}\to 0,\ \frac{d(u_n,p_n(M))}{\omega(\|u_n\|)}\to 0. \tag{1.3}$$

If in addition, $c>\max\{\Phi(u):u\in\Gamma_0(M_0)\}$, then there exists a sequence of paths $\{q_n\}\subset\Gamma$ such that $\sup_{t\in M}\Phi(q_n(t))\to c$ and $u_n\in q_n(M)$.

Remarks (1) As an example in [15] shows, for weights such that $\lim_{t\to\infty}\omega(t)=\infty$, there may be no sequence such that $\omega(\|u_n\|)\|\Phi'(u_n)\|_*\to 0$ and $\{d(u_n,p_n(M))\}$ remains bounded for a given sequence of paths in Γ satisfying $\sup_{t\in M}\Phi(p_n(t))\to c$. However, replacing p_n by q_n, we get $d(u_n,q_n(M))=0$ for all n, at least when $c>\max\{\Phi(u):u\in\Gamma_0(M_0)\}$.

(2) The condition (1.2)(II) implies that $c\geq\inf_{u\in W}\Phi(u)$, whereas for $W\subset\Phi_c$ we have that $\inf_{u\in W}\Phi(u)\geq c$. Hence, under the hypotheses of the theorem $\inf_{u\in W}\Phi(u)=c$.

(3) The set Φ_c is closed. If M is compact, the definition of c implies that (1.2)(II) is satisfied by $W=\Phi_c$ and that $c<\infty$. Thus, when M is compact, (1.2) holds for $W=\Phi_c$ if and only if $\max\{\Phi(u):u\in\Gamma_0(M_0)\}<c$.

The notion of linking provides a more intuitive framework for Theorem 1.1.

1.1 Linking in X

Consider $M_0 \subset M \subset X$ where $M_0 \neq \emptyset$ and set $H = \{p \in C(M,X) : p(t) = t \text{ for all } t \in M_0\}$. A subset A of X is said to link with (M, M_0) if (i) $M_0 \cap A = \emptyset$ and (ii) for all $p \in H, p(M) \cap A \neq \emptyset$.

Here are the usual examples, following Chapter 2.3 of [16]. For a subspace Y of X, let $B_Y(r) = \{y \in Y : \|y\| < r\}$ and $S_Y(r) = \{y \in Y : \|y\| = r\}$.

Example 1 For $e \in X \backslash \{0\}$, let $M = [0, e] = \{te : t \in [0, 1]\}$, $M_0 = \{0, e\}(= \partial M$ in $\text{span}\{e\})$, $A = S_X(r)$ where $r \in (0, \|e\|)$. Then A links with (M, M_0).

Example 2 Let $X = Y \oplus Z$ where $\dim Y < \infty$. Let $M = \overline{B_Y(r)}$ for some $r > 0$ and $M_0 = S_Y(r)$. Then Z links with (M, M_0).

Example 3 Let $X = Y \oplus Z$ where $\dim Y < \infty$. For some $\rho > r > 0$ and $z_0 \in Z$ with $\|z_0\| = r$, let $M = \{y + tz_0 : y \in Y, t \geq 0 \text{ and } \|y + tz_0\| \leq \rho\}$ and $M_0(= \partial M$ in $Y \oplus$ span $\{z_0\})$ $= \{y \in Y : \|y\| \leq \rho\} \cup \{y + tz_0 : \|y + tz_0\| = \rho \text{ and } t \geq 0\}$. Then $A = S_Z(r)$ links with (M, M_0).

In the context of Theorem 1.1, let M be a subset of X with the metric $\rho(u, v) = \|u - v\|$. For a non-empty, compact subset M_0 of M, let $q(t) = t$ for all $t \in M_0$ and set $\Gamma_0 = \{q\}$. Then $\Gamma_0(M_0) = M_0$ is compact and $\Gamma = \{p \in C(M,X) : p|_{M_0} \in \Gamma_0\} = \{p \in C(M,X) : p(t) = t \text{ for all } t \in M_0\}$. In this case, the condition (1.2) states that W links with (M, M_0).

Corollary 1.2 *Let $\Phi \in C^1(X, \mathbb{R})$ and let ω be an admissible weight. For $M_0 \subset M \subset X$ with $M_0 \neq \emptyset$ and compact, set $\Gamma = \{p \in C(M,X) : p(v) = v \text{ for all } v \in M_0\}$ and then $c = \inf_{p \in \Gamma} \sup_{v \in M} \Phi(p(v)) \, (\geq \sup_{v \in M_0} \Phi(v) > -\infty)$. Suppose that $c < \infty$ and let $\{p_n\} \subset \Gamma$ be such that $\sup_{v \in M} \Phi(p_n(v)) \to c$.*

Let W be closed subset of Φ_c such that W links with (M, M_0).

