Monografías de la Real Academia de Ciencias de Zaragoza **38:** 85–111, (2012)
ISSN 1132-6360

Energy Solutions of Evolution Equations with Nonstandard Growth Conditions

S. Antontsev* S. Shmarev†

This paper is dedicated to Prof. Monique Madaune Tort.

Abstract

We present an overview of the recent advances in the theory of parabolic equations with nonstandard anisotropic growth conditions. The presentation includes the existence theorems in the variable exponents Sobolev spaces and a description of the properties of propagation of disturbances from the data, intrinsic for solutions of such equations.

Keywords: Nonlinear parabolic equation, nonstandard growth conditions, anisotropic nonlinearity

AMSCode: 35K57, 35K65, 35K55

1 Introduction

This text is based on the lecture notes from the short course given by the second author at the Second Summer School on the p-Laplace operator held at Jaca, September 20-22 of 2010. The intention of the authors is to offer a brief overview of the recent advances in the theory of evolution equations of the $p(x,t)$-Laplace type.

In the recent years, the theory of PDEs with variable nonlinearity, often termed PDEs with nonstandard growth conditions, has been developing very actively and already accounts for a series of important results. In these notes we collect the results related to the homogeneous Dirichlet problem for the evolution $p(x,t)$-laplacian

$$u_t = \operatorname{div}\left(|\nabla u|^{p(x,t)-2}\nabla u\right) \tag{1.1}$$

*CMAF, University of Lisbon, Portugal

The author was partially supported by the Research Project PTDC/MAT/110613/2009, FCT, Portugal

†University of Oviedo, Spain

The author acknowledges the support of the Research Grants MTM2007-65088 and MTM2010-18427, MICINN, Spain

and its generalizations. We are specially interested in the case when the exponent of nonlinearity depends on the time variable t.

The presentation mostly follows the original results by the authors. In Section 2 we introduce the variable exponent Lebesgue and Sobolev spaces, recall the basic properties of the functions from these spaces, and prove some auxiliary results. In Section 3 we prove the existence theorems to the Dirichlet problem for an equation generalizing the evolution $p(x,t)$-laplacian. The proof uses the classical Faedo-Galerkin scheme and relies on the monotonicity of the diffusion part of the equation in a suitable function space prompted by the equation. Another approach to the proof of existence is presented in Section 4: a solution is obtained as the limit of a sequence of solutions to the singularly perturbed equations. Both methods work only in the case when the smooth functions are dense in the main variable exponent Sobolev space, which leads to certain continuity requirements on the variable exponents of nonlinearity. We provide extensions of the main existence theorem to equations with nonlinear reaction terms growing at infinity. For the rapidly growing reaction terms the solution need not exist globally in time, but can be continued to the maximal time interval where it remains bounded - see Section 5.

The solution is always understood as *energy solution* (see Definition 3.1 below). Solutions of this type possess intrinsic propagation properties not displayed by the solutions of equations with constant and isotropic nonlinearity. These properties are briefly discussed in Section 6.

In the concluding Section 7 we collect references to the works devoted to the questions of the theory of PDEs with variable nonlinearity which do not fall into the scope of these notes.

1.1 Formulation of the problem.

Let $\Omega \subset \mathbb{R}^n$ be a bounded simple-connected domain and $0 < T < \infty$. We consider the Dirichlet problem for the parabolic equation

$$\begin{cases} u_t - \sum_i D_i\left(|D_i u|^{p_i(z)-2} D_i u\right) + d(z,u) = 0 \quad \text{in } Q, \\ u = 0 \text{ on } \Gamma, \qquad u(x,0) = u_0(x) \text{ in } \Omega, \end{cases} \tag{1.2}$$

where $z = (x,t) \in Q \equiv \Omega \times (0,T]$, Γ is the lateral boundary of the cylinder Q, D_i denotes the partial derivative with respect to x_i. The term $d(\cdot,\cdot)$ is a Carathodory functions (defined for $(z,r) \in \overline{Q} \times \mathbb{R}$, measurable in z for every $r \in \mathbb{R}$, continuous in r for a.a. $z \in Q$). We assume that d is subject to the following growth conditions:

$$\forall\, (z,r) \in Q \times \mathbb{R} \qquad |d(z,r)| \le d_0 |r|^{\lambda-1} + h_d(z), \tag{1.3}$$

with positive constants d_0, $\lambda > 1$, and a function

$$h_d(z) \in L^{\lambda'}(Q), \quad \lambda' = \frac{\lambda}{\lambda - 1}. \tag{1.4}$$

The exponents of nonlinearity $p_i(z)$ are given functions, continuous in Q and separated away from zero and infinity:

$$p_i(z) \subset (p_i^-, p_i^+) \subseteq (p^-, p^+) \subset (1,\infty), \tag{1.5}$$

with finite constants $p^{\pm}$, $p_i^{\pm} > 1$.

For the sake of simplicity of presentation, we confine the further study to the model equation (1.2). Nonetheless, the arguments below are applicable to more complicated equations with variable coefficients.

2 The function spaces

2.1 Orlicz-Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$: definitions and basic properties

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, with Lipschitz–continuous boundary $\partial\Omega$ and assume that $p(x)$ is continuous with the logarithmic module of continuity:

$$\forall\, z, \zeta \in Q_T,\ |z-\zeta| < 1, \qquad \sum_i |p_i(z) - p_i(\zeta)| \leq \omega(|z-\zeta|), \tag{2.1}$$

where

$$\overline{\lim_{\tau\to 0^+}}\, \omega(\tau) \ln\frac{1}{\tau} = C < +\infty.$$

By $L^{p(\cdot)}(\Omega)$ we denote the space of measurable functions $f(x)$ on Ω such that

$$A_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}\, dx < \infty.$$

The space $L^{p(\cdot)}(\Omega)$ equipped with the norm (the Luxemburg norm)

$$\|f\|_{p(\cdot),\Omega} \equiv \|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 :\ A_{p(\cdot)}\left(f/\lambda\right) \leq 1\right\}$$

becomes a Banach space. The Banach space $W_0^{1,\,p(\cdot)}(\Omega)$ with $p(x) \in [p^-, p^+] \subset (1,\infty)$ is defined by

$$\begin{cases} W_0^{1,\,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) :\ |\nabla u|^{p(x)} \in L^1(\Omega),\ u = 0 \text{ on } \partial\Omega\right\}, \\ \|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{p(\cdot),\Omega}. \end{cases} \tag{2.2}$$

A thorough discussion of the variable exponent Lebesgue and Sobolev spaces can be found in the monograph [28]. We limit ourselves by mentioning the basic properties of the spaces $W_0^{1,p(\cdot)}(\Omega)$ used in the rest of this paper.

- The space $W^{1,p(\cdot)}(\Omega)$ is separable and reflexive, provided that $p(x) \in C^0(\overline{\Omega})$.
- If condition (2.1) is fulfilled, then $C_0^\infty(\Omega)$ is dense in $W_0^{1,\,p(\cdot)}(\Omega)$, which can be defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm (2.2). The density of smooth functions in the space $W_0^{1,p(\cdot)}(\Omega)$ is crucial for the further proceeding. The condition of log-continuity of $p(x)$ is the best known and the most frequently used sufficient condition for the density of C_0^∞ in $W_0^{1,p(x)}(\Omega)$ - [35, 38] (see also [27]). Although it is not necessary and can be substituted by

other conditions (see [30, 32, 38], for example), we keep it throughout the paper for the sake of simplicity of presentation.

• It follows directly from the definition of the norm that

$$\min\left(\|f\|_{p(\cdot)}^{p^-}, \|f\|_{p(\cdot)}^{p^+}\right) \le A_{p(\cdot)}(f) \le \max\left(\|f\|_{p(\cdot)}^{p^-}, \|f\|_{p(\cdot)}^{p^+}\right). \tag{2.3}$$

• Hölder's inequality. For all $f \in L^{p(\cdot)}(\Omega)$, $g \in L^{p'(\cdot)}(\Omega)$ with

$$p(x) \in (1,\infty), \quad p'(x) = \frac{p(x)}{p(x)-1}$$

the following inequality holds:

$$\int_\Omega |f\,g|\,dx \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)} \le 2\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}. \tag{2.4}$$

2.2 Spaces $L^{p(\cdot,\cdot)}(Q)$ and anisotropic space $\mathbf{W}(Q)$

For every fixed $t \in [0,T]$ we introduce the Banach space

$$\begin{aligned}
&\mathbf{V}_t(\Omega) = \left\{u(x) \in L^2(\Omega) \cap W_0^{1,1}(\Omega) : \ |D_i u(x)|^{p_i(x,t)} \in L^1(\Omega)\right\},\\
&\|u\|_{\mathbf{V}_t(\Omega)} = \|u\|_{2,\Omega} + \sum_i \|D_i u\|_{p_i(\cdot,t),\Omega},
\end{aligned}$$

and denote by $\mathbf{V}_t'(\Omega)$ its dual. For every $t \in [0,T]$ the inclusion

$$\mathbf{V}_t(\Omega) \subset \mathbf{X} = W_0^{1,p^-}(\Omega) \cap L^2(\Omega)$$

holds, which is why $\mathbf{V}_t(\Omega)$ is reflexive and separable as a closed subspace of $\mathbf{X}$.

By $\mathbf{W}(Q)$ we denote the Banach space

$$\begin{aligned}
&\mathbf{W}(Q) = \left\{u \in L^2(Q) : \ |D_i u|^{p_i(x,t)} \in L^1(Q), \ u(\cdot,t) \in \mathbf{V}_t(\Omega) \text{ for a.e. } t \in (0,T)\right\},\\
&\|u\|_{\mathbf{W}(Q)} = \sum_i \|D_i u\|_{p_i(\cdot),Q} + \|u\|_{2,Q}.
\end{aligned}$$

$\mathbf{W}'(Q_T)$ is the dual of $\mathbf{W}(Q_T)$ (the space of linear functionals over $\mathbf{W}(Q_T)$):

$$w \in \mathbf{W}'(Q) \iff \begin{cases} \exists\, w = (w_0, w_1, \dots, w_n), \quad w_0 \in L^2(Q), \quad w_i \in L^{p_i'(\cdot)}(Q),\\ \forall \phi \in \mathbf{W}(Q) \quad \langle\langle w, \phi\rangle\rangle = \displaystyle\int_Q \left(w_0\phi + \sum_i w_i D_i\phi\right) dz. \end{cases}$$

The norm in $\mathbf{W}'(Q)$ is defined by

$$\|v\|_{\mathbf{W}'(Q)} = \sup\left\{\langle\langle v, \phi\rangle\rangle \,|\, \phi \in \mathbf{W}(Q), \ \|\phi\|_{\mathbf{W}(Q)} \le 1\right\}.$$

2.3 Formulas of integration by parts in t

Set

$$\mathbf{V}_+(\Omega) = \left\{u(x)|\ u \in L^2(\Omega) \cap W_0^{1,1}(\Omega),\ |\nabla u| \in L^{p^+}(\Omega)\right\}.$$

Since $\mathbf{V}_+(\Omega)$ is separable, it is a span of a countable set of linearly independent functions $\{\psi_k\} \subset \mathbf{V}_+(\Omega)$. It is convenient to choose for $\{\psi_k\}$ the system of eigenfunctions of the operator

$$(\psi_j, w)_{H_0^s(\Omega)} = \lambda_j(\psi_j, w)_{L^2(\Omega)} \quad \forall\, w \in H_0^s(\Omega) \tag{2.5}$$

with

$$\frac{s-1}{n} \geq \frac{1}{2} - \frac{1}{p^+}.$$

For such s the embedding $H_0^s(\Omega) \subset W_0^{1,p^+}(\Omega)$ is continuous.

