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GIULIO RACAH, SPECTROSCOPY AND GROUP THEORETICAL METHODS IN PHYSICS

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The scientific influence of G. Racah can hardly be summarized in a few pages, as his methods and ideas dominate largely many of the modern techniques used in physics, notably those used in atomic and nuclear spectroscopy. This influence goes far beyond Racah's own publications (see list at the end of this article), as follows from the huge number of contributions of his students, collaborators and contemporary physicists outside the Hebrew University of Jerusalem. Although primarily associated to the algebra of tensor operators, an essential technique in theoretical physics, the ideas and methods of Racah are found in many different contexts, from the different branches of spectroscopy to labelling problems, classification schemes and crystal structures.[1]

The year 1949 can be considered as the origin of the modern application of Lie groups to the atomic shell theory, with the publication of the fourth seminal paper of Racah's series on complex spectra (papers 27, 30, 32, 35). The novelty of his approach was to introduce non-invariance Lie groups, in contrast to the usual symmetry groups used earlier and underlying the theory (specifically the rotation group $SO(3)$ in atomic spectroscopy).[2] The ansatz of Racah was also the first step towards a reconciliation of two opposed schools: the group theory defendants like Hermann Weyl (with his famous but opaque book "Gruppentheorie und Quantenmechanik") and the opponents of such methods, notably represented by Condon and Shortley with their important book "The Theory of Atomic Spectra".[3] Possibly this decline of interest concerning

[1] An extensive review on Racah's impact in crystal- and ligand-field theories written by Prof. M. R. Kibler can be found in this volume.

[2] Even if E. Wigner can be credited to be the first having recognized the value of labelling problems with his supermultiplet model $SU(4) \supset SU(2) \times SU(2)$ of 1937, this approach was not fully accepted until much later.

[3] It is however remarkable that in this book, the authors made implicitly use of group theory when chosing adequate linear combinations of Slater determinants that actually corresponded to irreducible

atomic spectroscopy (and basic research in it) in the late 30's was influenced by the increasing attention devoted to the structure of nuclei following the Hahn-Straßmann fission experiments [1], as well as the exigences of wartime for their immediate applications [2].

For a certain time after 1949, many of Racah's results remained unnoticed in atomic spectroscopy, although their importance in nuclear physics was rapidly recognized and applied to specific situations. With Racah's technique, to just mention an example, H. A. Jahn enumerated the orbital states arising from the filling of the nuclear d-shell in the Russel-Saunders coupling, obtaining a classification of the states according to the rotation group $SO(5)$.[4].

Racah's work prior to his emigration to Palestine in 1939 is mainly focused on various problems of convergence of the self-energy of the electron, where he developed various methods that generalize the work of Majorana on elementary particles and antiparticles (refs. 1-4, 12-20, 22, 25, 26). He also devoted various articles to the hyperfine structure (refs. 5-8) following Dirac's theory of the electron. Special mention deserves reference 8, where the analysis of the hyperfine structure of Thallium and Mercury led to the observation of a structure arising from a displacement of the atomic levels in different isotopes. This phenomenon would become a crucial point in the future, specially in connection with the spectra of lanthanides and actinides [3]. These research topics reflected the importance and relevance of the Italian school (with Fermi and Majorana among others) before its tragic dissolution in 1938. Besides these articles, Racah published a few others of more mathematical nature (refs. 9-11, 21, 23, 24), like those referring to isotropic tensors in few dimensions. In the latter two articles he obtained an interesting method to evaluate multiplicities of induced representations (of the rotation group in three dimensions) without using the classical character theory, and providing recursive formulae in dependence of linear independent tensors subjected to special symmetry constraints. In some sense, these articles show Racah's interest in algebraic methods and can be seen as the germ of the tensor operator theory developed from 1942 onwards. In 1939, an important year for Racah for both personal as well as scientific reasons, new research topics and interests emerge with his emigration to Jerusalem. Once there, he started a systematical and rigorous work on spectroscopy, which would absorb most of his attention for the rest of his life, and establishing him as one of the most influential theoretical physicists.

The now famous series of four papers "Theory of Complex Spectra" (TCS) in the Physical Review (refs. 27, 30, 32, 35) is the basis for the modern approach to spectroscopic analysis, specially concerning the computations of energy levels. A new powerful technique to express interactions in terms of scalar products of tensor operators simplified considerably the computation of matrix elements, in opposition to previous approaches, the computations of which rapidly became too cumbersome and involved to be carried out for more complex spectra. In addition, adequate and subsequently fundamental notions like fractional percentage and recoupling coefficients (later

representations of $SO(5)$.