Then there exists a sequence $\{u_n\} \subset X$ satisfying (1.3).

If in addition, $c > \max\{\Phi(v) : u \in M_0\}$, then there exists a sequence of paths $\{q_n\} \subset \Gamma$ such that $\sup_{v \in M} \Phi(q_n(v)) \to c$ and $u_n \in q_n(M)$.

Remarks (1) If W links with (M, M_0), it follows from the definition of c that $c \geq \inf_{u \in W} \Phi(u)$. Hence if $W \subset \Phi_c$ links with(M, M_0), $\inf_{u \in W} \Phi(u) = c$.

(2) Note that, in the context of this corollary, $c < \infty$ when M is compact and in that case, $W = \Phi_c$ links with (M, M_0) if and only if $c > \max_{v \in M_0} \Phi(v)$. To prove that $c > \max_{v \in M_0} \Phi(v)$, it is enough to show that there exists a subset A of X which links with (M, M_0) and for which $\inf_{u \in A} \Phi(u) > \max_{v \in M_0} \Phi(v)$.

(3) By the definition of c, we always have $c \geq \max_{v \in M_0} \Phi(v)$. In situations where $c = \max_{v \in M_0} \Phi(v)$ one must try to find a proper subset W of Φ_c which links with (M, M_0).

Note that in all three examples presented earlier the set M, and hence M_0, is compact.

Example 1(continued) For $e \in X \backslash \{0\}$, we have $M = [0, e] = \{te : t \in [0, 1]\}$ and $M_0 = \{0, e\}$. For these sets, Theorem 1.1 in [15] corresponds to Corollary 1.2 with the weight $\omega(t) = (1 + t)^k$, for some $k \in [0,1]$, yielding a sequence $\{u_n\}$ satisfying

$$\Phi(u_n) \to c, (1+\|u_n\|)^k \|\Phi'(u_n)\|_* \to 0, \frac{d(u_n, W)}{(1+\|u_n\|)^k} \to 0, \frac{d(u_n, p_n(M))}{(1+\|u_n\|)^k} \to 0. \tag{1.4}$$

By (2), $W = \Phi_c$ links with (M, M_0) if and only if $c > \max\{\Phi(0), \Phi(e)\}$, which is condition (MPG) (mountain pass geometry) in [15]. The stronger condition, there exists $r \in (0, \|e\|)$ such that $\inf_{u \in S(0,r)} \Phi(u) > \max\{\Phi(0), \Phi(e)\}$, referred to as (SMPG) (strong mountain pass geometry) in [15], implies (MPG) since $S(0,r)$ links with (M, M_0).

Example 2(continued) Let $X = Y \oplus Z$ where $\dim Y < \infty$, $M = \overline{B_Y(r)}$ for some $r > 0$ and $M_0 = S_Y(r)$. Since Z links with (M, M_0), $W = \Phi_c$ also links with (M, M_0) provided that $\inf_{z \in Z} \Phi(z) > \max_{v \in M_0} \Phi(v)$.

Example 3(continued) Here $M = \{y + tz_0 : y \in Y \text{ and } t \geq 0 \text{ with } \|y + tz_0\| \leq \rho\}$ and $M_0 = \partial V$ in $Y \oplus$ span $\{z_0\}$, for some $\rho > 0$ and $z_0 \in Z$ with $0 < r = \|z_0\| < \rho$. Since $S_Z(r)$ links with (M, M_0), $W = \Phi_c$ also links with (M, M_0) provided that $\inf_{z \in S_Z} \Phi(z) > \max_{v \in M_0} \Phi(v)$.

2 Admissible weights

Let ω be an admissible weight. Recall that the concavity of ω implies that for $0 \leq x < y < z$, we have

$$\frac{\omega(z) - \omega(x)}{z - x} \leq \frac{\omega(z) - \omega(y)}{z - y} \leq \frac{\omega(y) - \omega(x)}{y - x}. \tag{2.1}$$

Hence $\omega'(x+) = \lim_{z \to x+} \{\omega(z) - \omega(x)\}/(z - x)$ exists for all $x \geq 0$ and is finite for $x > 0$. Since we assume that ω is non-decreasing and $\omega'(0+) < \infty$, we have that $\omega'(x+)$ is non-negative and non-increasing on $[0, \infty)$. It follows easily that

$$\omega(x) \leq \omega(0) + \omega'(0+)x \text{ and } \frac{\omega(y) - \omega(x)}{y - x} \leq \omega'(0+) \text{ for } 0 \leq x < y,$$

from which we obtain

$$\omega(x) \leq A(1 + x) \text{ for all } x \geq 0 \text{ where } A = \max\{\omega(0), \omega'(0+)\} \tag{2.2}$$

and

$$\frac{\omega(y)}{\omega(x)} \leq 1 + \omega'(0+)\frac{|x - y|}{\omega(x)} \text{ for all } x, y \geq 0. \tag{2.3}$$

