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On the regularity for the Laplace equation and the Stokes system.

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This paper is dedicated to Prof. Monique Madaune Tort.

Abstract

The purpose of this work is to show a broad framework in which the theory of very weak solutions for the Dirichlet stationary problem for the Laplace and Stokes equations in bounded domains of $\mathbb{R}^n$, $n \geq 2$, could be developed. Broad in the sense of giving the more general spaces in which data can be taken in order to obtain a very weak solution and define properly the trace of such solution. Density arguments and a functional framework will be necessary, as well as classical regularity results in the L^p-Sobolev spaces that will be generalized here.

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Introduction and notations

We are interested in the boundary problem for the Laplace equation and the Stokes system. Recall that the Stokes systems is described by the following equations:

$$(S) \qquad -\Delta \boldsymbol{u} + \nabla q = \boldsymbol{f} \quad \text{and} \quad \nabla \cdot \boldsymbol{u} = h \quad \text{in } \Omega, \;\; \boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma,$$

where $\boldsymbol{u}$ denotes the velocity and q the pressure and both are unknown, $\boldsymbol{f}$ the external forces, h the compressibility condition and $\boldsymbol{g}$ the boundary condition for the velocity, the three functions being known.

We will consider Ω a bounded open set of $\mathbb{R}^n$, $n \geq 2$, with boundary Γ. The vector fields and matrix fields (and the corresponding spaces) defined over Ω or over $\mathbb{R}^n$ are respectively denoted by boldface Roman and special Roman.

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The different kinds of solutions defined for these two problems (and also for the Navier-Stokes system) have been widely studied in many works, specially weak and strong solutions. In the case of incompressible fluids, $h = 0$, it has been well-known since Leray [23] (see also [24]) that if $\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$ with $p \geq 2$ and for any $i = 0, \ldots, I$, verifying

$$\int_{\Gamma_i} \boldsymbol{g} \cdot \boldsymbol{n} \, d\sigma = 0, \tag{0.1}$$

where Γ_i denote the connected components of the boundary Γ of the open set Ω, then there exists a solution $(\boldsymbol{u}, q) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ satisfying (S).

The concept of very weak solution $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ for Stokes or Navier-Stokes equations, corresponding to very irregular data, has been developed in the last years by Giga [19] (in a domain Ω of class $\mathcal{C}^\infty$), Amrouche & Girault [5] (in a domain Ω of class $\mathcal{C}^{1,1}$) and more recently by Galdi et al. [18], Farwig et al. [17] (in a domain Ω of class $\mathcal{C}^{2,1}$, see also Schumacher [30]) and Kim [22] (in a domain of class $\mathcal{C}^2$). In this context, the boundary condition is chosen in $\mathbf{L}^p(\Gamma)$ (see Brown & Shen [13], Conca [15], Fabes et al. [16], Moussaoui [27], Shen [31], Savaré [29], Marusic-Paloka [26]) or more generally in $\mathbf{W}^{-1/p,p}(\Gamma)$. For the non-stationary case, the existence, uniqueness and regularity of very weak solutions for the Navier-Stokes equations have been investigated (among other authors) by Amann [2, 3].

In this work, (for Stokes) first we present a result of existence of very weak solution for the Stokes system in a bounded domain of $\mathbb{R}^n$, for $n \geq 2$. Before and after the study necessary in order to establish this kind of regularity, we will present the results of existence of weak and strong solutions in the $L^p(\Omega)$ Sobolev spaces. We use the method developed by Amrouche & Girault appearing for the Stokes problem in [4, 5], for a bounded open set, and those of Amrouche et al. in [7], for a half-space. However, the study will be made in a more general context, where the functional spaces, all the density lemmas and the nature of the boundary are different. The reason of this generalization is the necessity of using the Stokes results as a tool in the search of very weak solutions for the Oseen and Navier-Stokes equations. In these systems, the convection or non-linear term generate an "anisotropy" that can be collected in the space of solutions, and allows us to define rigorously the traces of the vector functions which are living in subspaces of $\mathbf{L}^p(\Omega)$ (see Lemma 2.5 and Lemma 2.6).

The case of $n = 3$ was completely developed for Stokes, Oseen and Navier-Stokes by the authors in [8, 9, 10], together with the result of existence of very weak solution for the Oseen and Navier-Stokes equations. Concretely, we prove existence and regularity of very weak solutions $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ (for $p \in (1, +\infty)$ in the case of the Stokes system) with arbitrary large data belonging to some Sobolev spaces of negative order, in a bounded connected open set of class $\mathcal{C}^{1,1}$. This regularity for the domain differs from that one appearing in [17] (for a result in domains of $\mathbb{R}^2$) and [18] (for a result in domains of $\mathbb{R}^3$), in which the authors consider a bounded domain of class $\mathcal{C}^{2,1}$. Moreover, our solution is obtained in the space $\mathbf{T}_{p,r}(\Omega)$ which has been clearly characterized contrary to the abstract spaces appearing there.

0.1 Functional spaces, norms and traces

In all this work, if we do not say anything else, Ω will be considered as a Lipschitz open bounded set of $\mathbb{R}^n$, $n \geq 2$. When Ω is connected, we will say Ω is a domain. We will only specify the regularity of Ω when it to be different from the regularity presented above.

In what follows, we will consider $s \in \mathbb{R}$, $p \in (1, +\infty)$ and p' stands for its conjugate: $1/p + 1/p' = 1$. We shall denote by m the integer part of s and by σ its fractional part: $s = m + \sigma$ with $0 \leq \sigma < 1$. The reflexive Banach space $W^{s,p}(\mathbb{R}^n)$ is the space of all distributions v defined in $\mathbb{R}^n$ such that:

- $D^\alpha v \in L^p(\mathbb{R}^n)$, for all $|\alpha| \leq m$, when $s = m$ is a nonnegative integer
- $v \in W^{m,p}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|D^\alpha v(x) - D^\alpha v(y)|^p}{|x-y|^{n+\sigma p}} \, \mathrm{d}x \, \mathrm{d}y < \infty$, for all $|\alpha| = m$, when $s = m + \sigma$ is nonnegative and is not an integer.

The space $W^{s,p}(\mathbb{R}^n)$ is equipped by the norm: $\|v\|^p_{W^{m,p}(\mathbb{R}^n)} = \left(\sum_{|\alpha| \leq m} \int_{\mathbb{R}^n} |D^\alpha v(x)|^p \, d\boldsymbol{x} \right)^{1/p}$, in the first case, and by the norm $\|v\|_{W^{s,p}(\mathbb{R}^n)} = \left(\|v\|^p_{W^{m,p}(\mathbb{R}^n)} + \sum_{|\alpha|=m} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|D^\alpha v(x) - D^\alpha v(y)|^p}{|x-y|^{n+\sigma p}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/p}$, in the second case. For $s < 0$, we denote by $W^{s,p}(\mathbb{R}^n)$ the dual space of $W^{-s,p'}(\mathbb{R}^n)$. In the special case of $p = 2$, we shall use the notation $H^s(\mathbb{R}^n)$ instead of $W^{s,2}(\mathbb{R}^n)$. We also consider the Sobolev space

$$H^{s,p}(\mathbb{R}^n) = \{ v \in L^p(\mathbb{R}^n); \ (I - \Delta)^{s/2} v \in \ L^p(\mathbb{R}^n) \}.$$

It is known that $H^{s,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$ if s is an integer or if $p = 2$. Furthermore, for $s \in \mathbb{R}$, we have that $W^{s,p}(\mathbb{R}^n) \hookrightarrow H^{s,p}(\mathbb{R}^n)$ if $p \leq 2$ and $H^{s,p}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n)$ if $p \geq 2$.

The definition of the space $W^{s,p}(\Omega)$ is exactly the same as in the case of the whole space. Because of $\mathcal{D}(\Omega)$ is not dense in $W^{s,p}(\Omega)$, the dual space of $W^{s,p}(\Omega)$ cannot be identified to a space of distributions in Ω. For this reason, we define $W^{s,p}_0(\Omega)$ as the closure of $\mathcal{D}(\Omega)$ in $W^{s,p}(\Omega)$ and we denote by $W^{-s,p'}(\Omega)$ its dual space.

For every $s > 0$, we denote by $W^{s,p}(\overline{\Omega})$ the space of all distributions in Ω which are restrictions of elements of $W^{s,p}(\mathbb{R}^n)$ and by $\widetilde{W}^{s,p}(\Omega)$ the space of functions $u \in W^{s,p}(\overline{\Omega})$ such that the extension $\widetilde{u}$ by zero outside of Ω belongs to $W^{s,p}(\mathbb{R}^n)$. Recall now some density results ([1, 20]):

i) The space $\mathcal{D}(\overline{\Omega})$ is dense in $W^{s,p}(\Omega)$ for any real s.

ii) The space $\mathcal{D}(\mathbb{R}^n)$ is dense in $W^{s,p}(\mathbb{R}^n)$ and in $H^{s,p}(\mathbb{R}^n)$ for any real s.

iii) The space $\mathcal{D}(\Omega)$ is dense in $\widetilde{W}^{s,p}(\Omega)$ for all $s > 0$.

iv) The space $\mathcal{D}(\Omega)$ is dense in $W^{s,p}(\Omega)$ for all $0 < s \leq 1/p$, that means that $W^{s,p}(\Omega) = W^{s,p}_0(\Omega)$.