[4] Jahn H A 1950 *Proc. Royal Society A, Math. Phys. Sciences* **201** 516; 1951 *Proc. Royal Society A, Math. Phys. Sciences* **205** 192.

named after him) were introduced, and their fundamental properties studied. The introduction of the new quantum number *seniority* v was the key to solve the designation problem of energy levels in configurations having several equivalent electrons. Although these notions were in principle applied to atomic spectra, they turned out to be also essential notions in nuclear physics (nuclear shell model), as was observed from the the mid fifties onwards. As a matter of fact, the quasi-spin introduced by Anderson, Wada, Takano and Fukuda in 1958, and adapted by Kerman to the study of pairing forces in nuclei was observed to be equivalent to a classification of states in terms of the seniority quantum number [4].[5] The fourth and last paper of the TCS series, appeared in 1949, not only extended to the f-shell the solution to the problem of designation of energy levels, but also constitutes a cornerstone for the use of group theoretical methods in physics. Specifically, the seniority scheme is defined in terms of irreducible representations of the orthogonal and symplectic groups[6]. The important role of the (quadratic) Casimir operators as natural candidates for labelling operators was shown in various reduction chains considered. Also for the first time an exceptional Lie group was used in the classification of f^n states. Even if the reduction involving G_2 was only a mathematical trick for the classification, it turned out to be the essential point that allowed to distinguish terms with the same quantum numbers L and S. The irreducible representations of the groups $SO\,(7)$ and G_2 were used to label the basis states within which the operators could be diagonalized. Selection rules arise then from the important Wigner-Eckart theorem. The reduction chain $SO(7) \supset G_2 \supset SO(3)$ is historically among the first examples of the so-called internal labelling problem, and in spite of later developments in the atomic shell theory, the sequences of groups used in the TCS series remained the central core around which most subsequent group chains have been constructed [6]

Although Racah's interest focused mainly in the interpretation of atomic spectra, he continued to work in the development of group theoretical methods. Following an invitation of R. Oppenheimer, he gave his famous lectures on Lie groups in Princeton in 1951 (ref. 67). The first three parts of these lectures contained the general theory on Lie groups, the classification of the (complex) semisimple algebras and the main facts concerning their representations. Even if the theory of classical Lie groups had been worked out earlier, most of the published texts were written in the opaque and difficult style of Cartan, Weyl and others, and were therefore of little attractive to the physicist. From the perspective of representation theory, various interesting results for physical applications were obtained. Among others, Racah showed that the total number of internal labels needed to distinguish states within an irreducible representation of a semisimple group G of rank l and dimension n was completely characterized by these quantities, *i.e.* $f = \frac{1}{2}\,(n-l)$. When using subgroups G' of G to label irreducible representations, these internal labelling split into three classes: those operators needed to characterize the representation and the components of its decomposition into G'-representations (Casimir operators of G and

[5] In fact, at that time many nuclear physicists were reproducing some of Racah's results without being aware of their existence, as observed by Igal Talmi.

[6] This approach was later systematically expanded by Lipkin in [5]. It turned out that this ansatz was also adequate for the classification schemes in elementary particles.

G'), the missing label operators (subgroup scalars solving the degeneracies of the decomposition) and G'-internal operators to separate the states within each multiplet.[7]