Furthermore, an admissible weight has the property that

$$\omega(1 + 2s) \leq K\omega(s) \text{ for all } s \geq 0 \text{ where } K = \max\{3, \frac{\omega(3)}{\omega(0)}\}. \tag{2.4}$$

Indeed, for $s \geq 1$, it follows from (2.1) with $x = 0, y = s, z = 1 + 2s$, that $\omega(1 + 2s) \leq (2 + \frac{1}{s})\omega(s) + (1 - \frac{1+2s}{s})\omega(0) \leq 3\omega(s)$, whereas for $0 \leq s \leq 1$, $\omega(1+2s) \leq \omega(3) \leq K\omega(0) \leq K\omega(s)$.

Examples For $k \in [0,1]$, $\omega(s) = (1 + s)^k$ is an admissible weight, as is $\ln(2 + s)$.

Lemma 2.1 *Let ω be an admissible weight. For a closed subset S of X and $\varepsilon > 0$, let*

$$S_\varepsilon = \{w \in X : \frac{\|w - u\|}{\omega(\|u\|)} \leq \varepsilon \text{ for some } u \in S\}.$$

(i) For any compact subset K of X such that $K \cap S = \emptyset$, there exists $\varepsilon > 0$ such that $K \cap S_\varepsilon = \emptyset$.
(ii) For all $z \in X$ and all $\varepsilon > 0$, $d(z, S_\varepsilon) + d(z, X \backslash S_{2\varepsilon}) > 0$.

Proof (i) Suppose that for every $n \in \mathbb{N}$, $K \cap S_{1/n} \neq \emptyset$. Then there exist $w_n \in K$ and $u_n \in S$ such that $\|w_n - u_n\| \leq \omega(\|u_n\|)/n \leq A(1 + \|u_n\|)/n$ by (2.2). But $\|u_n\| \leq \|w_n - u_n\| + \|w_n\| \leq \|w_n - u_n\| + C$ where $C = \max\{\|z\| : z \in K\} < \infty$ and hence $(1 - \frac{A}{n})\|u_n\| \leq \frac{A}{n} + C$. It follows that for $n \geq 2A$, $\|u_n\| \leq 1 + 2C$ and so $\|w_n - u_n\| \leq \omega(1 + 2C)/n$. By the compactness of K, there exist $w \in K$ and a subsequence w_{n_k} such that $\|w - w_{n_k}\| \to 0$. Since $\|w - u_{n_k}\| \leq \|w - w_{n_k}\| + \omega(1 + 2C)/n_k$ we also have $w \in \overline{S} = S$, contradicting the hypothesis $K \cap S = \emptyset$.

(ii) Choose $\varepsilon > 0$ and suppose that there exists $z \in X$ such that $d(z, S_\varepsilon) = d(z, X \backslash S_{2\varepsilon}) = 0$.

From $d(z, S_\varepsilon) = 0$ it follows that there exist sequences $\{w_n\}$ and $\{u_n\}$ such that $w_n \in S_\varepsilon$ with $\|z - w_n\| \to 0$ and $u_n \in S$ with $\|w_n - u_n\| \leq \varepsilon\omega(\|u_n\|)$. Hence $\frac{\|z-u_n\|}{\omega(\|u_n\|)} \leq \frac{\|z-w_n\|+\|w_n-u_n\|}{\omega(\|u_n\|)} \leq \frac{\|z-w_n\|}{\omega(0)} + \varepsilon$, showing that $z \in S_\delta$ for every $\delta > \varepsilon$.