Theorem 0.1. (Traces of functions living in $W^{s,p}(\Omega)$) *([1, 20]) Let Ω be a bounded open set of class $\mathcal{C}^{k,1}$, for some integer $k \geq 0$. Let s be real number such that $s \leq k+1, s - 1/p = m + \sigma$, where $m \geq 0$ is an integer and $0 < \sigma < 1$.*

i) The following mapping

$$\begin{array}{rcl}\gamma_0 : u & \mapsto & u_{|\Gamma} \\ W^{s,p}(\Omega) & \to & W^{s-1/p,p}(\Gamma)\end{array}$$

is continuous and surjective. When $1/p < s < 1 + 1/p$*, we have* $Ker(\gamma_0) = W_0^{s,p}(\Omega)$.

ii) For $m \geq 1$*, the following mapping*

$$\begin{array}{rcl}(\gamma_0, \gamma_1) : u & \mapsto & (u_{|\Gamma}, \frac{\partial u}{\partial \boldsymbol{n}}_{|\Gamma}) \\ W^{s,p}(\Omega) & \to & (W^{s-1/p,p}(\Gamma) \times W^{s-1-1/p,p}(\Gamma))\end{array}$$

is continuous and surjective. When $1+1/p < s < 2+1/p$*, we have* $Ker(\gamma_0, \gamma_1) = W_0^{s,p}(\Omega)$.

We recall also the following embeddings:

$$W^{s,p}(\Omega) \hookrightarrow W^{t,q}(\Omega) \quad \text{for } t \leq s,\ p \leq q \quad \text{such that } s - n/p = t - n/q$$

$$W^{s,p}(\Omega) \hookrightarrow \mathcal{C}^{k,\alpha}(\overline{\Omega}) \quad \text{for } k < s - n/p < k+1, \quad \alpha = s - k - n/p,$$

where k is a non negative integer.

1 The Laplace equation

We are interested here in the resolution of the problem

$$(L_D) \qquad -\Delta u = f \quad \text{in } \Omega \quad \text{and} \quad u = g \ \text{ on } \Gamma,$$

with data in some Sobolev spaces. Before starting our study, we recall some results concerning this problem. Recall that one consequence of the Calderon-Zygmund theory of singular integrals and boundary layer potential is that for every $f \in W^{m-2,p}(\Omega)$ and $g \in W^{m-1/p,p}(\Gamma)$, with m positive integer, the problem (L$_D$) has a unique solution $u \in W^{m,p}(\Omega)$ when Ω is of class $\mathcal{C}^{r,1}$ with $r = \max\{1, m-1\}$. If $f \in W^{s-2,p}(\Omega)$ and $g \in W^{s-1/p,p}(\Gamma)$, with $s > 1/p$, then $u \in W^{s,p}(\Omega)$ provided that Ω is of class $\mathcal{C}^{r,1}$ with $r = \max\{1, [s]\}$, where $[s]$ is the integer part of s. In [25], Lions and Magenes made a complete study for smooth domains and $p = 2$. Grisvard in [20] treats the case where Ω is of class $\mathcal{C}^{r,1}$.

Jerison & Kenig in [21] and many other authors study the case where Ω is only a bounded Lipschitz-continuous domain. First, we recall some results for $p = 2$.

i) If $f \in H^{-1/2+\varepsilon}(\Omega)$, for some $\varepsilon > 0$ or $f \in L^2(\Omega)$ and $g = 0$, then the unique solution u of (L$_D$) satisfies $u \in H^{3/2}(\Omega)$.

ii) If $f \in H^{-1+s}(\Omega)$, with $-1/2 < s < 1/2$ and $g = 0$, then $u \in H^{1+s}(\Omega)$.

iii) If $f = 0$ and $g \in H^{s+1/2}(\Gamma)$, with $-1/2 \leq s \leq 1/2$, then $u \in H^{1+s}(\Omega)$.

iv) The conclusion in point i) is not true for $\varepsilon = 0$: There exist a Lipschitz domain Ω and $f \in H^{-1/2}(\Omega)$ such that $u \notin H^{3/2}(\Omega)$.

v) The conclusion in point ii) is not true for $s > 1/2$: There exist a Lipschitz domain Ω and $f \in \mathcal{C}^\infty(\overline{\Omega})$ such that $u \notin H^{1+s}(\Omega)$.

In the case p arbitrary, we have the following result (see Jerison & Kenig, [21]).

Theorem 1.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, $N \geq 3$. There exists $\varepsilon \in\,]0, 1]$, depending only on the Lipschitz constant of Ω such that for every $f \in H^{s-2,p}(\Omega)$ and $g = 0$, there is a unique solution $u \in H^{s,p}(\Omega)$ to (L_D) provided one of the following holds:*

$$\begin{array}{lll} p_0 < p < p_0' & \text{and} & \frac{1}{p} < s < 1 + \frac{1}{p} \\ 1 < p \leq p_0 & \text{and} & \frac{3}{p} - 1 - \varepsilon < s < 1 + \frac{1}{p} \\ p_0' \leq p < \infty & \text{and} & \frac{1}{p} < s < \frac{3}{p} + \varepsilon \end{array}$$

where $1/p_0 = 1/2 + \varepsilon/2$ and $1/p_0' = 1/2 - \varepsilon/2$. Moreover, we have the estimate

$$\|u\|_{H^{s,p}(\Omega)} \leq C\|f\|_{H^{s-2,p}(\Omega)}$$

for all $f \in H^{s-2,p}(\Omega)$. When the domain is $\mathcal{C}^1$, the exponent p_0 may be taken to be 1. When $s = 1$, there is $p_1 > 3$ such that if $p_1' < p < p_1$, then the inhomogeneous Dirichlet problem has a unique solution $u \in H^{s,p}(\Omega)$.

As particular case of the third condition, for any $N \geq 3$ (and also $N = 2$), there exists a $\mathcal{C}^1$ domain Ω in $\mathbb{R}^N$ and $f \in H^{-1+1/p,p}(\Omega)$ for which the solution u of (L_D) with $g = 0$ does not belongs to $H^{1+1/p,p}(\Omega)$ for all $1 < p < \infty$.

As we said before, if Ω is an open set of class $\mathcal{C}^{1,1}$, for each $f \in W^{s-2,p}(\Omega)$ and $g \in W^{s-1/p,p}(\Omega)$, the problem (L_D) has a unique solution $u \in W^{s,p}(\Omega)$ assuming $1/p < s \leq 2$. In this work, we are interested in the search of very weak solutions, *i. e.* , solutions belonging to $W^{s,p}(\Omega)$ with $0 \leq s \leq 1/p$ and for a regular open set Ω, here of class $\mathcal{C}^{1,1}$. Moreover, we look for optimal conditions for the data f and g in order to obtain such solutions. With this aim, we introduce the space:

$$M_p(\Omega) = \left\{ v \in L^p(\Omega);\ \Delta v \in W^{-2+1/p,p}(\Omega) \right\},$$

which is reflexive Banach space for the norm

$$\|v\|_{M_p(\Omega)} = \|v\|_{L^p(\Omega)} + \|\Delta v\|_{W^{-2+1/p,p}(\Omega)}.$$

Lemma 1.2. *([10]) The space $\mathcal{D}\big(\overline{\Omega}\big)$ is dense in $M_p(\Omega)$.*

To study the traces of functions which belong to $M_p(\Omega)$, we have the following lemma.

Lemma 1.3. *Let Ω be a bounded open set of $\mathbb{R}^n$ of class $\mathcal{C}^{1,1}$. The linear mapping $\gamma_0 : v \longmapsto v|_\Gamma$ defined on $\mathcal{D}(\overline{\Omega})$ can be extended to a linear continuous mapping*

$$\gamma_0 : M_p(\Omega) \longrightarrow W^{-1/p,\,p}(\Gamma).$$

Moreover, we have the Green formula: $\forall v \in M_p(\Omega)$, $\quad \forall \varphi \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$,

$$\int_\Omega v \Delta\varphi \,\mathrm{d}x - \langle \Delta v, \varphi \rangle_{W^{-2+1/p,p}(\Omega) \times W_0^{2-1/p,p'}(\Omega)} = \left\langle v, \frac{\partial \varphi}{\partial \boldsymbol{n}} \right\rangle_{W^{-1/p,\,p}(\Gamma) \times W^{1/p,\,p'}(\Gamma)}. \tag{1.1}$$

Proof. Let $v \in \mathcal{D}(\overline{\Omega})$ and $\varphi \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$, then formula (1.1) obviously holds. For every $\mu \in W^{1/p,p'}(\Gamma)$, there exists $\varphi \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$ such that $\frac{\partial \varphi}{\partial \boldsymbol{n}} = \mu$ on Γ, with $\|\varphi\|_{W^{2,p'}(\Omega)} \leq C\, \|\mu\|_{W^{1/p,p'}(\Gamma)}$. Consequently,

$$\left| \langle v, \mu \rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)} \right| \leq C\, \|v\|_{M_p(\Omega)}\, \|\mu\|_{W^{1/p,p'}(\Gamma)}.$$

Thus

$$\|v\|_{W^{-1/p,p}(\Gamma)} \leq C\, \|v\|_{M_p(\Omega)}.$$

We can deduce that the linear mapping γ is continuous for the norm of $M_p(\Omega)$. Since $\mathcal{D}(\overline{\Omega})$ is dense in $M_p(\Omega)$, γ can be extended by continuity to $\gamma \in \mathcal{L}\left(M_p(\Omega);\ W^{-1/p,p}(\Gamma)\right)$ and formula (1.1) holds for all $v \in M_p(\Omega)$ and $\varphi \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$. □

We now can solve the Laplace equation with singular boundary condition.