The remaining lectures attempted to extend and generalize the methods used in the theory of complex spectra to the case of the nuclear shell model. Specifically, the classification and computation of nuclear states of mixed shells of nucleons with isospin and seniority in the LS-coupling was addressed. The final part presented a classification of nuclear interactions by means of tensorial methods, and applications to the computation of energy matrices. The typewritten version of these lectures became an essential reference with the years. Although these notes essentially contained the main facts to be used later in the classification schemes of elementary particles, the methods of these lectures remained largely unnoticed until the mid sixties, with the boom of group theory following the flavor classification of hadrons.[8] In this sense, in a short paper (refs. 41, 42) Racah showed the existence, for a semisimple Lie algebra of rank l, of exactly l independent operators in the generators that commute with the elements of the Lie algebras (i.e., the Casimir operators). This result, besides its importance in the labelling problem, expanded a well-known result of Casimir and Van der Waerden concerning the complete irreducibility of $SO(3)$-representations to all semisimple groups.[9] This work also presented an interesting approach to the invariant problem by means of differential operators, which would constitute in the future a powerful alternative to the formal computation in enveloping algebras. A beautiful application of these methods to nuclear physics can be found in Racah's contribution to the Farkas Memorial Volume (ref. 47), one of the fundamental references addressing to specific labelling problems in the analysis of nuclear structure.[10] In this paper, it was shown how the (quadratic) Casimir operator of a Lie group could be used to simplify the calculation of the levels of nuclear shells. Two cases were considered in detail: the LS-coupling with relevant reduction chain $SO(3) \subset SO(2l+1) \subset U(2l+1)$, and the jj-coupling, where the symplectic group $Sp(2l+1)$ emerged as the adequate tool to deal with the interactions. This fact was rapidly recognized and further developed by Flowers and Edmonds, among others.[11] It should be noted that, although Racah himself did not consider the symplectic groups in great detail, he implicitly used them by means of the seniority number scheme, as any v corresponds to an irreducible representation of $Sp(4l+2)$.[12] It was at this point that Racah realized that the notion of seniority v, introduced in atomic spectroscopy as a mere mathematical tool (like G_2 in the fourth part of

[7]This fact was generalized to arbitrary groups by R. T. Sharp and his collaborators in the 70s by means of an analytical approach to the labelling problem. With this method, many of the classical result could be reformulated in a general frame, even connecting it with the (vector) coherent state formalism.

[8]See e.g. the contribution of Y. Ne'eman in [7].

[9]H. Casimir, B.L. van der Waerden, *Math. Ann.* **111**, 1 (1935).

[10]In spite of the historical importance of this paper, it seems to be quite unknown, or at least to have been overseen for a long time.

[11]The different character of LS and jj-coupling actually arose clearly when the seniority notion was introduced. Although the dimension of the groups coincide, their structure clearly reflected the symmetry or antisymmetry of wave functions. See e.g. I. Talmi 1962 *Reviews Mod. Physics* **34**, 704 or chapter 5 in Lipkin's book [5].

[12]See for example the book by B. R. Judd cited later in this work.

the TCS series), actually had an important meaning in nuclear spectroscopy. However, as late as 1958 this fundamental fact had not been widely accepted, as we can read in the introduction of Racah's contribution to the Rehovoth Conference in Nuclear Structure (ref. 55).[13] According to Racah's close collaborators, he did not feel very confident with some of the physical implications of nuclear spectroscopy,[14] which, in addition to the few experimental data available at that time, prevented him of applying his methods beyond some generic ansatz. It seems, however, that a great opportunity was lost, specially in connection with the emerging group theoretical methods in elementary particle physics in the beginning 60's. As example, Racah analyzed in one paper (ref. 53), almost at the same time but independently from Murai,[15] the notion of isotopic-reflections for the physical interpretation of the quantum number U (number of "iso-fermions") introduced earlier by Prentki and D'Espagnat.[16]

Another relevant notion introduced by Racah was the so-called "model-interactions" (nowadays "effective interactions"), basing on an idea published in 1934 by Bacher and Goudsmit[17], where they showed that the terms of an l^n configuration could be expressed as linear combinations of the terms of l^2 configurations in such a way that perturbations of l^n by all configurations differing from it by the states of two electrons and lying far from it were described by suitable modifications of the terms of l^2, being therefore independent on n. Racah developed these ideas in order to obtain complete sets of two-body effective interactions for the d^n configurations (refs. 44, 50, 62, 65, 70). Effective interactions were actively studied in the Hebrew University in order to improve the fit between theory and experiment in atomic spectroscopy [8].

With the emergence of the first big computers an efficient way to handle and contrast experimental data became available, and many procedures that prior to their invention had been avoided for reasons of computational difficulty (like effective diagonalization of matrices in very high orders) were possible. A big concern to Racah was the lack of experimental data to contrast his methods, and the possibility of using computers to compare the theoretical predictions with the experimental results did not escape to his attention. Following a visit to the National Bureau of Standards in 1954, Racah developed a great interest in the application of computers to theoretical spectroscopic analysis. During two years he developed additional programs (see ref. 57) to diagonalize matrices, comparing the resulting diagonalizations with experimental data and to compute line strengths. These computer procedures allowed to chose the problems and spectra according to their interest rather than to their practical possibility on mechanical calculators.[18] With the help of computers, many of the relevant complex (second) spectra with filling d shells

[13]The footnote on page 159 of this note shows to what extent LS and jj-couplings were treated in the same way, without consideration to the important structural differences of the orthogonal and symplectic groups.

