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Frozen orbits around a prolate body

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Abstract

For a satellite orbiting a prolate body we determine the class of the so called frozen orbits, a kind of trajectories for which either the eccentricity and the argument of the perigee are close to be stationary. This is done after an averaging procedure that produces a one degree of freedom Hamiltonian system retaining the main qualitative features of the original one. This reduced system can be studied in a systematic way in order to calculate the equilibrium solutions, directly related to the frozen orbits.

1 Introduction

In the problem of mission design for artificial satellites there is a set of orbits of special interest for which the eccentricity and the argument of the perigee remain almost stationary [10]. These orbits are called *frozen orbits* and, according to [5], they are in correspondence with equilibrium solutions of an averaged system resulting from the original one.

For the main problem of the artificial satellite, when only the effect of the oblateness coefficient is taken into account, frozen orbits has been studied from the very beginning of the spacial era, pointing out a singularity at the so called critical inclination [3]. This singularity can be resolved when the problem is brought to normal form and the frozen orbits are viewed as relative equilibria [3, 5]. Under this point of view, the critical inclination is just a value for which a parametric bifurcation of the family of frozen orbits occurs. Numerical continuation of periodic orbits [9] and Poincaré surfaces of section [2] have confirmed the appearance of this bifurcation. In this way, the class of frozen orbits can be grouped into families in terms of the inclination, acting the critical inclination as a value that separates different families.

The main problem of the artificial satellite is linked to the Earth and the body around the satellite orbits is supposed oblate. In this way, no much is known for prolate bodies. However, recent missions to objects of the solar system with different shapes could find of interest an extension of the main problem for prolate bodies. This is, for instance, the point of view adopted in [8], where a special kind of halo orbits are derived. Nevertheless, the family of frozen orbits remains unexplored, at least from an analytical point of view, because they were presented in [4] as an application of the *paint by number* technique. The aim of this contribution is to determine the families of frozen orbits for the main problem of an artificial satellite around a prolate body analytically by means of the identification of critical inclinations, when frozen orbits suffer a bifurcation.

2 Problem formulation

The main problem of the artificial satellite is described by the Hamiltonian

$$\mathcal{H} = -\frac{1}{2a} - J_2 \frac{\mu}{2r} \left(\frac{\alpha}{r}\right)^2 (3\sin^2 I \sin^2(\omega + \nu) - 1), \tag{1}$$

where a is the semimajor axis of the orbit, r the distance of the satellite to the origin, that is supposed at the center of the attracting body, α is the equatorial radius of the attracting body, I the inclination of the orbit, ω the argument of perigee, ν the true anomaly, μ the mass parameter of the system and J_2 the oblateness coefficient (see for instance [1]).

Frozen orbits are referred as equilibrium solutions of an averaged system. This averaged system is obtained by performing a Delaunay normalisation [6] up to second order, due to the fact that at first order it exhibits a degeneration. Indeed, there is a set of non-isolated equilibria which corresponds to the critical inclination. Once the normalisation is carried out, we arrive to the new Hamiltonian function

$$\begin{aligned} \mathcal{H} = &-\tfrac{\mu}{2L^2} - J_2\,\alpha^2\mu^4\tfrac{G^2-3\,H^2}{4\,G^5 L^3} + 3\,J_2^2\,\alpha^4\,\mu^6\left(\tfrac{-5\,G^4+18\,G^2\,H^2-5\,H^4}{128\,G^9\,L^5} - \tfrac{\left(G^2-3\,H^2\right)^2}{32\,G^{10}\,L^4}\right. \\ &\left. + \tfrac{5\left(G^4-2\,G^2\,H^2-7\,H^4\right)}{128\,G^{11}\,L^3} + \tfrac{\left(G^2-15\,H^2\right)\left(G^2-H^2\right)\left(G^2-L^2\right)\cos(2\,g)}{64\,G^{11}\,L^5}\right), \end{aligned} \tag{2}$$

where (L, G, H, ℓ, g, h) are the Delaunay elements describing the orbit of the satellite. L is related with the semimajor axis, G is the total angular momentum, H the third component of the angular momentum, ℓ the mean anomaly, g the argument of perigee and h the argument of the node.

It is worth noting that ℓ and h are cyclic coordinates and then H and L are conserved quantities in this averaged model. Thus, the dynamics is reduced to that of the pair of variables (g, G), described by the canonical differential equations

$$\dot{g} = \frac{\partial \mathcal{H}}{\partial G}, \quad \dot{G} = -\frac{\partial \mathcal{H}}{\partial g}.$$

The equilibrium solutions of this system give rise to the families of frozen orbits and. They are obtained by setting to zero the right hand sides of the two equations. As it happens for a wide class of perturbed Keplerian systems [7], equilibria appear as the solutions of a nonlinear system of the form

$$\begin{aligned} &P_1(G, J_2)\sin 2g = 0, \\ &P_2(G, J_2) + P_3(G, J_2)\cos 2g = 0, \end{aligned} \tag{3}$$

such that $g \in [0, 2\pi)$, $G \in [|H|, L]$ and P_1, P_2 and P_3 are real polynomials in G whose coefficients are polynomial functions of J_2, as well as of α and μ. Nevertheless, a proper choice of units yields $\alpha = \mu = 1$.