Theorem 1.4. *Let Ω be a bounded open set of $\mathbb{R}^n$ of class $\mathcal{C}^{1,1}$. For any $f \in W^{-2+1/p,p}(\Omega)$ and $g \in W^{-1/p,p}(\Gamma)$, the Laplace equation (L_D) has a unique solution $u \in L^p(\Omega)$, with the estimate*

$$\|u\|_{M_p(\Omega)} \leq\ C\ (\|f\|_{W^{-2+1/p,p}(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma)}).$$

Proof. Thanks to the Green formula (1.1), it is easy to verify that $u \in L^p(\Omega)$ is solution of problem (L_D) is equivalent to the variational formulation: Find $u \in L^p(\Omega)$ such that

$$\begin{aligned} &\forall v \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega), \\ \int_\Omega u \Delta v \, \mathrm{d}x = -\, &\langle f, v \rangle_{W^{-2+1/p,p}(\Omega) \times W_0^{2-1/p,p'}(\Omega)} + \left\langle g, \frac{\partial v}{\partial \boldsymbol{n}} \right\rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)}. \end{aligned} \tag{1.2}$$

Indeed, let $u \in L^p(\Omega)$ be a solution to (L_D). Then, the Green formula (1.1) yields (1.2). Conversely, let $u \in L^p(\Omega)$ be a solution to (1.2). Taking v in $\mathcal{D}(\Omega)$, we obtain $-\Delta u = f$ in Ω and $u \in M_p(\Omega)$. Using this last relation and again the Green formula (1.1), we deduce that for all $v \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$,

$$\left\langle u, \frac{\partial v}{\partial \boldsymbol{n}} \right\rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)} = \left\langle g, \frac{\partial v}{\partial \boldsymbol{n}} \right\rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)}$$

and finally $u = g$ on Γ.

Let's then solve problem (1.2). We know that for all $F \in L^{p'}(\Omega)$, there exists a unique $v \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$ satisfying $-\Delta v = F$ in Ω, with the estimate

$$\|v\|_{W^{2,p'}(\Omega)} \leq C \|F\|_{L^{p'}(\Omega)}.$$

Then we have

$$\begin{aligned} &\left| \langle f, v \rangle_{W^{-2+1/p,p}(\Omega) \times W_0^{2-1/p,p'}(\Omega)} - \left\langle g, \frac{\partial v}{\partial \boldsymbol{n}} \right\rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)} \right| \\ &\leq\ C\ \|f\|_{W^{-2+1/p,p}(\Omega)} \|v\|_{W^{2-1/p,p'}(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma)} \|\frac{\partial v}{\partial \boldsymbol{n}}\|_{W^{1/p,p'}(\Gamma)} \\ &\leq\ C\ \left(\|f\|_{W^{-2+1/p,p}(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma)}\right) \|F\|_{L^{p'}(\Omega)}. \end{aligned}$$

In other words, we can say that the linear mapping

$$T : F \longmapsto \langle f, v\rangle_{W^{-2+1/p,p}(\Omega)\times W_0^{2-1/p,\ p'}(\Omega)} - \left\langle g, \frac{\partial v}{\partial \boldsymbol{n}}\right\rangle_{W^{-1/p,\,p}(\Gamma)\times W^{1/p,\,p'}(\Gamma)}$$

is continuous on $L^{p'}(\Omega)$, and according to the Riesz representation theorem, there exists a unique $u \in L^p(\Omega)$, such that

$$\forall F \in L^{p'}(\Omega),\ T(F) = \langle u, F\rangle_{L^p(\Omega)\times L^{p'}(\Omega)},$$

i.e u is solution of (L_D). □

Corollary 1.5. Let Ω be a bounded open set of $\mathbb{R}^n$ of class $\mathcal{C}^{1,1}$ and σ be a real number such that $0 \le \sigma \le 1$.

i) We assume that $f \in W^{-2+\sigma/p'+1/p,p}(\Omega)$ and $g \in W^{\sigma-1/p,p}(\Gamma)$. Then the solution u given by Theorem 1.4 belongs to $W^{\sigma,p}(\Omega)$ and satisfies the estimate

$$\|u\|_{W^{\sigma,p}(\Omega)} \le C\,(\|f\|_{W^{-2+\sigma/p'+1/p,p}(\Omega)} + \|g\|_{W^{\sigma-1/p,p}(\Gamma)}).$$

ii) If moreover $f \in W^{\sigma-1,p}(\Omega)$ and $g \in W^{\sigma+1/p',p}(\Gamma)$, then $u \in W^{\sigma+1,p}(\Omega)$ and satisfies the estimate

$$\|u\|_{W^{\sigma+1,p}(\Omega)} \le C\,(\|f\|_{W^{\sigma-1,p}(\Omega)} + \|g\|_{W^{\sigma+1/p',p}(\Gamma)}).$$

Proof. First, we observe that if $\sigma = 0$, the conclusion in point i) holds because Theorem 1.4 and the conclusion in point ii) is satisfied thanks to classical regularity of generalized solutions for Problem (L_D). If $\sigma = 1$, the point i) holds for the same reason and the second point due to the classical regularity of strong solutions for Problem (L_D). Hence, we can suppose that $0 < \sigma < 1$. In this case, it suffices to use interpolation argument (see [25], [32], [11]) and elliptic regularity problem for the generalized solutions. □

Remark 1.6.

i) The results of the second point are optimal unlike part i) which is optimal only when $f = 0$.

ii) We can reformulate the point ii) as follows. For any $f \in W^{-s,p}(\Omega)$ and $g \in W^{2-s-1/p,p}(\Gamma)$, with $0 \le s \le 1$, Problem (L_D) has a unique solution $u \in W^{2-s,p}(\Omega)$ satisfying $u = g$ on Γ.

Theorem 1.7. *Let Ω be a bounded open set of $\mathbb{R}^n$ of class $\mathcal{C}^{1,1}$, s be a real number such that $\frac{1}{p} < s \le 2$. We assume that $f \in W^{s-2,p}(\Omega)$ and $g \in W^{s-1/p,p}(\Gamma)$. Then Problem ($L_D$) has a unique solution $u \in W^{s,p}(\Omega)$ which satisfies the estimate*

$$\|u\|_{W^{s,p}(\Omega)} \le C\,(\|f\|_{W^{s-2,p}(\Omega)} + \|g\|_{W^{s-1/p,p}(\Gamma)}).$$

Proof. The theorem is proved by Corollary 1.5 point ii) if $1 \leq s \leq 2$. Let be then s a real number such that $\frac{1}{p} < s \leq 1$. Using Theorem 0.1, we can suppose $g = 0$. We known that $\mathcal{D}(\Omega)$ is dense in the space of functions of $W^{s,p}(\Omega)$ equal to zero on Γ, that means that

$$W_0^{s,p}(\Omega) = \{v \in W^{s,p}(\Omega);\ v = 0 \text{ on } \Gamma\}.$$

We have also the same relation for the space $W_0^{2-s,p'}(\Omega)$ because $1 \leq 2 - s < 1 + 1/p'$. Consequently $u \in W_0^{s,p}(\Omega)$ satisfies $-\Delta u = f$ in Ω if and only if: $\forall v \in W_0^{-s+2,p'}(\Omega)$,

$$\langle u,\ \Delta v\rangle_{W_0^{s,p}(\Omega)\times W^{-s,p'}(\Omega)} = -\langle f, v\rangle_{W^{s-2,p}(\Omega)\times W_0^{-s+2\,p'}(\Omega)} \tag{1.3}$$

Let's solve problem (1.3). By Remark 1.6 point ii), we know that for all $F \in W^{-s,p'}(\Omega)$, there exists a unique $v \in W_0^{-s+2,p'}(\Omega)$ satisfying $-\Delta v = F$ in Ω, with the estimate

$$\|v\|_{W^{-s+2,\,p'}(\Omega)} \leq C\|F\|_{W^{-s,p'}(\Omega)}.$$

Then, $\left|\langle f, v\rangle_{W^{s-2,p}(\Omega)\times W_0^{-s+2\,p'}(\Omega)}\right| \leq C\,\|f\|_{W^{s-2,p}(\Omega)}\|v\|_{W^{-s+2\,p'}(\Omega)} \leq C\,\|f\|_{W^{s-2,p}(\Omega)}\|F\|_{W^{-s,p'}(\Omega)}$.