Since $d(z, X \backslash S_{2\varepsilon}) = 0$, there exists a sequence $\{v_n\} \subset X \backslash S_{2\varepsilon}$ such that $\|z - v_n\| \to 0$. For all $u \in S$, we have that $\|v_n - u\| > 2\varepsilon\omega(\|u\|)$ and so $\frac{\|z-u\|}{\omega(\|u\|)} \geq \frac{\|u-v_n\|-\|z-v_n\|}{\omega(\|u\|)} > 2\varepsilon - \frac{\|z-v_n\|}{\omega(\|0\|)}$, showing that $z \notin S_\delta$ for whenever $\delta < 2\varepsilon$, contradicting the earlier conclusion that $z \in S_\delta$ for all $\delta > \varepsilon$. □

2.1 A Finsler structure induced by an admissible weight

Given an admissible weight ω on X the function $\delta : X \times X \to [0, \infty)$ defined by

$$\delta(u, v) = \inf\{\int_0^1 \frac{\|h'(t)\|}{\omega(\|h(t)\|)} dt : h \in C^1([0, 1]), X) \text{ with } h(0) = u \text{ and } h(1) = v\} \tag{2.5}$$

is a metric on X and (X, δ) is a complete metric space. Indeed, think of X as a Banach manifold modelled on itself with tangent space $T_u(X) = X$ at every $u \in X$. A Finsler structure (see [11] or [14]) $\| \cdot \| \to [0, \infty)$ on the tangent bundle $T(X)$ is defined by setting $\|v\|_u = \frac{\|v\|}{\omega(\|u\|)}$ for all $u \in X$ and $v \in T_u(X)$. Then δ is the associed Finsler metric on the manifold X and the completeness of (X, δ) is ensured by (2.2). The Finsler structure induced on the cotangent space is given by $\| \cdot \|^* : T^*(M) \to [0, \infty)$ where $\|f\|_u^* \equiv \sup\{f(v) : v \in T_u(X) \text{ with } \|v\|_u \leq 1\} = \omega(\|u\|)\|f\|_*$ for $u \in X$ and $f \in T_u^*(X) = X^*$ in this case.

Very general minimax results concerning the existence of sequences of approximate critical points of C^1- functionals on C^1-Finsler manifolds have been available for a long time, [7, 11]. In the context of our Theorem 1.1, results of this kind establish the existence of a sequence $\{u_n\} \subset X$ such that

$$\Phi(u_n) \to c, \omega(\|u_n\|)\|\Phi'(u_n)\|_* \to 0, D(u_n, W) \to 0 \text{ and } D(u_n, p_n(M)) \to 0 \tag{2.6}$$

where $D(u, A) = \inf\{\delta(u, v) : v \in A\}$ for $u \in X$ and $A \subset X$ and δ is the Finsler metric induced by the admissible weight, ω. Recently, P.J. Rabier has shown in [12] that, for any sequence of non-empty subsets $\{A_n\}$ of X,

$$D(u_n, A_n) \to 0 \Leftrightarrow \frac{d(u_n, A_n)}{\omega(\|u_n\|)} \to 0$$

and hence our conclusion (1.3) can be deduced from the much earlier results on Finsler manifolds. The approach in this paper provides a direct route to (1.3) avoiding the use of the Finsler metric.

3 A deformation lemma

We shall use the following well-known and elementary results from the theory of differential equations.

Lemma 3.1 (Gronwall) *Suppose that $a \in \mathbb{R}$ and $b, h \in C([0,\infty))$ satisfy the inequalities*

$$b(t) \geq 0 \text{ and } h(t) \leq a + \int_0^t b(s)h(s)ds \text{ for all } t \geq 0.$$

Then $h(t) \leq ae^{\int_0^t b(s)ds}$ for all $t \geq 0$.

Proposition 3.2 *Let $f : X \to X$ be locally Lipschitz continuous and suppose that there exists a constant A such that*

$$\|f(u)\| \leq A(1 + \|u\|) \text{ for all } u \in X. \tag{3.1}$$

Then, for every $u_0 \in X$, the initial value problem

$$\begin{cases} u'(t) = f(u(t)) & \text{for } t > 0 \\ u(0) = u_0 \end{cases}$$

has a unique solution $\eta(\cdot, u_0) \in C^1([0,\infty), X)$. Furthermore,

1. *$\eta \in C([0,\infty) \times X, X)$,*

2. *$\eta(t, \cdot) : X \to X$ is a homeomorphism for all $t \geq 0$,*

3. *$\eta(t, \eta(s, u)) = \eta(t + s, u)$ for all $t, s \geq 0$ and $u \in X$.*

The main result of this section is an appropriate version of what is usually referred to as a deformation lemma. We begin by recalling the definition and existence of a pseudo-gradient, [16].
Let $F \in C^1(X, \mathbb{R})$ and let $\Omega = \{u \in X : F'(u) \neq 0\}$. There is a locally Lipschitz continuous function $p : \Omega \to X$ such that, for every $u \in \Omega$,

$$\|p(u)\| \leq 2\|F'(u)\|_* \text{ and } \langle F'(u), p(u)\rangle \geq \|F'(u)\|_*^2.$$

Such a mapping p is called a pseudo-gradient for F. Note that $\|p(u)\| \geq \|F'(u)\|_*$ for all $u \in \Omega$.