Let ρ be the Friedrichs' mollifying kernel

$$\rho(s) = \begin{cases} \kappa\, \exp\left(-\frac{1}{1-|s|^2}\right) & \text{if } |s| < 1, \\ 0 & \text{if } |s| > 1, \end{cases} \qquad \kappa = const: \quad \int_{\mathbb{R}^{n+1}} \rho(z)\, dz = 1.$$

Given a function $v \in L^1(Q)$, we extend it to the whole $\mathbb{R}^{n+1}$ by a function with compact support (keeping the same notation for the continued function) and then define

$$v_h(z) = \int_{\mathbb{R}^{n+1}} v(s)\rho_h(z-s)\, ds \qquad \text{with } \rho_h(s) = \frac{1}{h^{n+1}}\rho\left(\frac{s}{h}\right), \quad h > 0.$$

Proposition 2.1 ([38]) *If $u \in \mathbf{W}(Q)$ with the exponents $p_i(z)$ satisfying* (2.1) *in Q, then*

$$\|u_h\|_{\mathbf{W}(Q)} \leq C\,\left(1 + \|u\|_{\mathbf{W}(Q)}\right) \text{ and } \|u_h - u\|_{\mathbf{W}(Q)} \to 0 \text{ as } h \to 0.$$

Proposition 2.2 *Let $p_i(z)$ satisfy condition* (2.1) *in Q. Then the set $\{\psi_k\}$ is dense in $\mathbf{V}_t(\Omega)$ for every $t \in [0,T]$.*

Proof. For every $u \in \mathbf{V}_t(\Omega)$ there is a sequence $u_\delta(\cdot,t) \in C^\infty(\Omega)$ such that $\operatorname{supp} u_\delta(\cdot,t) \subset\subset \Omega$ and $\|u - u_\delta\|_{\mathbf{V}_t(\Omega)} \to 0$ as $\delta \to 0$. Such a sequence is obtained via convolution of u with the Friedrics's mollifiers. Since $u_\delta \in C_0^\infty(\Omega) \subset \mathbf{V}_+(\Omega)$ and $\{\psi_m\}$ is dense in $\mathbf{V}_+(\Omega)$, one may choose constants c_m such that

$$u_\delta^{(k)} \equiv \sum_{m=1}^{k} c_m\psi_m(x) \to u_\delta \quad \text{strongly in } \mathbf{V}_+(\Omega) \text{ as } \delta \to 0.$$

Given an arbitrary $\epsilon > 0$, $\|u_\delta - u_\delta^{(k)}\|_{\mathbf{V}_+(\Omega)} < \epsilon$ for all $k \in \mathbb{N}$ from some $k(\epsilon)$ on. By (2.4)

$$\|u_\delta - u_\delta^{(k)}\|_{\mathbf{V}_t(\Omega)} \leq C\,\|u_\delta - u_\delta^{(k)}\|_{\mathbf{V}_+(\Omega)} \leq C\,\epsilon$$

with a constant $C = C(n, |\Omega|, p^\pm)$ independent of ϵ. It follows that for all sufficiently large k and small δ

$$\|u-u_\delta^{(k)}\|_{\mathbf{V}_t(\Omega)} \le \|u-u_\delta\|_{\mathbf{V}_t(\Omega)} + \|u_\delta - u_\delta^{(k)}\|_{\mathbf{V}_t(\Omega)} < 2\,\epsilon \quad \forall\, t\in[0,T].$$

□

Proposition 2.3 *For every $u\in \mathbf{W}(Q)$ there is a sequence $\{d_k(t)\}$, $d_k(t)\in C^1[0,T]$, such that*

$$\left\| u-\sum_{k=1}^m d_k(t)\psi_k(x)\right\|_{\mathbf{W}(Q)} \to 0 \quad \textit{as } m\to\infty.$$

Proof. (a) Let us first prove the assertion for $u\in C^\infty(0,T;C_0^\infty(\Omega))$. Let us represent

$$u=\sum_{i=1}^\infty u_i(t)\psi_i(x), \quad u_i(t)=(u(x,t),\,\psi_i(x))_{H_0^s(\Omega)}. \tag{2.6}$$

For every $t\in[0,T]$

$$\|u\|^2_{H_0^s(\Omega)}(t)=\sum_{i=1}^\infty u_i^2(t)<\infty. \tag{2.7}$$

Consider the sequence $u^{(m)}=\sum_{i=1}^m u_i(t)\psi_i(x)$. For every $t\in[0,T]$

$$\phi_m(t)\equiv \|u-u^{(m)}\|^2_{H_0^s(\Omega)}(t)=\sum_{i=m+1}^\infty u_i^2(t)\to 0 \quad \text{as } m\to\infty$$

because of (2.7). The sequences $\{\phi_m(t)\}$ and $\{\phi_m^{\frac{p^+}{2}}(t)\}$ are monotone decreasing, nonnegative, and tend to zero as $m\to\infty$. By the monotone convergence theorem

$$\|u-u^{(m)}\|^{\frac{p^+}{2}}_{L^{\frac{p^+}{2}}(0,T;H_0^s(\Omega))}=\int_0^T \phi_m^{\frac{p^+}{2}}(\tau)\,d\tau\to 0, \qquad \int_0^T \phi_m(t)\,dt\to 0 \quad \text{as } m\to\infty.$$

Let $v\in C^\infty([0,T];C_0^\infty(\Omega))$. Combining (2.3) with (2.4) and using the inequalities $p^-\le p_i^-\le p_i^+\le p^+$, we may estimate

$$\begin{aligned}
\|D_iv\|^{p^+}_{p_i(\cdot),Q} &\le C\max\left\{\int_Q |D_iv|^{p^+}\,dz,\ \left(\int_Q |D_iv|^{p^+}\,dz\right)^{\frac{p_i^-}{p_i^+}}\right\}\\
&\le C\begin{cases}\|D_iv\|^{p^-}_{p^+,Q} & \text{if } \|D_iv\|_{p^+,Q}<1,\\ \|D_iv\|^{p^+}_{p^+,Q} & \text{otherwise,}\end{cases}\\
&\le C\left(\int_0^T \|\nabla v\|^{p^+}_{p^+,\Omega}(t)\,dt+\left(\int_0^T\|\nabla v\|^{p^+}_{p^+,\Omega}(t)\,dt\right)^{\frac{p^-}{p^+}}\right),
\end{aligned} \tag{2.8}$$

whence

$$\begin{aligned}
\|v\|_{\mathbf{W}(Q)} &\le \|v\|_{2,Q}+C\left(\int_0^T \|v\|^{p^+}_{W_0^{1,p^+}(\Omega)}\,dt\right)^{\frac{p^-}{p^+}\frac{1}{p^+}}\left(1+\left(\int_0^T\|v\|^{p^+}_{W_0^{1,p^+}(\Omega)}\,dt\right)^{\frac{p^+-p^-}{(p^+)^2}}\right)\\
&\le C_1\|v\|_{L^2(0,T;H_0^s(\Omega))}+C_2\left(\int_0^T\|v\|^{p^+}_{H_0^s(\Omega)}\,dt\right)^{\frac{p^-}{p^+}\frac{1}{p^+}}\left(1+\left(\int_0^T\|v\|^{p^+}_{H_0^s(\Omega)}\,dt\right)^{\frac{p^+-p^-}{(p^+)^2}}\right).
\end{aligned}$$

Letting $v = u - u^{(m)}$ we finally have:

$$\|u - u^{(m)}\|_{\mathbf{W}(Q)} \le \widetilde{C} \int_0^T \phi_m(t)\,dt$$
$$+ \widetilde{C}_1 \left(\int_0^T \phi_m^{\frac{p^+}{2}}(t)\,dt\right)^{\frac{p^-}{(p^+)^2}} \left(1 + \left(\int_0^T \phi_m^{\frac{p^+}{2}}(t)\,dt\right)^{\frac{p^+-p^-}{(p^+)^2}}\right) \to 0 \quad \text{as } m \to \infty.$$

(b) Let now $u \in \mathbf{W}(Q)$, and let $\{u_\delta\}$ be the sequence of mollifications, $u_\delta \in C^\infty([0,T]; C_0^\infty(\Omega))$. Given an arbitrary $\epsilon > 0$, we take δ such that $\|u - u_\delta\|_{\mathbf{W}(Q)} < \epsilon$ and approximate u_δ using item (a):

$$\left\|u - u_\delta^{(m)}\right\|_{\mathbf{W}(Q)} \le \|u - u_\delta\|_{\mathbf{W}(Q)} + \|u_\delta - u_\delta^{(m)}\|_{\mathbf{W}(Q)} < 2\epsilon$$

with

$$u_\delta^{(m)} = \sum_{i=1}^m d_i(t)\psi_i(x), \qquad d_i(t) = (u_\delta, \psi_i)_{H_0^s(\Omega)} \in C^\infty[0,T].$$

□

Proposition 2.4 *Let in the conditions of Proposition* 2.1 $u_t \in \mathbf{W}'(Q)$. *Then* $(u_h)_t \in \mathbf{W}'(Q)$, *and for every* $\psi \in \mathbf{W}(Q)$

$$\langle\langle (u_h)_t, \psi \rangle\rangle \to \langle\langle u_t, \psi \rangle\rangle \quad \textit{as } h \to 0.$$

Proof. By the definition of $\mathbf{W}'(Q)$ there exist $\phi_0 \in L^2(Q)$, $\phi_i \in L^{p_i'(\cdot)}(Q)$ such that $\langle\langle u_t, \psi \rangle\rangle = (\phi_0, \psi)_{2,Q} + \sum_i (\phi_i, D_i\psi)_{2,Q}$, $\forall\, \psi \in \mathbf{W}(Q)$. Without loss of generality we assume that the functions u and ϕ_i $(i = 0, 1, \ldots, n)$ are continued to the whole of $\mathbb{R}^{n+1}$ by functions with compact supports. By the definition

$$|(u_t)_h| = |\langle\langle u_t,\, \rho_h \rangle\rangle| \le \|u_t\|_{\mathbf{W}'}\|\rho_h\|_{\mathbf{W}} \le C(h), \quad |(u_h)_t| = |(u, (\rho_h)_t)_2| \le \|u\|_2\|(\rho_h)_t\|_2 \le C(h),$$

whence $\partial_\tau\,(u(y,\tau)\,\rho_h(x-y, t-\tau)) \in L^1(\mathbb{R}^{n+1})$ for every $(x,t) \in \mathbb{R}^{n+1}$ and

$$\int_{\mathbb{R}^{n+1}} \partial_\tau\,(u(y,\tau)\,\rho_h(x-y,t-\tau))\;dyd\tau = 0.$$

It follows that for every $(x,t) \in \mathbb{R}^{n+1}$

$$(u_h)_t = \partial_t(u * \rho_h)(x,t) = u * (\rho_h)_t(x,t) = -\int u(y,\tau)(\rho_h)_\tau(x-y,t-\tau)\,dyd\tau$$
$$= -\int \partial_\tau(u\,\rho_h)\,dyd\tau + \int u_\tau(y,\tau)\,\rho_h(x-y,t-\tau)\,dyd\tau = (u_t * \rho_h)(x,t) = (u_t)_h.$$

For every $\psi \in \mathbf{W}(Q)$, continued by zero to $\mathbb{R}^{n+1}$,

$$\langle\langle (u_h)_t, \psi \rangle\rangle = \int (u_t)_h \psi\,dz = \int u_t \psi_h\,dz = \int \left(\phi_0 \psi_h + \sum_i \phi_i\, D_i \psi_h\right)\,dz$$
$$= \int \left((\phi_0)_h \psi + \sum_i (\phi_i)_h\, D_i \psi\right)\,dz \to \langle\langle u_t, \psi \rangle\rangle \quad \text{as } h \to 0$$

by virtue of Proposition 2.1. □

Proposition 2.5 (Integration by parts) *Let $v, w \in \mathbf{W}(Q)$ and $v_t, w_t \in \mathbf{W}'(Q)$ with the exponents $p_i(z)$ satisfying* (2.1) *in Q. Then*

$$\forall\,\text{a.e.}\, t_1, t_2 \in (0,T] \qquad \int_{t_1}^{t_2}\int_\Omega v\, w_t\, dz + \int_{t_1}^{t_2}\int_\Omega v_t\, w\, dz = \int_\Omega v\, w\, dx\Big|_{t=t_1}^{t=t_2}.$$