In other words, we can say that the linear mapping

$$T:\ F \longmapsto \langle f, v\rangle_{W^{s-2,p}(\Omega)\times W_0^{-s+2,\,p'}(\Omega)}$$

is continuous on $W^{-s,p'}(\Omega)$, and by the Riesz representation theorem, there exists a unique $u \in W_0^{s,p}(\Omega)$, such that

$$\forall F \in W^{-s,p'}(\Omega), \quad T(F) = \langle u, F\rangle_{W_0^{s,p}(\Omega)\times W^{-s,p'}(\Omega)},$$

i.e u is solution of (L_D) with $g = 0$. □

Remark 1.8. i) When $f \in W^{1/p-2,p}(\Omega)$, we can conjecture that $u \notin W^{1/p,p}(\Omega)$.

ii) If $1/p < s < 1$, $f \in W^{s-2,p}(\Omega)$ and $g \in W^{s-1/p,p}(\Gamma)$, then the solution u of (L_D) belongs to $W^{s,p}(\Omega)$. These assumptions are weaker than those of Corollary 1.5 i) because $W^{-2+s/p'+1/p,p}(\Omega) \hookrightarrow W^{s-2,p}(\Omega)$ if $1/p < s < 1$. Moreover, they are optimal.

iii) If $0 \leq s \leq 1/p$, Theorem 1.7 cannot be applied. Indeed, the trace mapping is not continuous (and not surjective) from $W^{s,p}(\Omega)$ into $W^{s-1/p,p}(\Gamma)$. If $s = 0$ and $g \in W^{-1/p,p}(\Gamma)$, we cannot expect to find a solution u more regular than $L^p(\Omega)$. Theorem 1.4 shows that it is possible if $f \in W^{-2+1/p,p}(\Omega)$. In the case of $0 < s \leq 1/p$ and $g \in W^{s-1/p,p}(\Gamma)$, we cannot expect either to find a solution u better than $W^{s,p}(\Omega)$. Corollary 1.5 point ii) shows that it is possible if $f \in W^{-2+s/p'+1/p,p}(\Omega)$, taking into account that $-2+s/p'+1/p > -2+s$.

Remark 1.9. In the case $p = 2$, we have proved in particular the following results which are naturally better than the case where Ω is considered only Lipschitz:

i) if $f \in H^{-1/2}(\Omega)$ and $g \in H^1(\Gamma)$, then $u \in H^{3/2}(\Omega)$,

ii) if $f \in H^{-1+s}(\Omega)$, with $-1/2 < s \leq 1$ and $g = 0$, then $u \in H^{1+s}(\Omega)$,

iii) if $f = 0$ and $g \in H^{s+1/2}(\Gamma)$, with $-1 \leq s \leq 1$ then $u \in H^{1+s}(\Omega)$.

2 The Stokes problem

2.1 Preliminary results

In the sequel, we will use the following spaces used in the following versions of De Rham's Theorem:

$$\boldsymbol{\mathcal{D}}_\sigma(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega);\ \nabla \cdot \boldsymbol{\varphi} = 0\}, \quad \boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}) = \{\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\overline{\Omega});\ \nabla \cdot \boldsymbol{\psi} = 0\}.$$

Lemma 2.1.

i) **De Rham's Theorem for distributions** *(See [28]): Let Ω be any open subset of $\mathbb{R}^n$ and let $\boldsymbol{f}$ be a distribution of $\boldsymbol{\mathcal{D}}'(\Omega)$ that satisfies:*

$$\forall \boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega), \qquad \langle \boldsymbol{f}, \boldsymbol{v}\rangle = 0.$$

Then, there exists a distribution π in $\mathcal{D}'(\Omega)$ such that $\boldsymbol{f} = \nabla\pi$.

ii) **De Rham's Theorem in $\mathbf{W}^{-m,p}(\Omega)$** *(See [5]): Let m be any integer, p any real number with $1 < p < \infty$. Let $\boldsymbol{f} \in \mathbf{W}^{-m,p}(\Omega)$ satisfy:*

$$\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega), \qquad \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle = 0.$$

Then, there exists $\pi \in W^{-m+1,p}(\Omega)$ such that $\boldsymbol{f} = \nabla\pi$. If in addition the set Ω is connected, then π is defined uniquely, up to an additive constant, and there exists a positive constant C, independent of $\boldsymbol{f}$, such that:

$$\inf_{K\in\mathbb{R}} \|\pi + K\|_{W^{-m+1,p}(\Omega)/\mathbb{R}} \leq C\,\|\boldsymbol{f}\|_{\mathbf{W}^{-m,p}(\Omega)}.$$

2.2 The new spaces

We begin by introducing some spaces: First,

$$\mathbf{X}_{r,p}(\Omega) = \{\boldsymbol{\varphi} \in \mathbf{W}^{1,r}_0(\Omega);\ \nabla \cdot \boldsymbol{\varphi} \in W^{1,p}_0(\Omega)\}, \quad 1 < r,\, p < \infty, \tag{2.1}$$

and we set $\mathbf{X}_{p,p}(\Omega) = \mathbf{X}_p(\Omega)$. Their dual spaces, $(\mathbf{X}_{r,p}(\Omega))'$ and $(\mathbf{X}_p(\Omega))'$, will be characterized in Lemma 2.3. Second, the solenoidal space:

$$\mathbf{H}_p(\Omega) = \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \nabla \cdot \boldsymbol{v} = 0\}. \tag{2.2}$$

And finally, the spaces:

$$\mathbf{T}_{p,r}(\Omega) = \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \Delta\boldsymbol{v} \in (\mathbf{X}_{r',p'}(\Omega))'\}, \quad \mathbf{T}_{p,r,\sigma}(\Omega) = \{\boldsymbol{v} \in \mathbf{T}_{p,r}(\Omega);\ \nabla \cdot \boldsymbol{v} = 0\}, \tag{2.3}$$

endowed with the topology given by the norm: $\|\boldsymbol{v}\|_{\mathbf{T}_{p,r}(\Omega)} = \|\boldsymbol{v}\|_{\mathbf{L}^p(\Omega)} + \|\Delta\boldsymbol{v}\|_{[\mathbf{X}_{r',p'}(\Omega)]'}$. Observe that when $p = r$, these spaces are denoted as $\mathbf{T}_p(\Omega)$ and $\mathbf{T}_{p,\sigma}(\Omega)$, respectively.

The proofs of the following lemmas are classical, although the functional spaces are changed. They can be seen in [10] for $n = 3$, but the proofs are also valid for any $n \geq 2$.

Lemma 2.2. *i) The space $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ is dense in $\mathbf{H}_p(\Omega)$.*

ii) The space $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\mathbf{X}_{r,p}(\Omega)$ and for all $q \in W^{-1,p}(\Omega)$ and $\boldsymbol{\varphi} \in \mathbf{X}_{r',p'}(\Omega)$, we have

$$\langle \nabla q,\ \boldsymbol{\varphi}\rangle_{[\mathbf{X}_{r',p'}(\Omega)]'\times \mathbf{X}_{r',p'}(\Omega)} = -\langle q,\ \nabla\ \cdot\ \boldsymbol{\varphi}\rangle_{W^{-1,p}(\Omega)\times W^{1,p'}_0(\Omega)}. \tag{2.4}$$

Lemma 2.3. *Let $\boldsymbol{f} \in (\mathbf{X}_{r,p}(\Omega))'$. Then, there exist $\mathbb{F}_0 = (f_{ij})_{1\le i,j\le n}$ such that $\mathbb{F}_0 \in \mathbb{L}^{r'}(\Omega)$, $f_1 \in W^{-1,p'}(\Omega)$ and satisfying:*

$$\boldsymbol{f} = \nabla\cdot\mathbb{F}_0 + \nabla f_1. \tag{2.5}$$

Moreover, $\|\boldsymbol{f}\|_{[\mathbf{X}_{r,p}(\Omega)]'} = \max\{\|f_{ij}\|_{L^{r'}(\Omega)}, 1\le i,j\le n,\ \|f_1\|_{W^{-1,p'}(\Omega)}\}$. Conversely, if $\boldsymbol{f}$ satisfies (2.5), then $\boldsymbol{f} \in (\mathbf{X}_{r,p}(\Omega))'$.

As consequence of Lemma 2.2 ii), we have the following embeddings if $\frac{1}{r} \le \frac{1}{p} + \frac{1}{n}$:

$$\mathbf{W}^{-1,r}(\Omega) \hookrightarrow (\mathbf{X}_{r',p'}(\Omega))' \hookrightarrow \mathbf{W}^{-2,p}(\Omega), \tag{2.6}$$

Giving a meaning to the trace of a very weak solution of a Stokes, Oseen or Navier-Stokes problem is not trivial. Remember that we are not in the classical variational framework. In this way, we need to introduce some spaces. First, we consider the space:

$$\mathbf{Y}_{p'}(\Omega) = \{\boldsymbol{\psi} \in \mathbf{W}^{2,p'}(\Omega);\ \boldsymbol{\psi}|_\Gamma = \mathbf{0},\ (\nabla\cdot\boldsymbol{\psi})|_\Gamma = 0\}$$

that can also be described (see [5]) as:

$$\mathbf{Y}_{p'}(\Omega) = \{\boldsymbol{\psi} \in \mathbf{W}^{2,p'}(\Omega);\ \boldsymbol{\psi}|_\Gamma = \mathbf{0},\ \frac{\partial\boldsymbol{\psi}}{\partial\boldsymbol{n}}\cdot\boldsymbol{n}\Big|_\Gamma = 0\}. \tag{2.7}$$

Observe that the range space of the normal derivative $\gamma_1 :\ \mathbf{Y}_{p'}(\Omega) \to \mathbf{W}^{1/p,p'}(\Gamma)$ is:

$$\mathbf{Z}_{p'}(\Gamma) = \{\boldsymbol{z} \in \mathbf{W}^{1/p,p'}(\Gamma);\ \boldsymbol{z}\cdot\boldsymbol{n} = 0\}.$$

We also introduce the space $\mathbf{H}_{p,r}(\mathrm{div};\Omega) = \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \nabla\cdot\boldsymbol{v} \in L^r(\Omega)\}$, which is equipped with the graph norm. The following lemma will help us to prove a trace result. The proof can be taken from [10], Lemmas 10 and 11, and it is also valid for the $n \ge 2$.