We now sharpen the deformation lemma proved in [2] in various ways required for the proof of Theorem 1.1.

Theorem 3.3 *Let $F \in C^1(X, \mathbb{R})$ and let ω be an admissible weight and set $A = \max\{\omega(0), \omega'(0+)\}$. For $c \in \mathbb{R}$ and $\delta \in (0, \delta_0]$ where $\delta_0 = (\frac{\ln 2}{16A})^2$, consider the set*
$N = \{u \in X : |F(u) - c| < 2\delta \text{ and } \omega(\|u\|)\|F'(u)\|_* > \sqrt{\delta}\}$.
There exists $\eta : [0,\infty) \times X \to X$ such that

(d1) *$\eta \in C([0,\infty) \times X, X)$ and*
$\eta(t, \cdot) : X \to X$ is a homeomorphism for all $t \geq 0$.

(d2) $\eta(0,u) = u$ *and* $\eta(t-s, \eta(s,u)) = \eta(t,u)$ *for all* $u \in X$ *and* $t \geq s \geq 0$.

(d3) $\eta(t,u) = u$ *for all* $t \geq 0$ *if* $u \notin N$.

(d4) $F(\eta(t,u)) \leq F(\eta(s,u)) \leq F(u)$ *for all* $t \geq s \geq 0$ *and* $u \in X$.

(d5) $F(u) - F(\eta(t,u)) \leq 4\delta$ *for all* $t \geq 0$ *and* $u \in X$.

(d6) $\|\eta(t,u) - \eta(s,u)\| \leq 16K\sqrt{\delta}\omega(\|\eta(s,u)\|)$ *for all* $t \geq s \geq 0$ *and* $u \in X$ *where* $K = \max\{3, \frac{\omega(3)}{\omega(0)}\}$.

(d7) $\|\eta(t,u) - \eta(s,u)\| \leq 32K\sqrt{\delta}\omega(\|\eta(t,u)\|)$ *for all* $t \geq s \geq 0$ *and* $u \in X$ *provided that* $16K\omega'(0+)\sqrt{\delta} \leq \frac{1}{2}$.

(d8) *Suppose that* $F(u) < c + \delta$. *Then, for all* $t \geq 8\delta$, *either* $F(\eta(t,u)) \leq c - \delta$ *or there exists* $\tau \in [0,t]$ *such that* $\omega(\|\eta(\tau,u)\|)\|F'(\eta(\tau,u))\|_* < 2\sqrt{\delta}$.

Proof Let $M = \{u \in X : |F(u) - c| \leq \delta \text{ and } \omega(\|u\|)\|F'(u)\|_* \geq 2\sqrt{\delta}\}$. We have $M \subset N$, M and $N^c = X\backslash N$ are closed, $M \cap N^c = \emptyset$. Set

$$\psi(u) = \frac{d(u, N^c)}{d(u,M) + d(u,N^c)}$$

Then $\psi : X \to [0,1]$ is locally Lipschitz continuous with $\psi(u) = \begin{cases} 1 & \text{for } u \in M \\ 0 & \text{for } u \in N^c. \end{cases}$

Let $p : \Omega = \{u \in X : F'(u) \neq 0\} \to X$ by a pseudo-gradient field for F.

Noting that $N \subset \Omega$, we define $f : X \to X$ by $f(u) = \begin{cases} -\frac{\psi(u)p(u)}{\|p(u)\|^2} & \text{for } u \in N \\ 0 & \text{for } u \notin N. \end{cases}$ Then $f : X \to X$ is locally Lipschitz continuous and, for $u \in N$,

$\|f(u)\| \leq \frac{1}{\|p(u)\|} \leq \frac{1}{\|F'(u)\|_*} \leq \frac{\omega(\|u\|)}{\sqrt{\delta}}$.

Since $f(u) = 0$ on N^c, we have that $\|f(u)\| \leq \frac{\omega(\|u\|)}{\sqrt{\delta}} \leq \frac{A}{\sqrt{\delta}}(1 + \|u\|)$ for all $u \in X$ by (2.2) where $A = \max\{\omega(0), \omega'(0+)\}$.