[14]See for example Prof. Unna's paper in this volume, where he recollects conversations with Racah at the Varenna school of 1960.

[15]Y. Murai 1956 *Nuclear Phys.* **1** 657. See also comments by A. Salam concerning these works.

[16]B. d'Espagnat, J Prentki 1956 *Nuclear Phys.* **1** 33

[17]Bacher RF and Goudsmit SA 1934 *Phys. Rev.* **46**, 948.

[18]These programmes were specifically designed for the WEIZAC, and very precise descriptions on the

(Iron, Palladium and Platinum groups) were analyzed in detail (see e.g. refs. 50, 58, 62, 63),[19] followed by the extremely complex spectra with filling f shells of rare earths (refs. 38, 49, 52, 61, 68, 69).[20] At the time of his death, the topic of the talk that Racah intended to give at the Zeeman Centennial Conference was to describe a new program to adjust calculated energy levels to observed ones, taking into account the g-factors of the levels. In the years after 1965, a combination of Racah's methods, powerful computer codes and more accurate instruments allowed to expand considerably the knowledge of the lanthanides, and opened new alternatives for finding the relative position of low-lying energy levels of opposite parities, as the infrared emission spectrum. The reference [3] discusses progress made from 1965 onwards, and describes the status and trends of lanthanide atomic spectra in 1978, when approximately 1000 complete or partial spectral analysis were available for 99 elements.[21]

In 1959 Racah and his cousin Ugo Fano wrote the monograph "Irreducible Tensorial Sets" (ref. 56),[22] which treated the algebra of tensor operators in great detail. In a first part, the theory was developed carefully from the first principles, paying special attention to subtle points like the consistent definition of phases of tensor operators. The second part dealt with quantum mechanical applications, although these were actually quite sporadic. Instead of analyzing concrete examples, the formalism was developed with great generality, leading to quite complicated formulae. These mere indications of applications was one of the major complaints about the book, accessible only for experts in the field. A quite puzzling fact about the book was the absence of group-theoretical argumentation.[23] Although this monograph was an important addition to the literature, it turned out to be mainly useful for experts already working with these techniques; for the beginners it constituted a quite hard reference. This however influential work was complemented with the books of A. R. Edmonds ("Angular Momentum in Quantum Mechanics", Princeton Univ. Press, Princeton, 1960) and B. R. Judd ("Operators Techniques in Atomic Spectroscopy", McGraw Hill Co., New York, 1963), written in a more direct style, making the algebraic formalism of Racah available to a more ample audience. Tabulations of angular momentum coupling coefficients [10, 11, 12], essential to the Racah algebra, further simplified computations and helped to derive new applications of the results. Special citation deserves B. G Wybourne book "Spectroscopic properties of Rare Earths", appeared in 1965, which stressed the importance of the Racah method for the analysis of lanthanides and actinides. Being himself

machine's capability and the programs were given in ref. 57. In 1963 the computer was replaced by a PHILCO 2000, and shortly before Racah's death by the IBM 7040 model.

[19] The papers written in collaboration with Y. Shadmi and N. Spector (refs. 62 and 63 respectively) opened alternative ways to understand the intrinsic properties of interaction parameters.

[20] For an interesting account on the rare earths spectroscopy, see e.g. Wybourne B G 2004 *J. Alloys Compounds* **380**, 96.

[21] See also the discussion and tables in the first chapter of Cowan's book [9].

[22] An updated and enlarged version of this book was published 1996 by Fano in collaboration with A.R.P. Rau ("Symmetries in Quantum Physics ", Academic Press, N.Y.). In contrast to the original book, the expanded version was written with the intention of being useful as a textbook.