Lemma 2.4. *i) The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\mathbf{T}_{p,r}(\Omega)$.*

ii) The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\mathbf{T}_{p,r}(\Omega) \cap \mathbf{H}_{p,r}(\mathrm{div};\Omega)$.

iii) The space $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ is dense in $\mathbf{T}_{p,r,\sigma}(\Omega)$.

2.3 The trace result

The following two lemmas prove that the tangential trace of functions $\boldsymbol{v}$ of $\mathbf{T}_{p,r,\sigma}(\Omega)$ belongs to the dual space of $\mathbf{Z}_{p'}(\Gamma)$, which is:

$$(\mathbf{Z}_{p'}(\Gamma))' = \{\boldsymbol{\mu} \in \mathbf{W}^{-1/p,p}(\Gamma);\ \boldsymbol{\mu}\cdot\boldsymbol{n} = 0\}. \tag{2.8}$$

Before, we recall that we can decompose $\boldsymbol{v}$ into its tangential, $\boldsymbol{v}_\tau$, and normal parts, that is: $\boldsymbol{v} = \boldsymbol{v}_\tau + (\boldsymbol{v}\cdot\boldsymbol{n})\,\boldsymbol{n}$.

Lemma 2.5. *Let Ω be a bounded open set of $\mathbb{R}^n$ of class $\mathcal{C}^{1,1}$. Let $1 < p < \infty$ and $r > 1$ be such that $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{n}$. The mapping $\gamma_\tau : \boldsymbol{v} \mapsto \boldsymbol{v}_\tau|_\Gamma$ on the space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping, still denoted by γ_τ, from $\mathbf{T}_{p,r}(\Omega)$ into $\mathbf{W}^{-1/p,p}(\Gamma)$, and we have the Green formula: for any $\boldsymbol{v} \in \mathbf{T}_{p,r}(\Omega)$ and $\boldsymbol{\psi} \in \mathbf{Y}_{p'}(\Omega)$,*

$$\langle \Delta \boldsymbol{v}, \boldsymbol{\psi} \rangle_{[\mathbf{X}_{r',p'}(\Omega)]' \times \mathbf{X}_{r',p'}(\Omega)} = \int_\Omega \boldsymbol{v} \cdot \Delta \boldsymbol{\psi} \, d\boldsymbol{x} - \left\langle \boldsymbol{v}_\tau, \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} \right\rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)}.$$

Proof. We start with the expression: let $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$, then

$$\left\langle \boldsymbol{v}_\tau, \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} \right\rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)} = \int_\Omega \boldsymbol{v} \cdot \Delta \boldsymbol{\psi} \, d\boldsymbol{x} - \langle \Delta \boldsymbol{v}, \boldsymbol{\psi} \rangle_{[\mathbf{X}_{r',p'}(\Omega)]' \times \mathbf{X}_{r',p'}(\Omega)} \tag{2.9}$$

which is valid for any $\boldsymbol{\psi} \in \mathbf{Y}_{p'}(\Omega)$. Observe that $\mathbf{Y}_{p'}(\Omega) \subset \mathbf{X}_{r',p'}(\Omega)$ because $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{n}$ and the normal trace of the functions of $\boldsymbol{\psi} \in \mathbf{Y}_{p'}(\Omega)$ belongs to the space $\mathbf{Z}_{p'}(\Gamma)$.

Let $\boldsymbol{\mu} \in \mathbf{W}^{1/p,p'}(\Gamma)$. Then, $\boldsymbol{\mu} = \boldsymbol{\mu}_\tau + (\boldsymbol{\mu} \cdot \boldsymbol{n})\boldsymbol{n}$. Since Ω is of class $\mathcal{C}^{1,1}$, we know that there exists $\boldsymbol{\psi} \in \mathbf{W}^{2,p'}(\Omega)$ such that $\boldsymbol{\psi} = \mathbf{0}$ and $\dfrac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} = \boldsymbol{\mu}_\tau$ on Γ and verifying:

$$\|\boldsymbol{\psi}\|_{\mathbf{W}^{2,p'}(\Omega)} \leq C \, \|\boldsymbol{\mu}_\tau\|_{\mathbf{W}^{1/p,p'}(\Gamma)} \leq C \, \|\boldsymbol{\mu}\|_{\mathbf{W}^{1/p,p'}(\Gamma)}.$$

Moreover, $\boldsymbol{\psi} \in \mathbf{Y}_{p'}(\Omega)$. Therefore, we can bound the boundary term as follows for such functions $\boldsymbol{\psi}$:

$$\left| \langle \boldsymbol{v}_\tau, \boldsymbol{\mu} \rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)} \right| = \left| \left\langle \boldsymbol{v}_\tau, \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} \right\rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)} \right|$$

$$\leq \|\boldsymbol{v}\|_{\mathbf{L}^p(\Omega)} \|\boldsymbol{\psi}\|_{\mathbf{W}^{2,p'}(\Omega)} + \|\Delta \boldsymbol{v}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} \|\boldsymbol{\psi}\|_{\mathbf{X}_{r',p'}(\Omega)} \leq C \, \|\boldsymbol{v}\|_{\mathbf{T}_{p,r}(\Omega)} \|\boldsymbol{\psi}\|_{\mathbf{Y}_{p'}(\Omega)}$$

Thus,

$$\|\boldsymbol{v}_\tau\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \leq C \, \|\boldsymbol{v}\|_{\mathbf{T}_{p,r}(\Omega)}.$$

Therefore, the linear continuous mapping $\boldsymbol{v} \mapsto \boldsymbol{v}_\tau|_\Gamma$ defined on $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is continuous for the norm of $\mathbf{T}_{p,r}(\Omega)$. Since $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\mathbf{T}_{p,r}(\Omega)$, then we can extend this mapping from $\mathbf{T}_{p,r}(\Omega)$ into $\mathbf{W}^{-1/p,p}(\Gamma)$, that is, the tangential trace of functions of $\mathbf{T}_{p,r}(\Omega)$ belongs to $\mathbf{W}^{-1/p,p}(\Gamma)$. □

Lemma 2.6. *i) The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\mathbf{H}_{p,r}(\mathrm{div};\Omega)$.*

ii) Let $1 < p < \infty$ and $r > 1$ be such that $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{n}$. The mapping $\gamma_{\boldsymbol{n}} : \boldsymbol{v} \mapsto \boldsymbol{v} \cdot \boldsymbol{n}|_\Gamma$ on the space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping, still denoted by $\gamma_{\boldsymbol{n}}$, from $\mathbf{H}_{p,r}(\mathrm{div};\Omega)$ into $W^{-1/p,p}(\Gamma)$, and we have the Green formula: for any $\boldsymbol{v} \in \mathbf{H}_{p,r}(\mathrm{div};\Omega)$ and $\varphi \in W^{1,p'}(\Omega)$,

$$\int_\Omega \boldsymbol{v} \cdot \nabla \varphi \; d\boldsymbol{x} + \int_\Omega \varphi \; \mathrm{div} \; \boldsymbol{v} \; d\boldsymbol{x} = \langle \boldsymbol{v} \cdot \boldsymbol{n}, \varphi \; \rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)}.$$

2.4 Very weak, weak and strong regularity

We treat the Stokes system under the compatibility condition:

$$\int_\Omega h(\boldsymbol{x})\, d\boldsymbol{x} = \langle \boldsymbol{g}\cdot\boldsymbol{n}, 1\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)}. \tag{2.10}$$

Basic results on weak and strong solutions of problem (S) for $n \geq 2$ may be summarized in the following theorem (see [5], [14]).

Theorem 2.7. *i) For every* $\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega)$, $h \in L^p(\Omega)$, $\boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$, *and satisfying the compatibility condition (2.10), the Stokes problem* (S) *has exactly one solution* $\boldsymbol{u} \in \mathbf{W}^{1,p}(\Omega)$ *and* $q \in L^p(\Omega)/\mathbb{R}$. *Moreover, there exists a constant* $C > 0$ *depending only on* p *and* Ω *such that:*

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|q\|_{L^p(\Omega)/\mathbb{R}} \leq C\,(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}). \tag{2.11}$$

ii) Moreover, if $\boldsymbol{f} \in \mathbf{L}^p(\Omega)$, $h \in W^{1,p}(\Omega)$, $\boldsymbol{g} \in \mathbf{W}^{2-1/p,p}(\Gamma)$, *then* $\boldsymbol{u} \in \mathbf{W}^{2,p}(\Omega)$, $q \in W^{1,p}(\Omega)$ *and there exists a constant* $C > 0$ *depending only on* p *and* Ω *such that:*

$$\|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} + \|q\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\,(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}). \tag{2.12}$$

We are interested here in the case of singular data satisfying precisely the following assumptions:

$$\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))',\ \ h \in L^r(\Omega),\ \ \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma),\ \text{with } \frac{1}{r} \leq \frac{1}{p} + \frac{1}{n} \text{ and } r \leq p. \tag{2.13}$$

Recall that the space $(\mathbf{X}_{r',p'}(\Omega))'$ is an intermediate space between $W^{-1,r}(\Omega)$ and $W^{-2,p}(\Omega)$ (see (2.6)).