Proof. Let $t_1 < t_2$. Take

$$\chi_k(t) = \begin{cases} 0 & \text{for } t \le t_1, \\ k(t-t_1) & \text{for } t_1 \le t \le t_1 + \frac{1}{k}, \\ 1 & \text{for } t_1 + \frac{1}{k} \le t \le t_2 - \frac{1}{k}, \\ k(t_2 - t) & \text{for } t_2 - \frac{1}{k} \le t \le t_2, \\ 0 & \text{for } t \ge t_2. \end{cases}$$

For every $k \in \mathbb{N}$ and $h > 0$

$$0 = \int_Q (v_h w_h \chi_k)_t\, dz \equiv \int_Q (v_h w_h)_t \chi_k\, dz - k \int_{\theta-\frac{1}{k}}^{\theta}\int_\Omega v_h w_h\, dz\Big|_{\theta=t_1}^{\theta=t_2}.$$

The last two integrals on the right-hand side exist because $v_h, w_h \in L^2(Q)$. Letting $h \to 0$, we obtain the equality

$$\lim_{h\to 0}\int_Q (v_h\,(w_h)_t + (v_h)_t w_h)\,\chi_k(t)\, dz = k\int_{t_2-\frac{1}{k}}^{t_2}\int_\Omega v\, w\, dz - k\int_{t_1}^{t_1+\frac{1}{k}}\int_\Omega v\, w\, dz.$$

According to Propositions 2.1, 2.4 $v_h \to v$ in $\mathbf{W}(Q)$, $(w_h)_t = (w_t)_h \to w_t$ weakly in $\mathbf{W}'(Q)$ as $h \to 0$, and $\|v\|_{\mathbf{W}}$, $\|(w_h)_t\|_{\mathbf{W}'}$ are uniformly bounded. It follows that

$$\begin{aligned} \lim_{h\to 0}\int_Q v_h\,(w_h)_t\,\chi_k(t)\, dz &= \lim_{h\to 0}\int_Q (v_h - v)(w_h)_t\chi_k(t)\, dz \\ &+ \lim_{h\to 0}\int_Q v\,((w_h)_t - w_t)\chi_k(t)\, dz + \int_Q v\, w_t\,\chi_k(t)\, dz = \int_Q v\, w_t\chi_k(t)\, dz. \end{aligned}$$

In the same way we check that

$$\lim_{h\to 0}\int_Q (v_h)_t\, w_h\,\chi_k(t)\, dz = \int_Q v\, w_t\chi_k(t)\, dz.$$

By the Lebesgue differentiation theorem

$$\forall\,\text{a.e.}\,\theta > 0 \qquad \lim_{k\to 0} k\int_{\theta-\frac{1}{k}}^{\theta}\left(\int_\Omega v\, w\, dx\right) dt = \int_\Omega v\, w\, dx,$$

whence for almost every t_1, $t_2 \in [0,T]$

$$\int_{t_1}^{t_2}\int_\Omega (v\, w_t + v_t w)\, dz = \lim_{k\to\infty}\int_Q (v\, w_t + v_t w)\chi_k(t)\, dz = \lim_{k\to\infty} k\int_{\theta-\frac{1}{k}}^{\theta}\int_\Omega v\, w\, dx\Big|_{\theta=t_1}^{t=t_2} = \int_\Omega v\, w\, dx\Big|_{\theta=t_1}^{t=t_2}.$$

$\square$

Corollary 2.1 *Let $u \in \mathbf{W}(Q)$ and $u_t \in \mathbf{W}'(Q)$ with the exponents $p_i(z)$ satisfying* (2.1)*. Then*

$$\forall \, \text{a.e.}\, t_1, t_2 \in (0,T] \qquad \int_{t_1}^{t_2} \int_\Omega u\, u_t \, dz = \frac{1}{2} \|u\|^2_{2,\Omega} \Big|_{t=t_1}^{t=t_2}.$$

3 The Faedo-Galerkin method

We are interested in the *energy solutions* of the problem

$$\begin{cases} u_t - \sum_i D_i \left(|D_i u|^{p_i(z)-2} D_i u \right) + d(z,u) = 0 \quad \text{in } Q, \\ u = 0 \text{ on } \Gamma_T, \qquad u(x,0) = u_0(x) \text{ in } \Omega, \end{cases} \tag{3.1}$$

understood in the following way.

Definition 3.1 (Energy solution) *A function $u : Q \mapsto \mathbb{R}$ is called weak (energy) solution of problem* (3.1) *if*

1. $u \in \mathbf{W}(Q) \cap C([0,T];\, L^2(\Omega))$, $u_t \in \mathbf{W}'(Q)$;
2. *for every test-function $\zeta \in \mathbf{Z} \equiv \{\eta(z) : \eta \in \mathbf{W}(Q) \cap C\left([0,T]; L^2(\Omega)\right),\ \eta_t \in \mathbf{W}'(Q)\}$*
$$\int_Q \Big(u_t\, \zeta + \sum_i |D_i u|^{p_i - 2} D_i u \cdot D_i \zeta + d(z,u)\zeta \Big)\, dz = 0. \tag{3.2}$$
3. *for every $\eta(x) \in C_0^\infty(\Omega)$*
$$\int_\Omega (u(x,t) - u_0(x))\, \eta(x)\, dx \to 0 \quad \textit{as } t \to 0.$$

Theorem 3.1 ([18]) a) *Let us assume that*

- *$d(z,r)$ satisfy the growth condition*
$$\begin{cases} |d(z,u)| \leq d_0 |u|^{\lambda - 1} + h_d(z), \quad \lambda = \textit{const} > 1, \quad d_0 = \textit{const} > 0, \\ \textit{with } \lambda = \max\{2, p^- - \delta\} \textit{ for some } \delta > 0, \\ h_d \in L^{\lambda'}(Q), \end{cases} \tag{3.3}$$
- *$p_i(z)$ take values in the interval $(p^-, p^+) \subset (1, \infty)$ and satisfy condition* (2.1) *of log-continuity.*

Then for every $u_0 \in L^2(\Omega)$ problem (3.1) *has at least one weak solution $u \in \mathbf{W}(Q)$ satisfying the energy estimate*

$$\|u\|^2_{L^\infty(0,T;L^2(\Omega))} + \int_Q \sum_i |D_i u|^{p_i} dz \leq M \left[\|u_0\|^2_{L^2(\Omega)} + \|h_d\|^{\lambda'}_{L^{\lambda'}(Q)} + 1 \right] \tag{3.4}$$

with a constant M independent of u. Moreover, $u_t \in \mathbf{W}'(Q)$.

b) *The assertion remains true if* (3.3) *is substituted by the weaker condition $\lambda = \max\{2, p^-\}$ and d_0 is appropriately small.*

3.1 Approximate solutions: existence and uniform a priori estimates

The proof of Theorem 3.1 is an imitation of the classical Faedo-Galerkin method. Define the operator $L: \mathbf{V}_t(\Omega) \mapsto \mathbf{V}'_t(\Omega)$:

$$\langle Lv, \phi\rangle_\Omega = \int_\Omega \Big(\sum_{i=1}^n |D_i v|^{p_i-2} D_i v \cdot D_i \phi + d(z,v)\phi \Big)\, dx, \quad \phi \in \mathbf{V}_t(\Omega).$$

Let $\{\psi_k\}$ be the orthonormal basis of $H^s_0(\Omega)$ composed of the eigenfunctions of problem (2.5). The approximate solutions to problem (1.2) are sought in the form

$$u^{(m)}(z) = \sum_{k=1}^m c_k^{(m)}(t)\psi_k(x),$$

where the coefficients $c_k^{(m)}(t)$ are defined from the relations

$$(u_t^{(m)}, \psi_k)_{L^2(\Omega)} = -\Big\langle Lu^{(m)}, \psi_k \Big\rangle_\Omega = 0, \quad k = 1, \dots, m. \tag{3.5}$$

Equalities (3.5) generate the system of m ordinary differential equations for the coefficients $c_k^{(m)}(t)$:

$$\begin{cases} \left(c_k^{(m)}\right)' = F_k\left(t, c_1^{(m)}(t), \dots, c_m^{(m)}(t)\right), \\ c_k^{(m)}(0) = \displaystyle\int_\Omega u_0(x)\psi_k\, dx \qquad k = 1, \dots, m. \end{cases} \tag{3.6}$$

Lemma 3.1 (Regular case) *Let the conditions of Theorem* 3.1, *a*) *be fulfilled. Then for every* $T < \infty$ *and* $m \in \mathbb{N}$ *system* (3.6) *has a solution* $\left\{c_k^{(m)}(t)\right\}_{k=1}^m$ *on the interval* $(0,T)$ *and the corresponding function* $u^{(m)}$ *satisfies the uniform estimate*

$$\|u^{(m)}(\cdot,t)\|^2_{L^\infty(0,T;L^2(\Omega))} + \sum_i \int_Q |D_i u^{(m)}|^{p_i} dz \le M\left[\|u_0\|^2_{L^2(\Omega)} + \|h_d\|^{\lambda'}_{\lambda',Q} + 1\right]. \tag{3.7}$$

Proof. By Peano's Theorem, for every finite m system (3.6) has a solution $c_i^{(m)}(t)$, $i = 1, \dots, m$, on an interval $(0, T_m)$. Multiplying each of equalities (3.5) by $c_k^{(m)}(t)$ and summing over $k = 1, \dots, m$ we arrive at the equality

$$\|u^{(m)}\|^2_{2,\Omega}\Big|_{t=0}^{t=\tau} + \int_{Q\cap\{t<\tau\}} \Big(\sum_i |D_i u^{(m)}|^{p_i} + d(z, u^{(m)})u^{(m)} \Big)\, dz = 0, \quad \tau \in [0, T_m]. \tag{3.8}$$

Using the growth condition (3.3) and applying Young's inequality, we find that for every $\epsilon > 0$

$$|d(z, u^{(m)})u^{(m)}| \le (d_0 + \epsilon)|u^{(m)}|^\lambda + C_\epsilon |h_d|^{\lambda'}.$$

Plugging this inequality into (3.8) we obtain:

$$\frac{1}{2}\,\|u^{(m)}\|^2_{2,\Omega}\Big|_{t=0}^{t=\tau}+\int_{Q\cap\{t<\tau\}}\sum_i |D_iu^{(m)}|^{p_i}\,dz\le C_\epsilon(d_0+\epsilon+1)\int_{Q\cap\{t<\tau\}}\left(|u^{(m)}|^\lambda+|h_d|^{\lambda'}\right)dz. \tag{3.9}$$

There are two possibilities.

(a) $\lambda=2$. Dropping the nonnegative term on the left-hand side of (3.9) we may write it in the form

$$\frac{d}{dt}\left(\|u^{(m)}\|^2_{2,Q\cap\{t<\tau\}}\right)\le(C_\epsilon+\epsilon+1)\,\left(\|u^{(m)}\|^2_{2,Q\cap\{t<\tau\}}+\|h_d\|^{\lambda'}_{\lambda',Q\cap\{t<\tau\}}\right)+\|u_0^{(m)}\|^2_{2,\Omega}.$$

By Gronwall's inequality $\|u^{(m)}\|^2_{2,Q_\tau}$ is uniformly bounded which yields the uniform estimate on $\|u^{(m)}(\cdot,\tau)\|^2_{2,\Omega}$, whence (3.7).