[23] See e.g. the review of this book by F. J. Dyson (*Physics Today* **13**, 38 (1960)).

an important contributor to the theory, Wybourne's treatment was everywhere closely related to the requirements of experimental spectroscopy. Most of the book is devoted to the spectra of free atoms and ions, with a final part dealing with rare-earth salts, showing how important these spectra were in the context of astrophysics and solid state physics. Approximately at the same time, an important and influential school was created in Vilnius, around specialists like Yutsis, Rudzikas and Vanagas, which would make serious efforts to motivate Racah's theory on physical grounds. An influential book[24] on the theory of angular momentum was written there [13], and consolidated the Vilnius school as one of the most relevant in the Soviet Union concerning Racah's methods. Starting from the classical reduction theory for the Kronecker products of irreducible representations by means of Clebsch-Gordan coefficients, arbitrary coupling schemes were considered, for which generalized Clebsch-Gordan, Wigner and Racah coefficients were defined. It seems that also in this book, the first graphical methods were developed, an approach that would evolve rapidly to simplify the formalism and provide new coupling schemes [14].[25]

Racah continued with calculations of atomic spectra until his death. With the members of his school and collaborators he systematically analyzed all known spectra in order to explain their structure. Specially for the rare earths, where the experimental data were rather scarce at the time, he tried to adapt and develop his methods to find some patterns in the high abundance of spectral lines and density of energy levels, in which no apparent regularity seemed to exist (ref. 61). Two factors were identified to be responsible for the complexity of these spectra, the competition between configurations and that of interactions. In the two posthumous references 68 and 69, the low levels of Erbium I and the 70 observed energy levels belonging to the subconfigurations $4f^6\left(^7F\right)4s4p$ of Samarium I were inspected in detail, correcting and enlarging recent work. Although Racah's collaborators continued with the this systematical analysis of spectra, his untimely death was a serious obstruction to the development of this programme.[26]

At the same time, he continued to develop general methods to nuclear spectroscopy. According to I. Talmi [15], shortly before his death he was working on a complete classification of the states in the seniority scheme of j^n configurations with nucleons. The group theoretical methods provided several degeneracies in the irreducible representations of $Sp\left(2j+1\right)$, with corresponding states having the same seniority quantum number, and a new quantum number to separate these degeneracies was required. How Racah's particular interests in this context would have evolved, finally focusing on problems he had not attacked directly, can only be a matter of speculation.

[24]The impact of this work can actually be recognized from the fact that the first translation was published in Israel (Israel Program for Scientific Translations, Jerusalem, 1962).

[25]For historical developments of the graphical methods, see also El Baz E and Castel B 1972 *Graphical Methods of Spin Algebras in Atomic, Molecular and Particle Physics* (Marcel Dekker, NY); Varshalovich D A, Moskalev A N, Kersonskii' V K 1975 *Kvantovaya Teoriya Uglovogo Momenta* (Nauka, Leningrad).

[26]See e.g. Z. B. Goldschmidt's contribution in [7] and references therein. In this work an unpublished collaboration of Goldschmidt, Racah and Y. Bordarier is listed. Further work by collaborators of Racah can be found in the NIST Atomic Spectra Database at the address http://physics.nist.gov/PhysRefData/ASD/Html/ref.html

Summarizing, the legacy of G. Racah has been very important for atomic, molecular and nuclear spectroscopy, as well as for the definitive standardization of group theoretical methods in physics. His ideas and methods motivated other authors to further generalize and develop these techniques, and to apply them to physical and chemical problems far beyond their original conception. Here we have only made a very superficial description of Racah's work. A very detailed discussion of his papers and ideas can be found in Talmi's paper [16]. It has not been the author's objective to review Racah's influence in establishing theoretical physics in Israel, nor enumerating the high number of students and work that emerged from that school. The reader interested in these aspects, as well as personal accounts, can find excellent reviews written by direct collaborators of Racah in the articles of N. Zeldes [8] and I. Unna [17], as well as in the extensive bibliographies given there.

Publications of G. Racah

For completion in the exposition, we enumerate the scientific publications of G. Racah. The following list has been taken from [15], where the last references (appeared posthumously) have been completed and updated. Minor mistakes in the citation of references 62 and 68 have been corrected.