Remark 2.8. If Ω is only a bounded Lipschitz domain, there exists $\varepsilon > 0$ depending only on the Lipschitz constant of Ω such that if $2 \leq p \leq 3+\varepsilon$, $\boldsymbol{f} = \mathbf{0}$, $h = 0$ and $\boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$ with $\int_\Gamma \boldsymbol{g}\cdot\boldsymbol{n} = 0$, the conclusion of the first part of Theorem 2.7 holds. The result is also valid under the assumptions $\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega)$, $h = 0$ and $\boldsymbol{g} = \mathbf{0}$, for a ε such that $(3+\varepsilon)/(2+\varepsilon) < p < 3+\varepsilon$ (see [13]).

We recall the definition and the existence result of very weak solution for the Stokes problem.

Definition 2.9 (Very weak solution for the Stokes problem). We say that $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ is a **very weak solution** of (S) if the following equalities hold: For any $\boldsymbol{\varphi} \in \mathbf{Y}_{p'}(\Omega)$ and $\pi \in W^{1,p'}(\Omega)$,

$$\begin{aligned} -\int_\Omega \boldsymbol{u}\cdot\Delta\boldsymbol{\varphi}\, d\boldsymbol{x} - \langle q, \nabla\cdot\boldsymbol{\varphi}\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p'}(\Omega)} &= \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_\Omega - \langle \boldsymbol{g}_\tau, \frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\rangle_\Gamma, \\ \int_\Omega \boldsymbol{u}\cdot\nabla\pi\, d\boldsymbol{x} &= -\int_\Omega h\,\pi\, d\boldsymbol{x} + \langle \boldsymbol{g}\cdot\boldsymbol{n}, \pi\rangle_\Gamma, \end{aligned} \tag{2.14}$$

where the dualities on Ω and Γ are defined by:

$$\langle\cdot,\cdot\rangle_\Omega = \langle\cdot,\cdot\rangle_{[\mathbf{X}_{r',p'}(\Omega)]'\times\mathbf{X}_{r',p'}(\Omega)},\ \langle\cdot,\cdot\rangle_\Gamma = \langle\cdot,\cdot\rangle_{\mathbf{W}^{-1/p,p}(\Gamma)\times\mathbf{W}^{1/p,p'}(\Gamma)}. \tag{2.15}$$

Note that $W^{1,p'}(\Omega) \hookrightarrow L^{r'}(\Omega)$ and $\mathbf{Y}_{p'}(\Omega) \hookrightarrow \mathbf{X}_{r',p'}(\Omega)$ if $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{n}$, that means that all the brackets and integrals have a sense.

Proposition 2.1. *Suppose that $\boldsymbol{f}$, h, $\boldsymbol{g}$ satisfy (2.13). Then the following two statements are equivalent:*

i) $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ is a very weak solution of (S),

ii) $(\boldsymbol{u}, q)$ satisfies the system (S) in the sense of distributions.

Proof. i) Let $(\boldsymbol{u}, q)$ be a very weak solution to problem (S). It is clear that $-\Delta \boldsymbol{u} + \nabla q = \boldsymbol{f}$ and $\nabla \cdot \boldsymbol{u} = h$ in Ω and consequently $\boldsymbol{u}$ belongs to $\mathbf{T}_{p,r}(\Omega)$. Using Lemma 2.6 point *ii)*, Lemma 2.5 and (2.4), we obtain

$$-\int_\Omega \boldsymbol{u} \cdot \Delta \boldsymbol{\varphi} \, d\boldsymbol{x} \quad + \quad \langle \boldsymbol{u}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)} - \langle q, \nabla \cdot \boldsymbol{\varphi} \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega .$$

Since for any $\boldsymbol{\varphi} \in \mathbf{Y}_{p'}(\Omega)$,

$$\langle \boldsymbol{u}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)} = \langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)},$$

we deduce that $\boldsymbol{u}_\tau = \boldsymbol{g}_\tau$ in $\mathbf{W}^{-1/p,p}(\Gamma)$. From the equation $\nabla \cdot \boldsymbol{u} = h$, we deduce that for any $\pi \in W^{1,p'}(\Omega)$, we have

$$\langle \boldsymbol{u} \cdot \boldsymbol{n}, \pi \rangle_\Gamma = \langle \boldsymbol{g} \cdot \boldsymbol{n}, \pi \rangle_\Gamma .$$

Consequently $\boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{g} \cdot \boldsymbol{n}$ in $W^{-1/p,p}(\Gamma)$ and finally $\boldsymbol{u} = \boldsymbol{g}$ on Γ.
ii) The converse is a simple consequence of Lemma 2.6 point *ii)*, Lemma 2.5 and (2.4). ☐

Observe that the following result is a variation from Proposition 4.11 in [5], which was made for $\boldsymbol{f} = \mathbf{0}$ and $h = 0$. Here, we focus on the aspect that the fact of taking $\boldsymbol{f} \neq \mathbf{0}$ and $h \neq 0$ make over the whole proof appearing there. In the case $r = p$, we have:

Proposition 2.2. *Let $\boldsymbol{f} \in (\mathbf{X}_{p'}(\Omega))'$, $h \in L^p(\Omega)$, $\boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$, and satisfying the compatibility condition (2.10). Then, the Stokes problem (S) has exactly one solution $\boldsymbol{u} \in \mathbf{L}^p(\Omega)$ and $q \in W^{-1,p}(\Omega)/\mathbb{R}$. Moreover, there exists a constant $C > 0$ depending only on p and Ω such that:*

$$\|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|q\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leq C \left\{ \|\boldsymbol{f}\|_{[\mathbf{X}_{p'}(\Omega)]'} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \right\}. \tag{2.16}$$

Moreover $\boldsymbol{u} \in \mathbf{T}_p(\Omega)$ and

$$\|\boldsymbol{u}\|_{\mathbf{T}_p(\Omega)} \leq C \left\{ \|\boldsymbol{f}\|_{[\mathbf{X}_{p'}(\Omega)]'} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \right\}.$$

Proof. In [5], the proof of Proposition 2.2 is made for $\boldsymbol{f} = \mathbf{0}$ and $h = 0$ (see Proposition 4.11 p. 132 [5]). It is on the aspects from the proof given in [5] were $\boldsymbol{f}$ and h take part on, where we focus on below.
i) First step: We suppose that $\mathbf{g} \cdot \boldsymbol{n} = 0$ on Γ and $\displaystyle\int_\Omega h(\boldsymbol{x}) \, d\boldsymbol{x} = 0$.
It remains to consider the following equivalent problem:

Find $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ such that: $\forall \boldsymbol{w} \in \mathbf{Y}_{p'}(\Omega)$, $\forall \pi \in W^{1,p'}(\Omega)$

$$\begin{aligned}\int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{w} + \nabla \pi)\, d\boldsymbol{x} \quad &- \quad \langle q, \nabla \cdot \boldsymbol{w} \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)} \\ &= \quad \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{[\mathbf{X}_{p'}(\Omega)]' \times \mathbf{X}_{p'}(\Omega)} - \langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \rangle_\Gamma - \int_\Omega h\, \pi\, d\boldsymbol{x}\end{aligned}$$

being $\mathbf{Y}_{p'}(\Omega)$ the space defined by (2.7) that verifies the embedding $\mathbf{Y}_{p'}(\Omega) \hookrightarrow \mathbf{X}_{p'}(\Omega)$. The duality brackets are given in (2.15).

We can prove (as in [5]) that for any pair $(\mathbf{F}, \varphi) \in \mathbf{L}^{p'}(\Omega) \times (W_0^{1,p'}(\Omega) \cap L_0^{p'}(\Omega))$, we have:

$$\Big| \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{[\mathbf{X}_{p'}(\Omega)]' \times \mathbf{X}_{p'}(\Omega)} - \Big\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \Big\rangle_\Gamma - \int_\Omega h\, \pi\, d\boldsymbol{x} \Big|$$

$$\leq C \left(\|\boldsymbol{f}\|_{[\mathbf{X}_{p'}(\Omega)]'} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Omega)} + \|h\|_{L^p(\Omega)} \right) \left(\|\mathbf{F}\|_{\mathbf{L}^{p'}(\Omega)} + \|\varphi\|_{W^{1,p'}(\Omega)} \right)$$

being $(\boldsymbol{w}, \pi) \in \mathbf{Y}_{p'}(\Omega) \times W^{1,p'}(\Omega)/\mathbb{R}$ the unique solution of the Stokes (dual) problem:

$$-\Delta \boldsymbol{w} + \nabla \pi = \mathbf{F} \quad \text{and} \quad \nabla \cdot \boldsymbol{w} = \varphi \quad \text{in } \Omega, \;\; \boldsymbol{w} = \mathbf{0} \quad \text{on } \Gamma.$$

Note that for any $k \in \mathbb{R}$,

$$\Big| \int_\Omega h\, \pi\, d\boldsymbol{x} \Big| = \Big| \int_\Omega h\, (\pi + k)\, d\boldsymbol{x} \Big| \leq \|h\|_{L^p(\Omega)} \|\pi\|_{L^{p'}(\Omega)/\mathbb{R}} \tag{2.17}$$

and

$$\|\boldsymbol{w}\|_{\mathbf{W}^{2,p'}(\Omega)} + \|\pi\|_{W^{1,p'}(\Omega)/\mathbb{R}} \leq C \left(\|\mathbf{F}\|_{\mathbf{L}^{p'}(\Omega)} + \|\varphi\|_{W^{1,p'}(\Omega)} \right).$$