Let $\eta(t,u)$ be the flow defined by the unique solution of initial value problem

$$\eta'(t) = f(\eta(t)) \text{ for } t > 0, \eta(0) = u.$$

By Theorem 3.2, $\eta \in C([0,\infty) \times X, X)$ and (d1),(d2) and (d3) are satisfied. Also, for $t > 0$ and $\eta = \eta(t,u)$,

$$\frac{d}{dt}F(\eta) = \langle F'(\eta), f(\eta)\rangle = \begin{cases} -\frac{\langle F'(\eta), \psi(\eta)p(\eta)\rangle}{\|p(\eta)\|^2} & \text{for } \eta \in N \\ 0 & \text{for } \eta \notin N \end{cases}$$

$$\leq \begin{cases} -\psi(\eta)\frac{\|F'(\eta)\|_*^2}{\|p(\eta)\|^2} & \text{for } \eta \in N \\ 0 & \text{for } \eta \notin N \end{cases} \leq \begin{cases} -\frac{1}{4}\psi(\eta) & \text{for } \eta \in N \\ 0 & \text{for } \eta \notin N \end{cases} = -\frac{1}{4}\psi(\eta) \leq 0,$$

proving (d4) and showing that

$$F(u) - F(\eta(t,u)) \geq \frac{1}{4}\int_0^t \psi(\eta(s,u))ds \text{ for all } t \geq 0 \text{ and } u \in X. \tag{3.2}$$

By (d3), (d5) is trivial for $u \notin N$. Considering $u \in N$, we have $F(u) - c < 2\delta$ and hence $F(u) - F(\eta(t,u)) < c + 2\delta - F(\eta(t,u)) \leq 4\delta$ if $F(\eta(t,u)) \geq c - 2\delta$. But $F(\eta(0,u)) = F(u) > c - 2\delta$ since $u \in N$ and so, if $F(\eta(t,u)) < c - 2\delta$, there exists $s \in (0,t)$ such that $F(\eta(s,u)) = c - 2\delta$ and we have $F(u) - F(\eta(s,u)) \leq 4\delta$. Since $\eta(s,u) \notin N$, $\eta(t,u) = \eta(t-s, \eta(s,u)) = \eta(s,u)$ for all $t \geq s \geq 0$ by (d2) and (d3). Thus $F(u) - F(\eta(t,u)) = F(u) - F(\eta(s,u)) \leq 4\delta$ in this case too. This proves (d5).

Combining (3.2) and (d5) we get

$$\int_0^t \psi(\eta(s,u))ds \leq 16\delta \text{ for all } t \geq 0 \text{ and } u \in X. \tag{3.3}$$

To prove (d6) we consider $t > 0$ and $u \in X$. Let $A(t,u) = (0,t) \cap \{s : \eta(s,u) \in N\}$. Then

$$\begin{aligned} \|\eta(t,u) - u\| &\leq \int_0^t \|\frac{d}{ds}\eta(s,u)\|ds = \int_0^t \|f(\eta(s,u))\|ds \\ &\leq \int_{A(t,u)} \|f(\eta(s,u))\|ds = \int_{A(t,u)} \frac{\psi(\eta(s,u))}{\|p(\eta(s,u))\|}ds \leq \int_{A(t,u)} \frac{\psi(\eta(s,u))}{\|F'(\eta(s,u))\|_*}ds \\ &\leq \int_{A(t,u)} \psi(\eta(s,u))\frac{\omega(\|\eta(s,u)\|)}{\sqrt{\delta}}ds \leq \int_0^t \psi(\eta(s,u))\frac{\omega(\|\eta(s,u)\|)}{\sqrt{\delta}}ds \end{aligned} \tag{3.4}$$

Hence

$$\begin{aligned} 1 + \|\eta(t,u)\| \leq 1 + \|u\| + \|\eta(t,u) - u\| &\leq 1 + \|u\| + \frac{1}{\sqrt{\delta}}\int_0^t \psi(\eta(s,u))\omega(\|\eta(s,u)\|)ds \\ &\leq 1 + \|u\| + \frac{A}{\sqrt{\delta}}\int_0^t \psi(\eta(s,u))[1 + \|\eta(s,u)\|]ds \end{aligned}$$

by (2.2) and this can be written as $h(t) \leq a(t) + \int_0^t b(s)h(s)ds$ where

$$h(t) = \|1 + \eta(t,u)\|, a(t) = 1 + \|u\|, b(t) = \frac{A\psi(\eta(t,u))}{\sqrt{\delta}}.$$

The Gronwall inequality yields

$$1 + \|\eta(t,u)\| \leq (1 + \|u\|)e^{\int_0^t b(s)ds}$$

where $\int_0^t b(s)ds \leq 16A\sqrt{\delta}$ by (3.3).