(b) $2<\lambda=p^- -\delta$. By the assumption $\lambda<p^-\delta<\dfrac{n(p^- -\delta)}{n-p^-+\delta}$. Applying Poincaré and Young inequalities we estimate

$$\begin{aligned}\int_\Omega |u^{(m)}|^\lambda dx &\le C(\lambda,p^-,n)\|\nabla u^{(m)}\|^\lambda_{p^-,\Omega}\le C\left(\int_\Omega|\nabla u^{(m)}|^{p^-}dx\right)^{\frac{\lambda}{p^-}}\\ &\le C\left[\left(\sum_i\int_\Omega|D_iu^{(m)}|^{p_i}dx\right)^{\frac{\lambda}{p^-}}+1\right]\le\epsilon\,\sum_i\int_\Omega|D_iu^{(m)}|^{p_i}dx+K\end{aligned} \tag{3.10}$$

with an arbitrary $\delta>0$ and a constant $K\equiv K(\epsilon,\delta,\Omega,a_0\ ,\ p^\pm)$. Choosing δ so small that $\delta\,C_\epsilon(d_0+\epsilon+1)\le\dfrac{1}{2}$, we obtain from (3.9)-(3.10)

$$\|u^{(m)}\|^2_{2,\Omega}\Big|_{t=0}^{t=\tau}+\sum_i\int_{Q\cap\{t<\tau\}}|D_iu^{(m)}|^{p_i}\,dz\le 2\,(K+\tau).$$

□

Lemma 3.2 (Borderline case) *If* $\lambda=\max\{2,p^-\}$ *and the constant* d_0 *is sufficiently small, then*

$$\|u^{(m)}(\cdot,t)\|^2_{L^\infty(0,T;L^2(\Omega))}+\sum_i\int_{Q\cap\{t<\tau\}}|D_iu^{(m)}|^{p_i}dz\le M\left[\|u_0\|^2_{L^2(\Omega)}+\|h_d\|^{\lambda'}_{\lambda',Q}+1\right].$$

Proof. We only have to study the case $\lambda=p^-$. The Poincaré inequality yields (see (3.10))

$$\int_\Omega|u^{(m)}|^\lambda dx\le C\int_\Omega|\nabla u^{(m)}|^{p^-}dx\le C\,\epsilon\sum_i\int_\Omega|D_iu^{(m)}|^{p_i}\,dx+R$$

with an independent of m constant R. Then

$$\frac{1}{2}\|u^{(m)}\|^2_{2,\Omega}\Big|_{t=0}^{t=\tau} + \sum_i \int_{Q\cap\{t<\tau\}} |D_i u^{(m)}|^{p_i}\,dz$$
$$\leq C'(d_0+\epsilon)\sum_i \int_{Q\cap\{t<\tau\}} |D_i u^{(m)}|^{p_i}\,dz + C''\,\|h_d\|^{\lambda'}_{\lambda',Q} + const. \tag{3.11}$$

The conclusion follows if we claim that $C'(d_0+\epsilon_d) < 1/2$. □

The uniform estimates of Lemmas 3.1, 3.2 allow one to continue the approximations $u^{(m)}$ to the time interval $(0,T)$.

Lemma 3.3 *Under the foregoing conditions, $u_t^{(m)} \in L^{(p^+)'}(0,T;H^{-s}(\Omega))$ and*

$$\|u_t^{(m)}\|_{L^{(p^+)'}(0,T;H^{-s}(\Omega))} \leq C'\,\left[1+\|h_d\|^{\lambda'}_{\lambda',Q}+\|u_0\|^2_{2,\Omega}\right].$$

Proof. Let

$$\mathbf{Z}_m = \left\{\eta(x,t)\,\middle|\,\eta=\sum_{k=1}^m d_k(t)\psi_k(x)\ ,\ d_k(t)\in C^1(0,T)\right\} \subset L^{p^+}(0,T;H^s_0(\Omega)) \subset \mathbf{W}(Q). \tag{3.12}$$

Given $\phi \in L^{p^+}(0,T;H^s_0(\Omega))$, we denote

$$\phi^{(m)} = \sum_{i=1}^m \phi_k(t)\psi_k(x) \in \mathbf{Z}_m.$$

Since $\{\psi_k\}$ are orthogonal in $L^2(\Omega)$, the definition of $u^{(m)}$ yields

$$-\int_Q u_t^{(m)}\phi\,dz = -\int_Q u_t^{(m)}\phi^{(m)}\,dz = \sum_{i=1}^n \int_Q |D_i u^{(m)}|^{p_i-2}D_i u^{(m)}\cdot D_i\phi^{(m)}\,dz + \int_Q d(z,u^{(m)})\phi^{(m)}\,dz.$$

Applying Hölder's inequality (2.4) and using (2.8) we estimate

$$\Big|\int_Q \sum_{i=1}^n |D_i u^{(m)}|^{p_i-2}D_i u^{(m)} D_i\phi^{(m)}\,dz\Big| \leq C\sum_{i=1}^n \||D_i u^{(m)}|^{p_i-1}\|_{p_i',Q}\|D_i\phi^{(m)}\|_{p_i,Q}$$
$$\leq C\|u^{(m)}\|_{\mathbf{W}(Q)}\left(\sum_{i=1}^n \|D_i\phi^{(m)}\|_{p_i(\cdot),Q}\right)$$
$$\leq C\,\|u^{(m)}\|_{\mathbf{W}(Q)}\left(\int_0^T \|\nabla\phi^{(m)}\|^{p^+}_{p^+,\Omega}(t)\,dt + \left(\int_0^T \|\nabla\phi^{(m)}\|^{p^+}_{p^+,\Omega}(t)\,dt\right)^{\frac{p^-}{p^+}}\right)^{\frac{1}{p^+}}$$
$$\leq C\,\|u^{(m)}\|_{\mathbf{W}(Q)}\left(\|\phi^{(m)}\|_{L^{p^+}(0,T;H^s_0(\Omega))} + \|\phi^{(m)}\|^{\frac{p^-}{p^+}}_{L^{p^+}(0,T;H^s_0(\Omega))}\right)$$
$$\leq C\,\|u^{(m)}\|_{\mathbf{W}(Q)}\left(\|\phi\|_{L^{p^+}(0,T;H^s_0(\Omega))} + \|\phi\|^{\frac{p^-}{p^+}}_{L^{p^+}(0,T;H^s_0(\Omega))}\right),$$

$$\left|\int_Q d\left(z,u^{(m)}\right)\phi^{(m)}\,dz\right| \leq \|d\|_{\lambda',Q}\|\phi^{(m)}\|_{\lambda,Q} \leq \left[C(d_0,\lambda)\||u^{(m)}|^{\lambda-1}\|_{\lambda',Q}+\|h_d\|_{\lambda',Q}\right]\|\phi^{(m)}\|_{\lambda,Q}$$
$$\leq C\left(1+\|u^{(m)}\|_{\lambda,Q}\right)\|\phi^{(m)}\|_{\lambda,Q} \leq C\left[1+\|u^{(m)}\|_{\mathbf{W}(Q)}\right]\|\phi^{(m)}\|_{\lambda,Q}$$
$$\leq C\left[1+\|u^{(m)}\|_{\mathbf{W}(Q)}\right]\|\phi^{(m)}\|_{L^{p^+}(0,T;H_0^s(\Omega))}$$
$$\leq C\left[1+\|u^{(m)}\|_{\mathbf{W}(Q)}\right]\|\phi\|_{L^{p^+}(0,T;H_0^s(\Omega))}$$

with independent of m constants C. For every $\phi \in L^{p^+}(0,T;H_0^s(\Omega))$ and arbitrary m

$$\left|\int_Q u_t^{(m)}\phi\,dz\right| \leq C\left[1+\|u^{(m)}\|_{\mathbf{W}(Q)}\right]\left(\|\phi\|_{L^{p^+}(0,T;H_0^s(\Omega))}+\|\phi\|^{\frac{p^-}{p^+}}_{L^{p^+}(0,T;H_0^s(\Omega))}\right)$$

and the assertion follows from the uniform estimates on $\|u^{(m)}\|_{\mathbf{W}(Q)}$. □

3.2 Compactness and passage to the limit

Let us show that the constructed sequence $\{u^{(m)}\}$ contains a convergent subsequence. The following inclusions hold:

$$\begin{cases} u^{(m)} \in \mathbf{W}(Q) \subseteq L^{p^-}(0,T;\, W_0^{1,p^-}(\Omega)), \\ u_t^{(m)} \in L^{(p^+)'}(0,T;\, H^{-s}(\Omega)), \\ W_0^{1,p^-}(\Omega) \subset L^2(\Omega) \subset H^{-s}(\Omega). \end{cases}$$

This yields ([36, Th.5]) that $\{u^{(m)}\}$ contains a subsequence which converges strongly in $L^q(Q)$ with some $q > 1$ and almost everywhere in Q to a function u. Using the uniform in m estimates of the previous section we extract from the sequence $\{u^{(m)}\}$ a subsequence (for the sake of simplicity we always assume that it merely coincides with the whole of the sequence) such that

$$\begin{cases} u^{(m)} \to u \text{ weakly in } \mathbf{W}(Q), \text{ strongly in } L^q(Q) \text{ with some } q>1, \text{ and a.e. in } Q, \\ u^{(m)}(x,T) \to U(x) \text{ weakly in } L^2(\Omega), \\ u^{(m)}(x,0) \to u_0 \text{ in } L^2(\Omega), \\ u_t^{(m)} \to u_t \quad \text{weakly in } L^{(p^+)'}(0,T;\, H^{-s}(\Omega)), \\ d\left(z,u^{(m)}\right) \to d(z,u) \text{ strongly in } L^{\lambda'}(Q), \\ \left|D_i u^{(m)}\right|^{p_i(z)-2} D_i u^{(m)} \to A_i(z) \text{ weakly in } L^{p_i'(\cdot)}(Q), \end{cases} \tag{3.13}$$

for some functions

$$U \in L^2(\Omega),\ \ A_i(z) \in L^{p_i'(\cdot)}(Q) \ \text{ and } u \in \mathbf{W}(Q).$$

Fix an arbitrary $l \in \mathbb{N}$. Each of $u^{(m)}$ with $m > l$ satisfies the identity

$$\int_Q \left[u_t^{(m)}\eta_l + \sum_i |D_i u^{(m)}|^{p_i-2}D_i u^{(m)}\cdot D_i\eta_l + d(z,u^{(m)})\,\eta_l\right]dxdt = 0, \qquad \eta_l \in \mathbf{Z}_l \subset \mathbf{Z}_m. \tag{3.14}$$

Letting $m \to \infty$ in (3.14) we conclude that for every fixed l

$$-\int_Q u_t \eta_l \, dz = \int_Q \left[\sum_i A_i(z) \cdot D_i \eta_l + d(z,u)\eta_l \right] dz, \qquad \forall\, \eta_l \in \mathbf{Z}_l. \tag{3.15}$$

Let $\eta \in \mathbf{W}(Q)$. Since $L^{p^+}(0,T; H_0^s(\Omega))$ is dense in $\mathbf{W}(Q)$, there is a sequence $\{\eta_l\} \in L^{p^+}(0,T; H_0^s(\Omega))$ such that $\|\eta_l - \eta\|_{\mathbf{W}(Q)} \to 0$ as $l \to \infty$. Then $D_i \eta_l \to D_i \eta$ weakly in $L^{p_i(\cdot)}(Q)$, which allows one to pass to the limit as $l \to \infty$ in the first term on the right-hand side of (3.15). Under the conditions of Theorem 3.1 $\|\eta_l\|_{\lambda,Q} \le C\,\|\eta_l\|_{\mathbf{W}(Q)}$, whence $\eta_l \to \eta$ weakly in $L^{\lambda}(Q)$ (up to a subsequence). Since $d(z,u) \in L^{\lambda'}(Q)$, we may now pass to the limit as $l \to \infty$ in the second term on the right-hand side of (3.15). This means that the left-hand side of (3.15) also has a limit as $l \to \infty$,

$$-\int_Q u_t \eta \, dz = \int_Q \left[\sum_i A_i(z) \cdot D_i \eta + d(z,u)\eta \right] dz \qquad \forall\, \eta \in \mathbf{W}(Q),$$

and $u_t \in \mathbf{W}'(Q)$. Moreover, by virtue of Lemmas 3.1, 3.2 and (3.13)

$$\left| \int_Q u_t \eta \, dz \right| \le \left(\sum_i \|A_i\|_{p_i'(\cdot),Q} + \|d\|_{\lambda',Q} \right) \|\eta\|_{\mathbf{W}(Q)} \le \widetilde{C}\,\|\eta\|_{\mathbf{W}(Q)}$$

with a constant C depending on the norms of u_0 and h_d, whence $\|u_t\|_{\mathbf{W}'(Q)} \le \widetilde{C}$.