From this bound, we deduce that the mapping

$$(\mathbf{F}, \varphi) \to \langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega - \langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \rangle_\Gamma - \int_\Omega h\, \pi\, d\boldsymbol{x}$$

defines an element of the dual space of $\mathbf{L}^{p'}(\Omega) \times (W_0^{1,p'}(\Omega) \cap L_0^{p'}(\Omega))$ with norm bounded by $C(\|\boldsymbol{f}\|_{[\mathbf{X}_{p'}(\Omega)]'} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Omega)})$. From Riesz' Representation Theorem we deduce that there exists a unique $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ solution of (S) satisfying the bound (2.16).

ii) Second step: Now, we suppose that $\int_\Omega h(\boldsymbol{x})\, d\boldsymbol{x} = \langle \mathbf{g} \cdot \boldsymbol{n}, 1 \rangle_\Gamma$ and consider the Neumann problem: Find $\theta \in W^{1,p}(\Omega)/\mathbb{R}$ such that:

$$(N) \quad \Delta \theta = h \;\; \text{in } \Omega, \;\; \frac{\partial \theta}{\partial \boldsymbol{n}} = \boldsymbol{g} \cdot \boldsymbol{n} \;\; \text{on } \Gamma,$$

which has a unique solution $\theta \in W^{1,p}(\Omega)/\mathbb{R}$ and verifies the estimate:

$$\|\theta\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C \left(\|h\|_{L^p(\Omega)} + \|\boldsymbol{g} \cdot \boldsymbol{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \right). \tag{2.18}$$

Set $\boldsymbol{u}_0 = \nabla \theta$. By step *i*), there exists a unique $(\boldsymbol{z}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ solution of problem:

$$-\Delta \boldsymbol{z} + \nabla q = \boldsymbol{f} + \nabla h \;\; \text{and} \;\; \nabla \cdot \boldsymbol{z} = 0 \;\; \text{in } \Omega, \;\; \boldsymbol{z} = \boldsymbol{g} - \boldsymbol{u}_0|_\Gamma \;\; \text{on } \Gamma,$$

where the characterization given by Lemma 2.3 implies that $\nabla h \in (\mathbf{X}_{p'}(\Omega))'$ and $\boldsymbol{g} - \boldsymbol{u}_0|_\Gamma$ satisfies the hypothesis of Step *i*). Finally, the pair of functions $(\boldsymbol{u}, q) = (\boldsymbol{z} + \boldsymbol{u}_0, q)$ is the required solution. $\square$

Theorem 2.10. *Let $\boldsymbol{f}$, h, $\boldsymbol{g}$ satisfy (2.13) and (2.10). Then, the Stokes problem (S) has exactly one solution $\boldsymbol{u} \in \mathbf{L}^p(\Omega)$ and $q \in W^{-1,p}(\Omega)/\mathbb{R}$. Moreover, there exists a constant $C > 0$ depending only on p and Ω such that:*

$$\|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|q\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leq C \left\{ \|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \right\} \tag{2.19}$$

Moreover $\boldsymbol{u} \in \mathbf{T}_{p,r}(\Omega)$ and

$$\|\boldsymbol{u}\|_{\mathbf{T}_{p,r}(\Omega)} \leq C \left\{ \|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \right\}.$$

In particular, if $\boldsymbol{f} \in \mathbf{W}^{-1,r_0}(\Omega)$ and $h \in L^{r_0}(\Omega)$ with $r_0 = 2p/(2+p)$, then $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ with the corresponding estimates.

Proof. If we want to use hypotheses $\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))'$ instead of $\boldsymbol{f} \in (\mathbf{X}_{p'}(\Omega))'$ and $h \in L^r(\Omega)$ instead of $h \in L^p(\Omega)$, appearing in Definition 2.9 and Proposition 2.2, then the differences on the proof are linked to:

- Instead of $\langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega$, we have: $\langle \boldsymbol{f}, \boldsymbol{w} \rangle_{[\mathbf{X}_{r',p'}(\Omega)]' \times \mathbf{X}_{r',p'}(\Omega)}$ for $\boldsymbol{w} \in \mathbf{Y}_{p'}(\Omega)$.
  Observe that $\mathbf{Y}_{p'}(\Omega) \subset \mathbf{X}_{r',p'}(\Omega)$ if $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$, which is the case defined in Lemma 2.5. Therefore, the same study can be made, only replacing the bound $\|\boldsymbol{f}\|_{[\mathbf{X}_{p'}(\Omega)]'}$ by $\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'}$.

- Now, we solve problem (N) with $h \in L^r(\Omega)$. Problem (N) is equivalent to the problem: Find $\theta \in W^{1,p}(\Omega)/\mathbb{R}$ such that:

$$\forall \varphi \in W^{1,p'}(\Omega), \qquad \int_\Omega \nabla\theta \cdot \nabla\varphi \, d\boldsymbol{x} = \langle \boldsymbol{g} \cdot \boldsymbol{n}, \varphi \rangle_\Gamma - \int_\Omega h \, \varphi \, d\boldsymbol{x}$$

  which is well defined for any $\varphi \in W^{1,p'}(\Omega)$ (observe that $W^{1,p'}(\Omega) \hookrightarrow L^{r'}(\Omega)$ if $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$). The mapping $\ell : \varphi \mapsto \langle \boldsymbol{g} \cdot \boldsymbol{n}, \varphi \rangle_\Gamma - \int_\Omega h \, \varphi \, d\boldsymbol{x}$ defines an element of the dual $(W^{1,p'}(\Omega)/\mathbb{R})'$ because $\langle \ell, 1 \rangle = 0$. Furthermore, an inf-sup condition is verified. Therefore, the problem (N) has a unique solution $\theta \in W^{1,p}(\Omega)/\mathbb{R}$ and satisfies the estimate:

$$\|\theta\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C \left(\|\boldsymbol{g} \cdot \boldsymbol{n}\|_{W^{-1/p,p}(\Gamma)} + \|h\|_{L^r(\Omega)} \right)$$

□

Remark 2.11.

i) Observe that in [18] Theorem 3, the domain in $\mathbb{R}^3$ considered is of class $\mathcal{C}^{2,1}$ instead of class $\mathcal{C}^{1,1}$. Moreover, the divergence term $h \in L^p(\Omega)$ instead of $h \in L^r(\Omega)$. The regularity considered for $\boldsymbol{f}$, taking into account Lemma 2.3, is the same as we consider ($\boldsymbol{f}$ is the divergence of a tensor in $\mathbb{L}^r(\Omega)$ because of the gradient part can be associated to the pressure). But for the divergence condition h, Galdi et al. consider $h \in L^p(\Omega)$, which is a space smaller than that considered in this work ($h \in L^r(\Omega)$ for $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$). Moreover, our solution is obtained in the space $\mathbf{T}_{p,r}(\Omega)$ which has been clearly characterized contrary to the space $\widehat{\mathbf{W}}^{1,p}(\Omega)$ appearing in [18] which is not characterized, is completely abstract and is obtained as the closure of $\mathbf{W}^{1,p}(\Omega)$ for the norm

$$\|\boldsymbol{u}\|_{\widehat{\mathbf{W}}^{1,p}(\Omega)} = \|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|A_r^{-1/2}\mathcal{P}_r\Delta\boldsymbol{u}\|_{\mathbf{L}^r(\Omega)},$$

where A_r is the Stokes operator with domain equal to $\mathbf{W}^{2,p}(\Omega)\cap\mathbf{W}_0^{1,p}(\Omega)\cap\mathbf{L}^p_\sigma(\Omega)$ and $\mathcal{P}_r$ is the Helmholtz projection operator from $\mathbf{L}^r(\Omega)$ onto $\mathbf{L}^r_\sigma(\Omega)$.

ii) The same type of consideration about the domain and the space of solutions $\mathbf{T}_{p,r}(\Omega)$ can be made in [17] Theorem 1.2. In this case, the regularity considered for $\boldsymbol{f}$, taking into account Lemma 2.3, is the same as we consider; and the space of regularity for $\boldsymbol{u}$ they obtained is the same of us. However, they say that condition over $\boldsymbol{f}$ can be weaken by $A_p^{-1}P_p\boldsymbol{f}\in\mathbf{L}^p_\sigma(\Omega)$ (see [17] Remark 1.6), but this condition is not clearly characterized.

Corollary 2.12. (See Corollary 3 in [10]) Let $\boldsymbol{f}$, h, $\boldsymbol{g}$ satisfy (2.10) and $\boldsymbol{f} = \nabla\cdot\mathbb{F}_0 + \nabla f_1$ with $\mathbb{F}_0\in\mathbb{L}^r(\Omega)$, $f_1\in W^{-1,p}(\Omega)$, $h\in L^r(\Omega)$, $\boldsymbol{g}\in\mathbf{W}^{1-1/r,r}(\Gamma)$. Then the solution $\boldsymbol{u}$ given by Theorem 2.10 belongs to $\mathbf{W}^{1,r}(\Omega)$. If moreover $f_1\in L^r(\Omega)$, then the solution q given by Theorem 2.10 belongs to $L^r(\Omega)$. In both cases, we have the corresponding estimates.