Hence $1 + \|\eta(t,u)\| \leq (1 + \|u\|)e^{16A\sqrt{\delta}} \leq 2(1 + \|u\|)$ for all $t \geq 0$ and $u \in X$ since $\delta \leq \delta_0$ and $e^{16A\sqrt{\delta_0}} = 2$. Consequently (3.4) yields

$$\|\eta(t,u) - u\| \leq \int_0^t \psi(\eta(s,u))\frac{\omega(2(1 + \|u\|) - 1)}{\sqrt{\delta}}ds.$$

Using (3.3) and (2.4) we now have that

$$\|\eta(t,u) - u\| \leq \frac{\omega(1 + 2\|u\|)}{\sqrt{\delta}}\int_0^t \psi(\eta(s,u))ds \leq 16K\sqrt{\delta}\omega(\|u\|)$$

for all $t \geq 0$ and $u \in X$, proving (d6) for $s = 0$. We get the complete conclusion with $s \neq 0$ by using (2).

To deduce (d7) from (d6), we observe that (2.3) yields

$$\frac{\omega(\|\eta(s,u)\|)}{\omega(\|\eta(t,u)\|)} \leq 1 + \omega'(0+)\frac{\|\eta(t,u) - \eta(s,u)\|}{\omega(\|\eta(t,u)\|)}$$

and hence we obtain

$$\frac{\|\eta(t,u)-\eta(s,u)\|}{\omega(\|\eta(t,u)\|)} \leq 16K\sqrt{\delta}\frac{\omega(\|\eta(s,u)\|)}{\omega(\|\eta(t,u)\|)} \leq 16K\sqrt{\delta}\{1+\omega'(0+)\frac{\|\eta(t,u)-\eta(s,u)\|}{\omega(\|\eta(t,u)\|)}\},$$

from which (d7) follows easily.

For (d8), we consider $t \geq 8\delta$ and u such that $F(u) < c+\delta$. Suppose that $F(\eta(t,u)) > c-\delta$. By (d4), this implies that $c-\delta < F(\eta(s,u)) \leq F(u) < c+\delta$ for all $s \in [0,t]$. If $\omega(\|\eta(s,u)\|)\|F'(\eta(s,u))\|_* \geq 2\sqrt{\delta}$ for all $s \in [0,t]$, we have that $\eta(s,u) \in M$ for all $s \in [0,t]$ and so, by (3.2) and the definition of ψ,

$$F(u) - F(\eta(t,u)) \geq \frac{1}{4}\int_0^t ds = \frac{t}{4}.$$

Hence $c-\delta < F(\eta(t,u)) \leq F(u) - \frac{t}{4} < c+\delta-\frac{t}{4}$ and so $8\delta > t$. □

4 The proof of Theorem 1.1

As before, let $A = \max\{\omega(0), \omega'(0+)\}$ and $K = \max\{3, \frac{\omega(3)}{\omega(0)}\}$.
Since W is closed and $\Gamma_0(M_0)$ is compact with $\Gamma_0(M_0) \cap W = \emptyset$, it follows from Lemma 2.1 that there exists $\varepsilon_0 > 0$ such that $\Gamma_0(M_0) \cap W_\varepsilon = \emptyset$ for all $\varepsilon \in (0, \varepsilon_0]$. For $\delta > 0$, let $\varepsilon(\delta) = 32K\sqrt{\delta}$. There exists $\delta_0 > 0$ such that, for all $\delta \in (0, \delta_0]$, we have

$\Gamma_0(M_0) \cap W_{2\varepsilon(\delta)} = \emptyset$, $8\delta < 1$, $\delta < (\frac{\ln 2}{16A})^2$ and $32K\omega'(0+)\sqrt{\delta} < 1$.
Consider $\delta \in (0, \delta_0]$. By Lemma 2.1(ii) we can define a function $T_\delta : X \to [0,1]$ by

$$T_\delta(u) = \frac{d(u, X\backslash W_{2\varepsilon(\delta)})}{d(u, W_{\varepsilon(\delta)}) + d(u, X\backslash W_{2\varepsilon(\delta)})}$$

and it follows that T_δ is locally Lipschitz continuous with $T_\delta(u) = 0$ for all $u \notin W_{2\varepsilon(\delta)}$ and $T_\delta(u) = 1$ for all $u \in W_{\varepsilon(\delta)}$.

By (1.2)(II), $m_n \geq c$ for all n and, since $\{p_n\}$ is an optimal sequence, there exists $n_0 \in \mathbb{N}$ such that $m_n < c + \delta_0$ for all $n \geq n_0$. For each $n \geq n_0$, choose some $\delta_n \in (m_n - c, \delta_0)$ in such a way that $\delta_n \to 0$ as $n \to 0$. For example, we could use $\delta_n = m_n - c + \frac{1}{n}$ for all large enough n. Then, to simplify the notation, set $T^n = T_{\delta_n}$.