Lemma 3.4 *Let $u \in \mathbf{W}(Q)$, $u_t \in \mathbf{W}'(Q)$. There exists $\widehat{u} \in C([0,T]; L^2(\Omega)) \cap \mathbf{W}(Q)$ such that $\|u - \widehat{u}\|_{2,\Omega}(t) = 0$ for a.e. $t \in (0,T)$.*

Proof. Let $u_h \in C^\infty(Q)$ be the sequence of mollifications of u. By Propositions 2.1, 2.4 $u_h \to u$ in $\mathbf{W}(Q)$, $(u_h)_t \to u_t$ in $\mathbf{W}'(Q)$ as $h \to 0$. For every $h_1, h_2 > 0$ and every $t \in [0,T]$

$$\|u_{h_1} - u_{h_2}\|_{2,\Omega}^2(t) = \|u_{h_1} - u_{h_2}\|_{2,\Omega}^2(0) + 2 \int_0^t \int_\Omega (u_{h_1} - u_{h_2})(u_{h_1} - u_{h_2})_t \, dz,$$

whence

$$\begin{aligned} \sup_{t \in (0,T)} \|u_{h_1} - u_{h_2}\|_{2,\Omega}^2(t) &\le \|u_{h_1} - u_{h_2}\|_{2,\Omega}^2(0) + 2 \int_Q (u_{h_1} - u_{h_2})(u_{h_1} - u_{h_2})_t \, dz \\ &\le \|u_{h_1} - u_{h_2}\|_{2,\Omega}^2(0) + 2\,\|u_{h_1} - u_{h_2}\|_{\mathbf{W}(Q)} \|(u_{h_1} - u_{h_2})_t\|_{\mathbf{W}'(Q)} \to 0 \quad \text{as } h_1, h_2 \to 0. \end{aligned}$$

It follows that $\{u_h\} : [0,T] \mapsto L^2(\Omega)$ is a Cauchy sequence in $C([0,T]; L^2(\Omega))$ and contains a subsequence which converges to a function $\widehat{u} \in C([0,T]; L^2(\Omega))$. On the other hand, $u_h \to u \in \mathbf{W}(Q)$, which means that $\|u - \widehat{u}\|_{2,\Omega}(t) = 0$ for a.e. $t \in (0,T)$. $\square$

From now on we identify the limit function $u \in \mathbf{W}(Q)$ in (3.13) with its continuous representative $\widehat{u} \in C([0,T]; L^2(\Omega)) \cap \mathbf{W}(Q)$, in particular, $U(x) = \widehat{u}(x,T)$ and $\widehat{u}(x,0) = u_0(x) \in L^2(\Omega)$. Notice that by virtue of Lemma 3.4 the inclusion $\eta \in \mathbf{Z}$ yields $\eta \in C([0,T]; L^2(\Omega))$. Applying Lemma 3.4 and Proposition 2.5 we have then that for every $t_1, t_2 \in [0,T]$

$$\int_\Omega u\,\eta \, dx \Big|_{t=t_1}^{t=t_2} + \int_{Q \cap \{t \in (t_1,t_2)\}} \left[-u\,\eta_t + \sum_{i=1}^n A_i(z) \cdot D_i \eta + d(z,u)\,\eta \right] dz = 0 \quad \forall\, \eta \in \mathbf{Z}. \tag{3.16}$$

Since $u \in \mathbf{Z}$, it can be taken for the test-function in (3.16), which leads to the energy equality

$$\forall\, t_1, t_2 \in [0,T] \qquad \frac{1}{2}\int_{\Omega} u^2\,dx\Big|_{t=t_1}^{t=t_2} + \int\limits_{Q\cap\{t_1<t<t_2\}} \left[\sum_i A_i(z)\cdot D_i u + d(z,u)\,u\right] dz = 0. \quad (3.17)$$

The initial condition for the function $u \in \mathbf{Z}\cap C(0,T;L^2(\Omega))$ immediately follows from (3.16) with $\eta \in C_0^\infty(\Omega)$ and $t_1 = 0$: for every $t \in [0,T]$

$$\int_{\Omega}(u(x,t)-u_0(x))\,\eta(x)\,dx = -\int_0^t\int_{\Omega}\left[\sum_i A_i(z)\cdot D_i\eta + d(z,u)\,\eta\right] dz \to 0 \quad \text{as } t\to 0.$$

It remains to identify the functions $A_i(z)$.

Lemma 3.5 *For a.a. $z \in Q$ $A_i(z) = |D_i u|^{p_i(z)-2}D_i u$, $(i = 1,\dots,n)$.*

Proof. The proof is an adaptation of the proof given in [33, pp.158-161]. We rely on the monotonicity of the operator $\mathcal{M}(s) = |s|^{p-2}s$: $\forall\,\xi,\eta\in\mathbb{R}^n$

$$(\mathcal{M}(\xi)-\mathcal{M}(\eta))\,(\xi-\eta) \geq \begin{cases} 2^{-p}\,|\xi-\eta|^p & \text{if } 2\leq p<\infty,\\ (p-1)\,|\xi-\eta|^2\,(|\xi|^p+|\eta|^p)^{\frac{p-2}{p}} & \text{if } 1<p<2. \end{cases} \quad (3.18)$$

According to (3.18), for every $\zeta\in\mathbf{Z}_m$

$$X_m^{(i)} := \int_Q \left(|D_i u^{(m)}|^{p_i-2}D_i u^{(m)} - |D_i\zeta|^{p_i-2}D_i\zeta\right)\,D_i(u^{(m)}-\zeta)\,dz \geq 0.$$

By virtue of (3.14) with $\eta_l = u^{(m)}$

$$\frac{1}{2}\|u^{(m)}(\cdot,t)\|_{2,\Omega}^2\Big|_{t=0}^{t=T} + \int_Q\left[\sum_i |D_i u^{(m)}|^{p_i} + u^{(m)}\,d(z,u^{(m)})\right] dz = 0.$$

Notice that for every $\zeta\in\mathbf{Z}_s$ with $s\leq m$

$$\begin{aligned} |D_i u^{(m)}|^{p_i} &= |D_i u^{(m)}|^{p_i-2}D_i u^{(m)}\cdot D_i(u^{(m)}-\zeta) + |D_i u^{(m)}|^{p_i-2}D_i u^{(m)}\cdot D_i\zeta\\ &= \left(|D_i u^{(m)}|^{p_i-2}D_i u^{(m)} - |D_i\zeta|^{p_i-2}D_i\zeta\right)\cdot D_i(u^{(m)}-\zeta)\\ &\qquad + |D_i\zeta|^{p_i-2}D_i\zeta\cdot D_i(u^{(m)}-\zeta) + |D_i u^{(m)}|^{p_i-2}D_i u^{(m)}\cdot D_i\zeta \end{aligned}$$

and write the previous equality in the form

$$\begin{aligned} 0 \leq \sum_i X_m^{(i)} &= -\frac{1}{2}\|u^{(m)}(\cdot,T)\|_{2,\Omega}^2 + \frac{1}{2}\|u^{(m)}(\cdot,0)\|_{2,\Omega}^2 - \int_Q u^{(m)}d(z,u^{(m)})\,dz\\ &\quad -\sum_i\int_Q\left(|D_i u^{(m)}|^{p_i-2}D_i u^{(m)} - |D_i\zeta|^{p_i-2}D_i\zeta\right)\cdot D_i\zeta\,dz\\ &\quad -\sum_i\int_Q |D_i\zeta|^{p_i-2}D_i\zeta\cdot D_i u^{(m)}\,dz. \end{aligned}$$

By the property of lower semicontinuity of the norm

$$-\liminf_{m\to\infty}\|u^{(m)}(\cdot,T)\|^2_{2,\Omega} \leq -\|u(\cdot,T)\|^2_{2,\Omega}.$$

Letting $m\to\infty$ we now find that

$$0 \leq \frac{1}{2}\|u_0\|^2_{2,\Omega} - \frac{1}{2}\|u(\cdot,T)\|^2_{2,\Omega} - \int_Q \left[\sum_i A_i(z)\cdot D_i u + u\, d(z,u)\right] dz$$
$$+\sum_i \int_Q A_i(z)\cdot D_i(u-\zeta)\,dz - \sum_i \int_Q |D_i\zeta|^{p_i-2} D_i\zeta \cdot D_i(u-\zeta)\,dz.$$

Using (3.17) with $t_1 = 0$, $t_2 = T$, we finally have:

$$\sum_i \int_Q \left[A_i(z) - |D_i\zeta|^{p_i-2}D_i\zeta\right]\cdot D_i(u-\zeta)\,dz \geq 0 \quad \forall\,\zeta\in \mathbf{Z}_s,\ s\in\mathbb{N}.$$

We may now take $\zeta = u\pm\epsilon\,\eta$ with arbitrary $\epsilon>0$ and $\eta\in\mathbf{W}(Q)$. Simplifying and letting $\epsilon\to 0$ we arrive at the inequalities

$$\pm\sum_i \int_Q \left[A_i(z) - |D_i u|^{p_i(z)-2}D_i u\right]\cdot D_i\eta\,dz \geq 0 \qquad \forall\,\eta\in\mathbf{W}(Q),$$

which are impossible unless $A_i(z) = |D_i u|^{p_i(z)-2}D_i u$ a.e. in Q. □

The proof of Theorem 3.1 is completed.

Corollary 3.1 *For every $t_1, t_2 \in [0,T]$ the solution of problem (3.1) satisfies the identity*

$$\int_{Q\cap\{t_1<t<t_2\}} \left[-u\zeta_t + \sum_i |D_i u|^{p_i-2}D_i u\cdot D_i\zeta + d\,\zeta\right] dz = -\int_\Omega u\,\zeta\,dx\Big|_{t=t_1}^{t=t_2} \quad \forall\,\zeta\in\mathbf{Z} \tag{3.19}$$

and the energy relation

$$\frac{1}{2}\int_\Omega u^2\,dx\Big|_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2}\int_\Omega \left[\sum_i |D_i u|^{p_i(z)} + d(z,u)\,u\right] dz = 0. \tag{3.20}$$

4 Singular perturbation of the $p(x,t)$-Laplace operator

A solution of problem (3.1) can be obtained as the limit of the sequence of solutions of the perturbed equations. Let us assume that the exponents $p_i(z)$ are measurable, bounded and satisfy the log-continuity condition (2.1). For the sake of simplicity we assume here that $d\equiv 0$. Set $\mathbf{s} = (s_1,\dots,s_n)\in\mathbb{R}^n$ and define the operator

$$\mathbf{A}(z,\mathbf{s}) = (A_1(z,s_1),\dots,A_n(z,s_n)) : \,Q\times\mathbb{R}^n\mapsto\mathbb{R}^n,$$
$$A_i(z,r) = |r|^{p_i(z)-2}r \text{ for all } r\in\mathbb{R} \text{ and a.a. } z\in Q.$$