Remark 2.13. The space $\mathbf{T}_{p,r}(\Omega)$ is an intermediate space between $\mathbf{W}^{1,r}(\Omega)$ and $\mathbf{L}^p(\Omega)$, because of

$$\mathbf{W}^{1,r}(\Omega)\hookrightarrow\mathbf{T}_{p,r}(\Omega)\quad\text{when}\quad\frac{1}{r}\le\frac{1}{p}+\frac{1}{n}.$$

Remark 2.14. i) First, we have as consequence of Proposition 2.2 the following Helmholtz decomposition: for any $\boldsymbol{f}\in(\mathbf{X}_{p'}(\Omega))'$, there exist $\boldsymbol{\psi}\in\mathbf{W}^{-1,p}(\Omega)$ and $q\in W^{-1,p}(\Omega)$ such that

$$\boldsymbol{f} = \mathbf{curl}\;\boldsymbol{\psi} + \nabla q,\ \operatorname{div}\boldsymbol{\psi} = 0\quad\text{in }\Omega.$$

ii) In the same way, suppose that $\boldsymbol{f} = \nabla\cdot\mathbb{F}$ with $\mathbb{F}\in\mathbb{L}^p(\Omega)$, $h\in L^p(\Omega)$ and $\boldsymbol{g}\in\mathbf{W}^{1-1/p,p}(\Gamma)$ verifying the compatibility condition (2.10). Then, the solution $(\boldsymbol{u},q)\in\mathbf{L}^p(\Omega)\times W^{-1,p}(\Omega)$ given by Theorem 2.10 satisfies $(\boldsymbol{u},q)\in\mathbf{W}^{1,p}(\Omega)\times L^p(\Omega)$ with the appropriate estimate.

Corollary 2.15. Let $h\in L^r(\Omega)$ and $\boldsymbol{g}\in\mathbf{W}^{-1/p,p}(\Gamma)$ verifying the compatibility condition (2.10) with $\frac{1}{r}\le\frac{1}{p}+\frac{1}{n}$ and $r\le p$. Then, there exists at least one solution $\boldsymbol{u}\in\mathbf{T}_{p,r}(\Omega)$ verifying

$$\nabla\cdot\boldsymbol{u} = h\quad\text{in }\Omega,\qquad\boldsymbol{u} = \boldsymbol{g}\quad\text{on }\Gamma.$$

Moreover, there exists a constant $C = C(\Omega,p,r)$ such that:

$$\|\boldsymbol{u}\|_{\mathbf{T}_{p,r}(\Omega)}\le C\left(\|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}\right).$$

The following corollary gives Stokes solutions $(\boldsymbol{u},q)$ in fractionary Sobolev spaces of type $\mathbf{W}^{\sigma,p}(\Omega)\times W^{\sigma-1,p}(\Omega)$, with $0<\sigma<2$.

Corollary 2.16. Let s be a real number such that $0\le s\le 1$.

i) Let $\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1$, h and $\boldsymbol{g}$ satisfy the compatibility condition (2.10) with

$$\mathbb{F}_0 \in \mathbf{W}^{s,r}(\Omega), \quad f_1 \in W^{s-1,p}(\Omega), \ \boldsymbol{g} \in \mathbf{W}^{s-1/p,p}(\Gamma), \ h \in W^{s,r}(\Omega),$$

with $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{2}$ and $r \leq p$. Then, Stokes Problem (S) has exactly one solution $(\boldsymbol{u},\ q) \in \mathbf{W}^{s,p}(\Omega) \times W^{s-1,p}(\Omega)/\mathbb{R}$ satisfying the estimate

$$\|\boldsymbol{u}\|_{\mathbf{W}^{s,p}(\Omega)} + \|q\|_{W^{s-1,p}(\Omega)/\mathbb{R}} \leq C \left(\|\mathbb{F}_0\|_{\mathbf{W}^{s,r}(\Omega)} + \|f_1\|_{W^{s-1,p}(\Omega)} + \|h\|_{W^{s,r}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{s-1/p,p}(\Gamma)}\right)$$

ii) Assume that $\boldsymbol{f} \in \mathbf{W}^{s-1,p}(\Omega)$, $\boldsymbol{g} \in \mathbf{W}^{s+1-1/p,p}(\Gamma)$, $h \in W^{s,p}(\Omega)$, with the compatibility condition (2.10). Then, Stokes Problem (S) has exactly one solution $(\boldsymbol{u},\ q) \in \mathbf{W}^{s+1,p}(\Omega) \times W^{s,p}(\Omega)/\mathbb{R}$ with

$$\|\boldsymbol{u}\|_{\mathbf{W}^{s+1,p}(\Omega)} + \|q\|_{W^{s,p}(\Omega)/\mathbb{R}} \leq C \left(\|\boldsymbol{f}\|_{\mathbf{W}^{s-1,p}(\Omega)} + \|h\|_{W^{s,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{s+1-1/p,p}(\Gamma)}\right)$$

Remark 2.17. We can reformulate the point ii) as follows. For any

$$\boldsymbol{f} \in \mathbf{W}^{-s,p'}(\Omega), \ h \in W^{-s+1,p'}(\Omega), \ \boldsymbol{g} \in \mathbf{W}^{2-s-1/p',p'}(\Gamma),$$

with $0 \leq s \leq 1$, then problem (S) has a unique solution $(\boldsymbol{u},\ q) \in \mathbf{W}^{2-s,p'}(\Omega) \times W^{1-s,p'}(\Omega)/\mathbb{R}$.

The following theorem gives solutions for external forces $\boldsymbol{f} \in \mathbf{W}^{s-2,p}(\Omega)$ and divergence condition $h \in W^{s-1,p}(\Omega)$ with $1/p < s < 2$. If $p = 2$, we can obtain solutions in $\mathbf{H}^{1/2+\varepsilon}(\Omega) \times H^{1/2+\varepsilon}(\Omega)$, $0 < \varepsilon \leq 3/2$.

Theorem 2.18. *Let s be a real number such that $\frac{1}{p} < s \leq 2$. Let $\boldsymbol{f}$, h and $\boldsymbol{g}$ satisfy the compatibility condition (2.10) with*

$$\boldsymbol{f} \in \mathbf{W}^{s-2,p}(\Omega), \ h \in W^{s-1,p}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{s-1/p,p}(\Gamma).$$

Then, the Stokes problem (S) has exactly one solution $(\boldsymbol{u},\ q) \in \mathbf{W}^{s,p}(\Omega) \times W^{s-1,p}(\Omega)/\mathbb{R}$ satisfying the estimate

$$\|\boldsymbol{u}\|_{\mathbf{W}^{s,p}} + \|q\|_{W^{s-1,p}/\mathbb{R}} \leq C \left(\|\boldsymbol{f}\|_{\mathbf{W}^{s-2,p}(\Omega)} + \|h\|_{W^{s-1,p}} + \|\boldsymbol{g}\|_{\mathbf{W}^{s-1/p,p}(\Gamma)}\right) \tag{2.20}$$

Remark 2.19. i) Remark 1.8 point ii) and iii) holds.

ii) If $n = 2$, Ω is a convex polygon, with $\Gamma = \cup \Gamma_i, \Gamma_i$ linear segments, $\boldsymbol{f} = \mathbf{0}$, $h = 0$ and $\boldsymbol{g} \in H^s(\Gamma_i)$, for $i = 1, \ldots, I_0$, $-1/2 < s < 1/2$, then $\boldsymbol{u} \in \mathbf{H}^r(\Omega)$ for any $r < s + 1/2$ and $q \in H^{s-1/2}(\Omega)$ ([27]).

iii) If Ω is a simply connected domain of $\mathbb{R}^2$, a result of existence of a $\boldsymbol{u}$ verifying the Stokes equations for $\boldsymbol{f} = \mathbf{0}$, $h = 0$ and $\boldsymbol{g} \in \mathbf{L}^2(\Gamma)$ (with $\int_\Gamma \boldsymbol{g} \cdot \boldsymbol{n} = 0$) can be seen in [12]. An analogous result is also presented when $\boldsymbol{g} \in \mathbf{L}^\infty(\Gamma)$.

iv) When Ω is bounded Lipschitz domain in $\mathbb{R}^n$, with $n \geq 3$, $\boldsymbol{f} = \mathbf{0}$, $h = 0$, $\boldsymbol{g} \in \mathbf{L}^2(\Gamma)$ (respectively $\boldsymbol{g} \in \mathbf{W}^{1,2}(\Gamma)$) , with $\int_\Gamma \boldsymbol{g} \cdot \boldsymbol{n} = 0$, then $\boldsymbol{u} \in \mathbf{H}^{1/2}(\Omega)$ (respectively $\boldsymbol{u} \in \mathbf{H}^{3/2}(\Omega)$ and $q \in H^{-1/2}(\Omega)$ (respectively $q \in H^{1/2}(\Omega)$) (see Fabes et al. [16]). If $\boldsymbol{g} \in \mathbf{L}^p(\Gamma)$, there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that if $2 - \varepsilon \leq p \leq 2 + \varepsilon$, then $\boldsymbol{u} \in \mathbf{W}^{1-1/p,p}(\Omega)$ and $q \in W^{-1/p,p}(\Omega)$.

v) When $\Omega \subset \mathbb{R}^3$ is only a bounded Lipschitz domain, with connected boundary, the same result has been proved by [31] with $\boldsymbol{f} = \boldsymbol{0}$ and $h = 0$ for any $p \geq 2$. The case of $\Omega \subset \mathbb{R}^n$ for $n \geq 4$ suppose that for $\boldsymbol{f} = \boldsymbol{0}$, $h = 0$ and $\boldsymbol{g} \in L^p(\Omega)$, $p \in \left[2, \dfrac{2(n-1)(n-2)}{n(n-3)}\right]$ there exists a unique $\boldsymbol{u} \in L^{p_1}(\Omega)$ for $p_1 = \dfrac{np}{p-1}$.

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