Let N_n and η_n be the set and the deformation given by Theorem 3.3 for $F = \Phi$ and $\delta = \delta_n$ where $n \geq n_0$. Define $g_n : M \to X$ by

$$g_n(v) = \eta_n(T^n(p_n(v)), p_n(v)) \text{ for } v \in M.$$

Clearly $g_n \in C(M,X)$ and, since $p_n(v) \in \Gamma_0(M_0)$ for all $v \in M_0$, it follows that $p_n(v) \notin W_{2\varepsilon(\delta_n)}$ and so $T^n(p_n(v)) = 0$. Hence, for $v \in M_0$, $g_n(v) = \eta_n(0, p_n(v)) = p_n(v)$ showing that $g_n \in \Gamma$. It now follows from (1.2) that there exists $v_n \in M \backslash M_0$ such that $g_n(v_n) \in W$. Setting $w_n = p_n(v_n)$, we have that $g_n(v_n) = \eta_n(T^n(w_n), w_n)$. By property (d7) of η_n, it follows that

$$\frac{\|g_n(v_n) - p_n(v_n)\|}{\omega(\|g(v_n)\|)} = \frac{\|\eta_n(T^n(w_n), w_n) - w_n\|}{\omega(\|\eta_n(T^n(w_n), w_n)\|)} \leq 32K\sqrt{\delta_n}$$

and so $w_n = p_n(v_n) \in W_{\varepsilon(\delta_n)}$. Thus $T^n(w_n) = 1$ and $g_n(v_n) = \eta_n(1, w_n)$. But $g_n(v_n) \in W \subset \Phi_c$ which implies that

$$c \le \Phi(\eta_n(1, w_n)) \le \Phi(w_n) = \Phi(p_n(v_n)) \le m_n < c + \delta_n.$$

Using property (d8) of η_n (with $t = 1$ and recalling that $0 < \delta_n \le \delta_0 < 1/8$), there exists $\tau_n \in [0, 1]$ such that

$$\omega(\|\eta_n(\tau_n, w_n)\|)\|\Phi'(\eta_n(\tau_n, w_n))\|_* < 2\sqrt{\delta_n}.$$

By property (d6) of η_n we also have that

$$\frac{\|\eta_n(1, w_n) - \eta_n(\tau_n, w_n)\|}{\omega(\|\eta_n(\tau_n, w_n)\|)} \le 16K\sqrt{\delta_n}$$

and so

$$\frac{d(\eta_n(\tau_n, w_n), W)}{\omega(\|\eta_n(\tau_n, w_n)\|)} \le 16K\sqrt{\delta_n} \text{ since } \eta_n(1, w_n) = g_n(v_n) \in W.$$

On the other hand, by property (d7) of η_n,

$$\frac{d(\eta_n(\tau_n, w_n), p_n(M))}{\omega(\|\eta_n(\tau_n, w_n)\|)} \le \frac{\|\eta_n(\tau_n, w_n) - p_n(v_n)\|}{\omega(\|\eta_n(\tau_n, w_n)\|)} = \frac{\|\eta_n(\tau_n, w_n) - w_n\|}{\omega(\|\eta_n(\tau_n, w_n)\|)} \le 32K\sqrt{\delta_n}.$$

Finally we observe that

$$c \le \Phi(\eta_n(1, w_n)) \le \Phi(\eta_n(\tau_n, w_n)) \le \Phi(w_n) < c + \delta_n$$

and that $\delta_n \to 0$ as $n \to \infty$.
Setting $u_n = \eta_n(\tau_n, w_n)$, we see that the sequence $\{u_n\}$ has all the required properties.

Suppose now that $c > \max_{v\in\Gamma_0(M_0)} \Phi(v)$. In this case, we can add to the conditions on δ_0 imposed at the beginning of the proof the requirement that $c - \max_{v\in\Gamma(M_0)} \Phi(v) > 2\delta_0$. This ensures that $\Gamma_0(M_0) \cap N_n = \emptyset$ for all n. Now consider the function q_n defined by $q_n(v) = \eta_n(\tau_n, p_n(v))$ for $v \in M$. Clearly $q_n \in C(M, X)$ and, since $\Gamma_0(M_0) \cap N_n = \emptyset$, it follows from (d3) that, for $v \in M_0$, $q_n(v) = p_n(v)$ and so $q_n \in \Gamma$. Furthermore, for all $v \in M$, by (d4),

$$\Phi(q_n(v)) = \Phi(\eta_n(\tau_n, p_n(v)) \le \Phi(p_n(v_n)) \le m_n$$

and so $\{q_n\}$ is an optimal sequence of paths with $u_n = \eta_n(\tau_n, p_n(v_n)) \in q_n(M)$. □

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