We want to construct a solution of the problem

$$
\begin{cases} u_t = \operatorname{div}\,(\mathbf{A}(z,\nabla u)) \quad \text{in } Q, \\ u = 0 \text{ on } \Gamma, \quad u(x,0) = u_0 \text{ in } \Omega \end{cases} \tag{4.1}
$$

approximating the nonlinear operator $\mathbf{A}$ with variable nonlinearity by singularly perturbed operators with constant nonlinearity exponents defined on a function space smaller than the natural energy space $\mathbf{W}(Q)$ prompted by the operator $\mathbf{A}$. Let us introduce the family of the perturbed operators

$$
\mathbf{A}_\epsilon(z,\nabla u) := \epsilon\,|\nabla u|^{p^+-2}\nabla u + \mathbf{A}(z,\nabla u), \quad \epsilon > 0.
$$

It is straightforward to check that $\mathbf{A}_\epsilon(z,\,\nabla u)$ is defined for every $\epsilon > 0$ for the elements of the Banach space

$$
\mathbf{W}_+(Q) := L^2(Q) \cap L^{p^+}(0,T;\, W_0^{1,p^+}(\Omega)), \qquad \|u\|_{\mathbf{W}_+(Q)} = \|u\|_{2,Q} + \|\nabla u\|_{p^+,Q}.
$$

For every $\epsilon > 0$ $\mathbf{A}_\epsilon(\cdot)$ is a Leray–Lions operator:

- $\mathbf{A}_\epsilon(z,\nabla u):\ \mathbf{W}_+(Q) \times Q \mapsto \mathbf{W}'_+(Q)$: $\forall\, u,\, \phi \in \mathbf{W}_+(Q)$,
- $\mathbf{A}_\epsilon(z,\nabla u) : \mathbf{W}_+(Q) \times Q \mapsto \mathbf{W}'_+(Q)$ is bounded,
- $\mathbf{A}_\epsilon$ is monotone,
- $\mathbf{A}_\epsilon(z,\nabla u)$ is hemi-continuous: $\forall\, u, v, w \in \mathbf{W}_+(Q)$ the mapping

$$
\mathbb{R} \ni \mu \mapsto \int_Q \mathbf{A}_\epsilon(z,\nabla(u+\mu\, v)) \cdot \nabla w\, dz \in \mathbb{R} \quad \text{is continuous.}
$$

Let us consider the sequence of problems

$$
\begin{cases} u_t = \operatorname{div}\,(\mathbf{A}_\epsilon(z,\nabla u)) \text{ in } Q, \quad \epsilon > 0, \\ u = 0 \text{ on } \Gamma, \quad u(x,0) = u_0(x) \in L^2(\Omega). \end{cases} \tag{4.2}
$$

Definition 4.1 *A function u is called weak solution of problem* (4.2) *if $u \in L^\infty(0,T;\, L^2(\Omega)) \cap \mathbf{W}_+(Q)$, $\partial_t u \in \mathbf{W}'_+(Q)$, for every test-function $\zeta \in \mathbf{W}_+(Q)$ with $\zeta_t \in \mathbf{W}'_+(Q)$*

$$
\int_Q \left(\zeta\, u_t + \mathbf{A}_\epsilon(z,\nabla u)\cdot\nabla\zeta\right) dz = 0, \tag{4.3}
$$

and $u(x,t) \to u_0(x)$ as $t \to 0$ weakly in $L^2(\Omega)$.

Theorem 4.1 ([33], Ch.2, Sec.1, Th.1.2) *For every $\epsilon > 0$ and $u_0 \in L^2(\Omega)$ problem* (4.2) *has a unique weak solution in the sense of Definition* 4.1*, and for every $t_1, t_2 \in [0,T]$*

$$
\frac{1}{2}\int_\Omega u_\epsilon^2\, dx\Big|_{t=t_1}^{t=t_2} + \epsilon \int_Q |\nabla u_\epsilon|^{p^+}\, dz + \sum_i \int_Q |D_i u_\epsilon|^{p_i(z)}\, dz = 0. \tag{4.4}
$$

The function $u_\epsilon : [0,T] \to L^2(\Omega)$ is continuous after possible redefining on a set of zero measure.

It remains to show that the family of solutions of the perturbed problems (4.2) with $\epsilon > 0$ contains a subsequence which converges to a solution of problem (4.1). Due to the uniform estimate (4.4) there exist $u \in \mathbf{W}(Q)$ and $\mathbf{A}^* \in \mathbf{W}'(Q)$, $B_i \in L^{p_i(\cdot)}(Q)$, such that (up to a subsequence)

$$u_\epsilon \to u \text{ weakly in } L^2(Q) \text{ and a.e. in } Q, \quad D_i u_\epsilon \to B_i \text{ weakly in } L^{p_i(\cdot)}(Q), \tag{4.5}$$

Lemma 4.1 *There exists a subsequence $\{u_\epsilon\}$ such that*

$$\mathbf{A}_\epsilon(z, \nabla u_\epsilon) \to \mathbf{A}^* = (B_1, \dots, B_n) \text{ with some } B_i \in L^{p'_i(\cdot)}(Q) \quad \text{and} \quad \partial_t u_\epsilon \to \partial_t u \text{ weakly in } \mathbf{W}'(Q).$$

Proof. Relation (4.4) allows us to choose a subsequence such that $u_\epsilon \to u$ weakly in $L^2(Q)$, $D_i u_\epsilon \to B_i$ weakly in $L^{p_i(\cdot)}(Q)$ and $\epsilon^{\frac{1}{p^+}} |\nabla u_\epsilon|$ are uniformly bounded in $L^{p^+}(Q)$. Observe that

$$\epsilon \, |\nabla u_\epsilon|^{p^+-2} \nabla u_\epsilon \to 0 \text{ weakly in } L^{(p^+)'}(Q). \tag{4.6}$$

Indeed: by virtue of (4.4) with $t_1 = 0$, $t_2 = T$ for every $\phi \in \mathbf{W}_+(Q)$

$$\epsilon \left| \int_Q |\nabla u_\epsilon|^{p^+-2} \nabla u_\epsilon \cdot \nabla \phi \, dz \right| \leq \epsilon^{\frac{1}{p^+}} \left(\epsilon \int_Q |\nabla u_\epsilon|^{p^+} \, dz \right)^{1-\frac{1}{p^+}} \|\nabla \phi\|_{p^+,Q} \to 0 \quad \text{as } \epsilon \to 0.$$

Let $\phi \in \mathbf{W}(Q)$ and $\phi_k \in \mathbf{W}_+(Q)$ be such that $\|\phi_k - \phi\|_{\mathbf{W}(Q)} \to 0$ as $k \to \infty$. Accept the notation

$$\int_Q \mathbf{A}_\epsilon(z, \nabla u_\epsilon) \cdot \nabla \phi_k \, dz = \underbrace{\epsilon \int_Q |\nabla u_\epsilon|^{p^+-2} \nabla u_\epsilon \cdot \nabla \phi_k \, dz}_{=J_1} + \sum_i \underbrace{\int_Q |D_i u_\epsilon|^{p_i(z)-2} D_i u_\epsilon \cdot D_i \phi_k \, dz}_{=J_{2,i}}.$$

According to (4.6) $J_1 \to 0$ as $\epsilon \to 0$, while $J_{2,i} \to \displaystyle\int_Q B_i D_i \phi_k \, dz$ with $B_i \in L^{p'_i(\cdot)}(Q)$. Passing in (4.3) with $\zeta = \phi_k$ to the limit as $\epsilon \to 0$ and then letting $k \to \infty$ we obtain: $\displaystyle\int_Q \mathbf{A}^* \cdot \nabla \phi_k \, dz \to \int_Q \mathbf{A}^* \cdot \nabla \phi \, dz$ for every $\phi \in \mathbf{W}(Q)$. By virtue of (4.3) and (4.6) $\|\partial_t u_\epsilon\|_{\mathbf{W}'_+(Q)} \leq C$ uniformly with respect to ϵ, which means that $\partial_t u_\epsilon \to \partial_t u$ weakly in $\mathbf{W}'_+(Q)$ (up to a subsequence). Moreover, $\partial_t u \in \mathbf{W}'(Q)$ because

$$\int_Q \phi_k \partial_t u \, dz = -\int_Q \mathbf{A}^* \cdot \nabla \phi_k \, dz \to -\int_Q \mathbf{A}^* \cdot \nabla \phi \, dz \quad \text{as } k \to \infty.$$

□

Corollary 4.1 $\epsilon \nabla u_\epsilon \to 0$ *weakly in* $\mathbf{W}_+(Q)$*: for every* $\phi \in \mathbf{W}_+(Q)$

$$\epsilon \left| \int_Q |\nabla \phi|^{p^+-2} \nabla \phi \cdot \nabla u_\epsilon \, dz \right| \leq \epsilon^{1-\frac{1}{p^+}} \left(\epsilon \int_Q |\nabla u_\epsilon|^{p^+} \, dz \right)^{\frac{1}{p^+}} \|\nabla \phi\|_{p^+,Q} \to 0 \quad \text{as } \epsilon \to 0.$$

Corollary 4.2 *By Lemma* 3.4 $u \in C([0,T]; L^2(\Omega))$. *Since*

$$\int_Q [u_t\phi + \mathbf{A}^* \cdot \nabla u]\, dz = 0 \qquad \forall\, \phi \in \mathbf{W}(Q),$$

it follows from Corollary 2.1 *that*

$$\frac{1}{2}\int_\Omega u^2(x,t)\, dx\Big|_{t=0}^{t=T} + \int_Q \mathbf{A}^* \cdot \nabla u\, dz = 0. \tag{4.7}$$

To identify the limit $\mathbf{A}^*$ we follow the proof of Lemma 3.5. Taking in (4.3) $\zeta = u_\epsilon$,

$$\frac{1}{2}\int_\Omega u_\epsilon^2\, dx\Big|_{t=0}^{t=T} + \int_Q \mathbf{A}_\epsilon(z, \nabla u_\epsilon) \cdot \nabla u_\epsilon\, dz = 0,$$

and using the monotonicity of $\mathbf{A}_\epsilon$ we find that for every $\phi \in C^\infty([0,T]; C_0^\infty(\Omega))$

$$-\int_Q \mathbf{A}_\epsilon(z, \nabla u_\epsilon) \cdot \nabla \phi\, dz - \int_Q \mathbf{A}_\epsilon(z, \nabla \phi) \cdot \nabla(u_\epsilon - \phi)\, dz - \frac{1}{2}\int_Q u_\epsilon^2\, dx\Big|_{t=0}^{t=T} \geq 0.$$

Letting $\epsilon \to 0$ and using Lemma 4.1 together with Corollary 4.1 we arrive at the inequality

$$-\int_Q \mathbf{A}^* \cdot \nabla\phi\, dz - \int_Q \mathbf{A}(z, \nabla\phi) \cdot \nabla(u - \phi)\, dz + \frac{1}{2}\int_Q u_0^2\, dx \geq \frac{1}{2}\int_\Omega u^2(x,T)\, dx.$$

Plugging (4.7) we then have:

$$\forall\, \phi \in C^\infty([0,T]; C_0^\infty(\Omega)) \qquad \int_Q (\mathbf{A}^* - \mathbf{A}(z, \nabla\phi)) \cdot \nabla(u - \phi)\, dz \geq 0.$$

Since the smooth functions are dense in $\mathbf{W}(Q)$, the same is true for every $\phi \in \mathbf{W}(Q)$, which allows us to take $\phi = u \pm \delta\,\psi$ with an arbitrary $\psi \in \mathbf{W}(Q)$, $\delta \in \mathbb{R}$. Simplifying and letting $\delta \to 0$ we arrive at the conclusion that $\mathbf{A}^* = \mathbf{A}(z, \nabla u)$ a.e. in Q.