1. G. Racah and B. Rossi. *A proposito di un'osservazione di Stark sulla realta del moto assoluto*, Nuovo Cimento **6** 317 (1929)

2. G. Racah. *Sopra un esempio di trattazione quantistica di un fenomeno di interferenza*, Rendiconti della R. Academia Nazionale dei Lincei **11** 837 (1930) [Nuovo Cimento **7** 330 (1930)]

3. G. Racah. *Sopra l'elettrodinamica quantistica*, Rendiconti della R. Academia Nazionale dei Lincei **11** 1100 (1930) [Nuovo Cimento **7** 335 (1930)]

4. G. Racah. *Caratteristiche delle equazione di Dirac e principio de indeterminazione*, Rendiconti della R. Academia Nazionale dei Lincei **13** 424 (1931) [Nuovo Cimento **9** 28 (1932)]

5. G. Racah. *Sopra le strutture iperfini*, Nuovo Cimento **8** 178 (1931)

6. G. Racah. *Zur Theorie der Hyperfeinstruktur*, Zeitschrift für Physik **71** 431 (1931)

7. G. Racah. *Il Convegno di Fisica Nucleare*, Ricerca Scientifica **2** 416 (1931)

8. G. Racah. *Isotopic displacement and hyperfine structure*, Nature **129** 723 (1932)[27]

9. G. Racah. *Determinazione del numero dei tensori isotropi indipendenti di rango n*, Rendiconti della R. Academia Nazionale dei Lincei **17** 386 (1933)

[27] This paper was later included in the book "Atomic Spectra" by W. R. Hindmarsh (Pergamon press, N.Y. 1967).

10. G. Racah. *Numero dei tensori isotropi e emisotropi in spazi a piu dimensioni*, Rendiconti della R. Academia Nazionale dei Lincei **18** 135 (1933)

11. G. Racah. *Numero dei tensori isotropi e emisotropi in spazi a piu dimensioni*, *Nota II*, Rendiconti della R. Academia Nazionale dei Lincei **20** 109 (1934)

12. G. Racah. *Sopra l'rradiazione nell'urto di particelle veloci, Nuovo Cimento* **11** 461 (1934)

13. G. Racah. *Sulla nascita degli elettroni positivi*, Nuovo Cimento **11** 477 (1934)

14. G. Racah. *Sul cosiddeto momento elettrico dell'elettrone,* Rendiconti della R. Academia Nazionale dei Lincei **20** 39 (1934)

15. G. Racah. *Sopra l'effeto Zeeman quadratico*, Nuovo Cimento **11** 723 (1934)

16. G. Racah. *Bemerkung zur Arbeit von Herrn Gamow: Empirische Stabilitätsgrenzen von Atomkernen*, Zeitschrift für Physik **93** 704 (1935)

17. G. Racah. *Vedute generali sulla constituzione dell nucleo atomico*, Bolletino di Matematica **31** 1 (1935)

18. G. Racah. *Production of electron pairs*, Nature **136** 393 (1935)

19. G. Racah. *Sulla nascita di coppie per urti di particelle elettrizzate*, Nuovo Cimento **13** 66 (1936)

20. G. Racah. *Sulla nascita di coppie per urti di particelle elettrizzate, Nota II*, Nuovo Cimento **14** 93 (1937)

21. G. Racah. *Sopra alcuni integrali collegati con gli integrali ellittici*, Rendiconti del R. Istituto Lombardo di Scienze y Lettere **70** 340 (1937)

22. G. Racah. *Sulla forma lagrangiana delle forze elettromagnetiche*, Rendiconti della R. Academia Nazionale dei Lincei **25** 233 (1937)

23. G. Racah. *Sopra i tensori isotropi che presentano particolari simmetrie, Nota I,* Rendiconti della R. Academia Nazionale dei Lincei **25** 475 (1937)

24. G. Racah. *Sopra i tensori isotropi che presentano particolari simmetrie, Nota II,* Rendiconti della R. Academia Nazionale dei Lincei **25** 615 (1937)

25. G. Racah. *Sulla simmetria tra particelle e antiparticelle*, Nuovo Cimento **14** 322 (1937)

26. G. Racah. *Ricerche moderne sulle teorie nucleari*, Societa Italiana per il progresso delle scienze **16** 82 (1938)

27. G. Racah. *Theory of complex spectra I*, Physical Reviews **61** 186 (1942)

28. G. Racah. *On a new type of vector coupling in complex spectra*, Physical Review **61** 537 (1942)

29. G. Racah. *On the deep configuration of Cobalt*, Physical Review **61** 537 (1942)

30. G. Racah. *Theory of complex spectra II*, Physical Review **62** 438 (1943)

31. G. Racah. *On the configuration ll's*, Physical Review **62** 523 (1943)

32. G. Racah. *Theory of complex spectra III*, Physical Review **63** 367 (1943)

33. G. Racah. *On the structure of* $\mathrm{Mo(CN)}_8^{-4}$, J. Chemical Physics **11** 214 (1943)

34. G. Racah. *On the self-energy of the electron*, Physical Review **70** 406 (1946)

35. G. Racah. *Theory of complex spectra IV*, Physical Review **76** 1352 (1949)

36. G. Racah. *On the decomposition of tensors by contraction*, Reviews Modern Physics **21** 494 (1949)

37. G. Racah. *On the calculation of nuclear levels*, Helvetica Physica Acta **23** 229 (1950)

38. G. Racah. *On the spectrum of* Th III, Physica **16** 651 (1950)

39. G. Racah. *Nuclear coupling and shell model*, Physical Review **78** 622 (1950)

40. G. Racah and N. Zeldes. *On the spins of* Li^6 and B^{10}, Physical Review **79** 1012 (1950)

41. G. Racah. *Sulla caratterizzazione delle rappresentazione irriducibili dei gruppi semisimplici di Lie*, Rendiconti della R. Academia Nazionale dei Lincei **8** 108 (1950)

42. G. Racah. *On the characterization of the rows and columns of the representations of the semisimple Lie groups,* Proceedings of the International Congress of Mathematicians 1950, Vol 1, p. 467

43. G. Racah. *Directional correlations of successive nuclear radiations,* Physical Review **84** 910 (1951)

44. G. Racah. $L\,(L+1)$ *correction in the spectra of the Iron group*, Physical Review **85** 381 (1952)

45. G. Racah. *Term values of the* f^3 electron configuration, Current Science 21 67 (1952)

46. G. Racah and I. Talmi. *The pairing property of nuclear interactions,* Physica **18** 1097 (1952)

47. G. Racah. *Nuclear levels and Casimir operators,* Farkas Memorial Volume, Research Council of Israel, Jerusalem 1952, p.294

48. G. Racah. *The competition between the low configurations in the spectra of the Iron group,* Bulletin of the Research Council of Israel III 290 (1954)

49. G. Racah. *Some new levels in the spectrum of* Yb II, Bulletin of the Research Council of Israel III 437 (1954)

50. G. Racah. *The present state and problems of the theory of atomic spectra,* Proceedings of the Rydberg Centennial Conference on Atomic Spectroscopy, Lund 1955, p.31

51. G. Racah. *A search for new quantum numbers in nuclear configurations*, Proceedings of the 1954 Glasgow Conference on Nuclear and Meson Physics, Pergamon Press, London and New York 1955, p.126

52. G. Racah. *Intercombination in the spectrum of* CeII, Bulletin of the Research Council of Israel **5A** 78 (1955/56)

53. G. Racah. *Remarques sur une formulation mathématique du modèle de Gell-Mann,* Nuclear Physics **1** 302 (1956)

54. G. Racah. *Coupling schemes in atomic spectra,* Bulletin of the Research Council of Israel **7F** 188 (1957/58)

55. G. Racah. *The seniority quantum number and its applications to nuclear spectroscopy,* Proceedings of the Rehovoth Conference on Nuclear Structure, North Holland Publishing Co, Amsterdam 1958, p.155

56. G. Racah and U. Fano. *Irreducible tensorial sets,* Academic Press, New York 1959

57. G. Racah. *The use of the Weizac in theoretical spectroscopy,* Bulletin of the Research Council of Israel **8F** 1 (1959)

58. G. Racah and Y. Shadmi. *The configurations $(3d+4s)^n$ in the second spectra of the Iron group*, Bulletin of the Research Council of Israel **8F** 15 (1959)

59. G. Racah. *Theory of Lie Groups*, Nuovo Cimento Supplemento **14** 67 (1959)

60. G. Racah. *Four dimensional orthogonal groups,* Nuovo Cimento Supplemento **14** 75 (1959)

61. G. Racah. *Vector coupling and notations in the rare-earths spectra*, Journal of the Optical Society of America **50** 408 (1960)

62. G. Racah and Y. Shadmi. *Q correction in the spectra of the Iron group*, Physical Review **119** 156 (1960)

63. G. Racah and N. Spector. *The configurations* $3d^n4p$ *in the second spectra of the Iron group*, Bulletin of the Research Council of Israel **9F** 75 (1960)

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