Theorem 4.2 *Let the exponents $p_i(z)$ satisfy conditions* (2.1). *Then for every $u_0 \in L^2(\Omega)$ problem* (4.1) *has at least one weak solution $u \in \mathbf{W}(Q)$ such that $u_t \in \mathbf{W}'(Q)$, $u : [0,T] \mapsto L^2(\Omega)$ is weakly continuous, and for every $\zeta \in \mathbf{W}(Q)$ with $\zeta_t \in \mathbf{W}'(Q)$*

$$\int_Q (\zeta\, u_t + \mathbf{A}(z, \nabla u) \cdot \nabla\zeta)\, dz = 0.$$

5 Global L^∞ bounds, local existence theorem

Solvability of problem (3.1) can be established under a weaker restriction on the growth of the term $d(z,u)$ and a stronger regularity assumption on the initial function u_0. To be precise, we assume that

$$\begin{cases} |d(z,u)| \leq d_0|u|^{\lambda-1} + h_d(z) \text{ with } \lambda > 2,\ \ h_d \in L^1(0,T; L^\infty(\Omega)) \cap L^{p'(\cdot)}(Q), \\ u_0 \in L^\infty(\Omega). \end{cases} \tag{5.1}$$

Let us begin with the study of equation (3.1) under the assumption that the term $d(z,u)$ has the linear growth at infinity.

Theorem 5.1 ([18], Th.4.1) *Let in the conditions of Theorem* 3.1

$$\forall\, s \in \mathbb{R},\ z \in Q \qquad |d(z,s)| \le d_0|s| + h_d(z), \quad d_0 = const > 0. \tag{5.2}$$

If $\|u_0\|_{\infty,\Omega} < \infty$, *then the energy solution of problem* (3.1) *is bounded and satisfies the estimate*

$$\|u(\cdot,t)\|_{\infty,\Omega} \le \mathrm{e}^{d_0 t}\|u_0\|_{\infty,\Omega} + \mathrm{e}^{d_0 t}\int_0^t \mathrm{e}^{-d_0 t}\|h_d(\cdot,t)\|_{\infty,\Omega}\, d\tau. \tag{5.3}$$

Sketch of the proof. Let us fix $k \in \mathbb{N}$ and consider the auxiliary problem

$$\begin{cases} u_t - \sum_i D_i\left(|D_i u|^{p_i-2}D_i u\right) + d_k(z,u) = 0 \quad \text{in } Q, \\ u = 0 \text{ on } \Gamma_T, \qquad u(x,0) = u_0(x) \text{ in } \Omega \end{cases} \tag{5.4}$$

with

$$d_k(z,u) \equiv d(z, \min\{|u|;\, k\}\operatorname{sign} u).$$

Since for every finite k

$$|d\left(z, \min\{|u|;\, k\}\operatorname{sign} u\right)| \le d_0\, k^{\lambda-1} + h_d(z) \quad \text{with } \lambda = 2,$$

by Theorem 3.1 problem (5.4) has a weak solution $u(z)$ in the sense of Definition 3.1. Let us introduce the function

$$u_k = \min\{|u|,\, k\}\operatorname{sign} u \equiv \begin{cases} k & \text{if } u > k, \\ u & \text{if } |u| \le k, \\ -k & \text{if } u < -k \end{cases}$$

and take u_k^{2m-1} with $m \in \mathbb{N}$ for the test–function in (3.19). Let in (3.19) $t_2 = t + h$, $t_1 = t$, with $t, t+h \in [0,T]$. Observe that $d_k(z,u) = d(z,u_k)$. Then

$$\begin{aligned} &\frac{1}{2m}\int_t^{t+h} \frac{d}{dt}\left(\int_\Omega u_k^{2m}(x,t)\,dx\right)dt \\ &\quad + (2m-1)\sum_i \int_t^{t+h}\int_\Omega u_k^{2(m-1)}|D_i u_k|^{p_i}\,dxdt + \int_t^{t+h}\int_\Omega d(z,u_k)u_k^{2m-1}\,dxdt = 0. \end{aligned}$$

This relation leads to the linear differential inequality for the function $y_k(t) = \|u_k(\cdot,t)\|_{2m,\Omega}$:

$$\frac{dy_k}{dt}(t) \le d_0\, y_k(t) + \|h_d(\cdot,t)\|_{2m,\Omega}.$$

Integrating this inequality and letting $m \to \infty$, we have

$$\|u_k(\cdot,t)\|_{\infty,\Omega} \le \mathrm{e}^{d_0 t}\|u_0\|_{\infty,\Omega} + \mathrm{e}^{d_0 t}\int_0^t \mathrm{e}^{-d_0\tau}\|h_d(\cdot,\tau)\|_{\infty,\Omega}\,d\tau := K. \tag{5.5}$$

The right–hand side of this estimate does not depend on k. Let us choose now $k \geq K+1$. Under this choice

$$u_k \equiv \min\{|u|;\, k\} \operatorname{sign} u = u, \qquad d\,(z, u_k) \equiv d_k(z,u) \equiv d(z,u),$$

which means that the solution of problem (5.4) with $k \geq K+1$ is, in fact, a solution of problem (3.1) which satisfies estimate (5.3). □

We proceed to study problem (3.1) with the term $d(z,u)$ satisfying the growth condition

$$|d(z,u)| \leq d_0|u|^{\lambda-1} + h_d(z), \quad \lambda = const > 2. \tag{5.6}$$

For $0 \leq \lambda \leq 2$ the existence of a global bounded solution to problem (1.2) is proved in Theorem 3.1.

Theorem 5.2 (Local in time existence, [18], Th.4.3) *Let us assume that in the conditions of Theorems* 3.1 *and* 5.1 *the growth condition on the function* $d(z,u)$ *is substituted by* (5.6). *Then for every* $u_0 \in L^\infty(\Omega)$ *there exists* $\theta \in (0,T]$ *depending on*

$$\lambda,\ d_0,\ \|u_0\|_{L^\infty(\Omega)} \text{ and } \|h_d\|_{L^1(0,\theta;L^\infty(\Omega))}$$

such that in the cylinder $Q_\theta \equiv Q \cap \{t < \theta\}$ *problem* (1.2) *has at least one weak solution* $u \in \mathbf{W}(Q_\theta)$ *such that* $u_t \in \mathbf{W}'(Q_\theta)$ *and* $\|u\|_{\infty,Q_\theta} < \infty$. *The solution can be continued to the maximal interval* $[0,T^*]$,

$$T^* = \sup\{\theta \in [0,T]:\ \|u\|_{\infty,Q_\theta} < \infty\}.$$

Proof. Let us consider the auxiliary problem

$$\begin{cases} u_t - \sum\limits_i D_i\left(|D_i u|^{p_i(z)-2} D_i u\right) + d_r(z,u) = 0 \quad \text{in } Q \\ u = 0 \text{ on } \Gamma, \qquad u(x,0) = u_0(x) \text{ in } \Omega \end{cases} \tag{5.7}$$

with the right–hand side

$$d_r(z,u) = d(z, \min\{|u|, r\} \operatorname{sign} u), \quad r = const > 1. \tag{5.8}$$

As in the proof of Theorem 5.1, we will make use of the fact that

$$|d_r(z,u)| \leq d_0 r^{\lambda-1} + h_d(z), \quad d_r(z,u) = d(z,u) \quad \text{if } r \geq u.$$

By Theorems 3.1, 5.1, for every $r > 1$ the regularized problem (5.7) has a global bounded weak solution $u(z)$. Let us show that the function $w(t) = \|u(\cdot,t)\|_{\infty,\Omega}$ can be estimated by a constant which does not depend on r. Following the proof of Theorem 5.1 we find that the solution of (5.7) satisfies inequality (5.3) with h_d substituted by $h_d + d_0 r^{\lambda-1}$:

$$\|u(\cdot,t)\|_{\infty,\Omega} \leq \|u_0\|_{\infty,\Omega} + \int_0^t \|h_d(\cdot,t)\|_{\infty,\Omega}\,dt + t\,d_0 r^{\lambda-1} := \mathcal{R}(r,t).$$

For every fixed $r > 1$

$$\mathcal{R}(r,t) \to \|u_0\|_{\infty,\Omega} \quad \text{as } t \to 0,$$

whence for every $r \geq \|u_0\|_{\infty,\Omega}$ there is $t \equiv t(r)$ such that

$$\forall\, t \in [0, t(r)] \qquad \|u(\cdot,t)\|_{\infty,\Omega} \leq r.$$

It follows that for r and $t(r)$ chosen in this way we have $\|u(\cdot,t)\|_{\infty,\Omega} \leq r$ for all $t \leq t(r)$, i.e., the constructed solution of the regularized problem (5.7) is a weak solution of problem (1.2) in the cylinder $Q_{t(r)}$. The possibility of continuation of this solution to the maximal interval $[0, T^*]$ follows from the fact that the function $u(x, t(r))$ possesses the same properties as the initial function u_0. □

By Theorem 5.2 the solution of problem (3.1) with the term $d(z,u)$ subject to the growth condition (5.1) exists as long as its L^∞-norm remains bounded. Sufficient conditions of the finite time blow-up of the solution to (3.1) were established in [20, 21, 34].

6 Propagation of disturbances from the data

The study of propagation of disturbances from the data in solutions of parabolic equations is traditionally based on comparison with suitable explicit solutions or sub/supersolutions. For the equations with variable nonlinearity this method wouldn't work because such equations may not have explicit solutions. The known results on this issue were obtained by means of analysis of the local energy functions associated with the solution under study [25]. Besides the properties typical for the solutions of equations of the same structure as (1.2) but with constant exponents of nonlinearity, the solutions of equations with variable nonlinearity display certain intrinsic properties.

A detailed discussion of the propagation properties of the energy solutions and the rigorous proofs can be found in [16, 17, 19, 22]. Let us illustrate these properties by the example of the model equation with two independent space variables:

$$\begin{cases} u_t = \left(|u_x|^{p(z)-2}u_x\right)_x + \left(|u_y|^{q(z)-2}u_y\right)_y - c_0|u|^{\sigma(z)-2}u + f(z) \quad \text{in } Q, \\ u = 0 \text{ on } \Gamma_T, \\ u(x,0) = u_0(x) \text{ in } \Omega = (0,a)\times(0,a). \end{cases} \tag{6.1}$$

By agreement, we use the notations

$$\phi^+(t) = \sup_\Omega \phi(\cdot,t), \qquad \phi^-(t) = \inf_\Omega \phi(\cdot,t) \quad \text{for } \phi \in C^0(Q).$$

- *Vanishing in a finite time.* Let $f(z) \equiv 0$ for all $t \geq t_f$ and $c_0 > 0$. Then every (energy) solution of problem (6.1) vanishes in a finite time $t_* \geq t_f$ if

$$\frac{1}{\nu^+(t)} \equiv \frac{1}{\sigma^+(t)} + \frac{1}{2}\left(\frac{1}{p^+(t)} + \frac{1}{q^+(t)}\right) > 1 \quad \text{in } Q$$

and the oscillation of the variable exponents of nonlinearity in Ω is appropriately small: for every $t \in (0,T)$

$$\frac{1}{\nu^-(t)} \equiv \frac{1}{\sigma^-(t)} + \frac{1}{2}\left(\frac{1}{p^-(t)} + \frac{1}{q^-(t)}\right) \leq \frac{3}{2}.$$

- *Vanishing at a prescribed moment.* Let us additionally assume that in the above conditions

$$\int_\Omega |f(x,y,t)|^{\sigma(x,y,t)}\,dxdy \begin{cases} \equiv 0 & \text{as } t \geq t_f, \\ \leq\ C\,(t_f - t)^\mu & \text{for } t \in (0,t_f), \end{cases} \qquad C,\ \mu = const > 0,$$

with a suitably big exponent μ. Then every (energy) solution of problem (6.1) vanishes at the instant $t = t_f$, provided that C and $\|u_0\|_{2,\Omega}$ are sufficiently small.

- *Vanishing of solutions of eventually linear equations.* Although the effect of finite time vanishing is never displayed by the solutions of the linear parabolic equations, it may happen that equation (6.1) with variable nonlinearity transforms into the linear one as $t \to \infty$ and nonetheless possesses localized in time solutions. A condition sufficient for such an effect can be formulated as a restriction on the rate of vanishing of $1 - \nu^+(t)$ as $t \to \infty$: if $f \equiv 0$, $c_0 > 0$ and $\|u_0\|_{2,\Omega} \leq 1$, then every solution of problem (6.1) vanishes at a finite moment, provided that

$$\int_0^\infty \|u_0\|_{2,\Omega}^{2(\nu^+(t)-1)}\,dt = \infty \qquad \text{and} \qquad \int_0^\infty \frac{dt}{e^{t(1-\nu^+(t))}} < \infty.$$

Notice that these conditions are surely fulfilled if $\nu^+(t) \leq \nu_0 < 1$.

- *Vanishing due to the anisotropy of the diffusion term.* The effect of finite time vanishing may be caused by the anisotropy of the diffusion operator. Let $c_0 = 0$ and

$$q^+(t) > 1, \qquad \frac{q^+(t)}{q^+(t) - 1} > p^+(t) > 1.$$

Then every solution of problem (6.1) vanishes in a finite time, provided that $f(x,t) \equiv 0$ for all $t \geq t_f$. It is easy to see that these restrictions on the exponents allow the diffusion to be "fast" or linear in the direction y, $q(z) \in (1,2]$ and "slow" in the direction x, $p(z) > 2$.

- *Nonpropagation of disturbances due to the anisotropy.* It turns out that the anisotropy of the diffusion part of the equation causes as well localization of solutions in the direction of the slowest diffusion. Let in equation (6.1) p and q be constant. If $1 < q < p$ and $u_0(x,y) \equiv 0$ for all $x > x_*$, then even in the absence of the absorption term ($c_0 = 0$)

there exists $x^* \geq x_*$ such that $u(x, y, t) \equiv 0$ for $x \geq x^*$ and all $t \geq 0$. Combining the last two properties we conclude that the anisotropic equation (6.1) admits solutions which are simultaneously localized in space and time.

7 Final comments

Other notions of solution. Besides the energy solutions, the parabolic equations with variable nonlinearity admit solutions of other types. Papers [26, 37] deal with the renormalized and entropy solutions of isotropic equations (1.2) with the exponents of nonlinearity independent of t; the exponents $p(x)$ are assumed to satisfy the log-continuity condition in Ω. The authors of [29] study the Dirichlet problem for the system of equations of the type (1.2) with $d \equiv 0$ and under the assumption of logarithmic continuity of the exponent $p(z)$ in Q; their proof is based on the theory of monotone operators. One more approach to the study of evolution $p(x)$-laplacian is proposed in [1], where equation (1.2), with the exponent p independent of t, is considered as an evolution equation governed by subdifferential operators. Different notions of weak solution to equation (1.1) are discussed in [3, 4, 5]. Papers [4, 5] are, apparently, the only works which neither rely on the condition of log-continuity of the exponents $p(x, t)$, nor use the possibility of approximation of the elements of the variable exponent Sobolev space by smooth functions. The proof of existence given in [4, 5] is also based on the theory of monotone operators but in a completely different way than in [29] and the solution is understood in a very weak sense.

Uniqueness Uniqueness of energy solutions of the Dirichlet problem for equation (1.2) is proved in [18] under suitable assumptions on the behavior of $d(z, u)$ and continuity of the coefficients $a_i(z, u)$ (which are not needed for the proof of existence). The proof of uniqueness in [18] follows some ideas from [13, 14].

Further regularity of the solution. The question of the regularity of solutions and its dependence on the regularity of the data is partially studied in papers [24] and [18, Sec.6]. It is proved in [24] that the solutions of equation (1.1) with the exponent $p(z)$ being log-continuous and bounded, $\dfrac{2n}{n+2} < p(z) < \infty$, are locally bounded and $|\nabla u|^{p(x,t)+\epsilon}$ is integrable with some $\epsilon > 0$. Hölder-continuity and the Harnack inequality for the solutions of (1.1) were proved in [2]. It is shown in [18] that the regularity of solutions to (1.2) improves if the data are subject to stronger regularity assumptions. However, the proof of this assertion requires also some additional monotonicity assumptions on the exponents of nonlinearity in (1.2).

Some related equations. Hyperbolic equations with variable nonlinearity are studied in [12, 31, 34]; paper [34] deals also with nonlocal equations with $p(x)$-growth. Existence of energy solutions to doubly nonlinear parabolic equations is established in [23]. The series of works [6, 7, 8, 9, 10, 11] is devoted to the questions of existence, uniqueness and homogenization of weak solutions to elliptic and parabolic equations with nonstandard growth. Parabolic equations generalizing the porous media equation to the case of variable exponents of nonlinearity are discussed in [15, 16].

References

[1] G. Akagi and K. Matsuura, *Well-posedness and large-time behaviors of solutions for a parabolic equation involving $p(x)$-laplacian*, AIMS Proceedings, to appear, (2011).

[2] Y. Alkhutov, *The harnack inequality and the Hölder property of solutions of nonlinear elliptic equations with a nonstandard growth condition*, J.Differential Equations, 33 (1998), pp. 1653–1663.

[3] Y. Alkhutov, S. Antontsev, and V. Zhikov, *Parabolic equations with variable order of nonlinearity*, Zb. Pr. Inst. Mat. NAN Ukr., 2009 (6), pp. 23–50.

[4] Y. Alkhutov and V. Zhikov, *Existence theorems and qualitative properties of solutions of parabolic equations with a variable order of nonlinearity*, Dokl. Akad. Nauk, 430 (2010), pp. 295–299.

[5] Y. A. Alkhutov and V. V. Zhikov, *Existence theorems for solutions of parabolic equations with a variable order of nonlinearity*, Tr. Mat. Inst. Steklova, 270 (2010), pp. 21–32.

[6] B. Amaziane, S. Antontsev, L. Pankratov, and A. Piatnitski, *Homogenization of a class of nonlinear elliptic equations with nonstandard growth*, Comptes Rendus Mecanique, 335 (2007), pp. 138–143.

[7] ——, *Γ-convergence and homogenization of functionals in Sobolev spaces with variable exponents*, J. Math. Anal. Appl., 342 (2008), pp. 1192–1202.

[8] ——, *Homogenization of p-laplacian in perforated domain*, Ann.I.H. Poincare-AN, 26 (2009), pp. 2457–2479.

[9] ——, *Homogenization of immiscible two-phase flow in porous media: Application to gas migration in a nuclear waste repository*, SIAM, Multiscale Model. Simul, 8 (2010), p. 2023.

[10] B. Amaziane, L. Pankratov, and A. Piatnitski, *Homogenization of variational functionals with nonstandard growth in perforated domains*, Netw. Heterog. Media, 5 (2010), pp. 189–215.

[11] B. Amaziane, L. Pankratov, and V. Prytula, *Nonlocal effects in homogenization of $p_\epsilon(x)$-Laplacian in perforated domains*, Nonlinear Anal., 71 (2009), pp. 4078–4097.

[12] S. Antontsev , *Wave Equation with $p(x,t)$-Laplacian and Damping Term: Existence and Blow-up*, In Proceedings "Modern Problems of Applied Mathematics and Mechanics: Theory, Experiment and Applications", 2011, {http://conf.nsc.ru/files/conferences/niknik-90/fulltext/39157/46117/NIK_NIK_90_Antontsev.pdf}

[13] S. Antontsev and M. Chipot, *Anisotropic equations: uniqueness and existence results*, Differential Integral Equations, 21 (2008), pp. 401–419.

[14] S. Antontsev, M. Chipot, and Y. Xie, *Uniqueness results for equations of the $p(x)$-Laplacian type*, Adv. Math. Sci. Appl., 17 (2007), pp. 287–304.

[15] S. Antontsev and S. Shmarev, *A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions*, Nonlinear Anal., 60 (2005), pp. 515–545.

[16] ——, *Existence and uniqueness of solutions of degenerate parabolic equations with variable exponents of nonlinearity*, Fundam. Prikl. Mat., 12 (2006), pp. 3–19.

[17] ——, *Parabolic equations with anisotropic nonstandard growth conditions*, in Free boundary problems, vol. 154 of Internat. Ser. Numer. Math., Birkhäuser, Basel, 2007, pp. 33–44.

[18] ——, *Anisotropic parabolic equations with variable nonlinearity*, Publ. Mat., 53 (2009), pp. 355–399.

[19] ——, *Localization of solutions of anisotropic parabolic equations*, Nonlinear Anal., 71 (2009), pp. e725–e737.

[20] ——, *Blow-up of solutions to parabolic equations with nonstandard growth conditions*, J. Comput. Appl. Math., 234 (2010), pp. 2633–2645.

[21] ——, *On the blow-up of solutions to anisotropic parabolic equations with variable nonlinearity*, Proceedings of the Steklov Institute of Mathematics, 270 (2010), pp. 27–42.

[22] ——, *Vanishing solutions of anisotropic parabolic equations with variable nonlinearity*, J. Math. Anal. Appl., 361 (2010), pp. 371–391.

[23] ——, *Parabolic equations with double variable nonlinearities*, Mathematics and Computers in Simulation, 81 (2011), pp. 2018–2032.

[24] S. Antontsev and V. Zhikov, *Higher integrability for parabolic equations of $p(x,t)$-Laplacian type*, Adv. Differential Equations, 10 (2005), pp. 1053–1080.

[25] S. N. Antontsev, J. I. Díaz, and S. Shmarev, *Energy methods for free boundary problems*, Progress in Nonlinear Differential Equations and their Applications, 48, Birkhäuser Boston Inc., Boston, MA, 2002. Applications to nonlinear PDEs and fluid mechanics.

[26] M. Bendahmane, P. Wittbold, and A. Zimmermann, *Renormalized solutions for a nonlinear parabolic equation with variable exponents and L^1-data*, J. Differential Equations, 249 (2010), pp. 1483–1515.

[27] L. Diening, *Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$*, Math. Inequal. Appl., 7 (2004), pp. 245–253.

[28] L. Diening, P. Harjulehto, P. Hästö, and M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents*, vol. 2017 of Lecture Notes in Mathematics, Springer, 2011.

[29] P. Diening, L. Nägele and M. Ruzicka, *Monotone operator theory for unsteady problems in variable exponent spaces*, CVEE, to appear, (2011).

[30] D. E. Edmunds and J. Rákosník, *Density of smooth functions in $W^{k,p(x)}(\Omega)$*, Proc. Roy. Soc. London Ser. A, 437 (1992), pp. 229–236.

[31] J. Haehnle and A. Prohl, *Approximation of nonlinear wave equations with nonstandard anisotropic growth conditions*, J. Mathematics of Computation, 79 (2010), pp. 189–208.

[32] P. A. Hästö, *On the density of continuous functions in variable exponent Sobolev space*, Rev. Mat. Iberoam., 23 (2007), pp. 213–234.

[33] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, 1969.

[34] J. P. Pinasco, *Blow-up for parabolic and hyperbolic problems with variable exponents*, Nonlinear Anal., 71 (2009), pp. 1094–1099.

[35] S. Samko, *Denseness of $C_0^\infty(\mathbf{R}^N)$ in the generalized Sobolev spaces $W^{M,P(X)}(\mathbf{R}^N)$*, in Direct and inverse problems of mathematical physics (Newark, DE, 1997), vol. 5 of Int. Soc. Anal. Appl. Comput., Kluwer Acad. Publ., Dordrecht, 2000, pp. 333–342.

[36] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl. (4), 146 (1987), pp. 65–96.

[37] C. Zhang and S. Zhou, *Renormalized and entropy solutions for nonlinear parabolic equations with variable exponents and L^1 data*, J. Differential Equations, 248 (2010), pp. 1376–1400.

[38] V. V. Zhikov, *On the density of smooth functions in Sobolev-Orlicz spaces*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 310 (2004), pp. 67–81, 226.