For the case considered here, polynomials P_1, P_2 and P_3 are given by

$$\begin{aligned} P_1 \equiv\ & (G-H)\ (G+H)\ (G^2 - 15\,H^2)\ (G-L)\ (G+L)\,, \\ P_2 \equiv\ & -32\,L^2\,G^8 + (160\,H^2\,L^2 - 25\,J_2)\,G^6 - 24\,J_2\,L\,G^5 \\ & + (126\,H^2\,J_2 + 35\,J_2\,L^2)\,G^4 + 192\,H^2\,J_2\,L\ \ G^3 \\ & - (45\,H^4\,J_2 + 90\,H^2\,J_2\,L^2)\,G^2 - 360\,H^4\,J_2\,L\,G \\ & - 385\,H^4\,J_2\,L^2, \\ P_3 \equiv\ & 10\,J_2\,G^6 - (224\,H^2\,J_2 + 14\,J_2\,L^2)\,G^4 \\ & + (270\,H^4\,J_2 + 288\,H^2\,J_2\,L^2)\,G^2 - 330\,H^4\,J_2\,L^2. \end{aligned} \tag{4}$$

3 Bifurcations and families of frozen orbits

In order to establish the solutions of system (3) we observe that they can be divided into different classes, attending to the structure of the first equation in (3). On the one hand we have the solutions for which $P_1(G, J_2) = 0$. They are

$$G = |H|, \quad G = L, \quad G = \sqrt{15}|H|.$$

The two first correspond with equatorial and circular orbits, respectively, and they constitute equilibrium solutions that must be studied in a different way, as Delaunay variables are singular for these kind of orbits. Thus, we are left with the third case $G = \sqrt{15}|H|$. A direct substitution of G into the second equation of (3) yields

$$54000H^4L^2 + J_2\left(2835H^2 - 307L^2 + 144\sqrt{15}L|H| + 42(15H^2 - L^2)\cos 2g\right) = 0.$$

Thus, if g_0 is a solution of the last equation we have the equilibrium point $(g_0, \sqrt{15}|H|)$. This point exists whenever $-1 \leq \cos 2g_0 \leq 1$. As a consequence, we obtain two bifurcation surfaces when $\cos 2g$ reaches the extremum values, namely:

$$\begin{aligned} B_1 &\equiv 54000H^4L^2 + J_2(3465H^2 + 144\sqrt{15}L|H| - 349L^2) = 0, \\ B_2 &\equiv 54000H^4L^2 + J_2(3465H^2 + 144\sqrt{15}L|H| - 265L^2) = 0. \end{aligned} \tag{5}$$

The other solutions of system (3) come from $\sin 2g = 0$. In this case, $\cos 2g = \pm 1$ and G must be a root of one of the two following polynomials

$$
\begin{aligned}
\mathcal{P}_+ &= 32L^2G^8 + (15J_2 - 160H^2L^2)G^6 + 24J_2LG^5 + (98H^2 - 21L^2)J_2G^4 \\
&\quad - 192J_2H^2LG^3 - (225H^4 + 198H^2L^2)J_2G^2 + 360J_2H^4LG \\
&\quad + 715J_2H^4L^2, \\
\mathcal{P}_- &= 32L^2G^8 + (35J_2 - 160H^2L^2)G^6 + 24J_2LG^5 - (350H^2 + 49L^2)J_2G^4 \\
&\quad - 192J_2H^2LG^3 + (315H^4 + 378H^2L^2)J_2G^2 + 360J_2H^4LG \\
&\quad + 55J_2H^4L^2,
\end{aligned}
\tag{6}
$$

where $\mathcal{P}_+$ and $\mathcal{P}_-$ are obtained from $\cos 2g = 1$ and $\cos 2g = -1$ respectively.

An explicit calculation of the roots of $\mathcal{P}_+$ and $\mathcal{P}_-$ is not possible, and even the task of counting the number of roots in the interval $[|H|, L]$ is very difficult. Nevertheless, we satisfy ourselves by determining a change in the number of roots in the interval $[|H|, L]$. In this way we consider the following two situations:

1. one of the roots reaches the extrema of the interval,
2. two or more roots join to produce a multiple root.

We emphasize that the first item is straightforward to compute, but not the second one. Indeed, there is a multiple root if and only if the polynomial's discriminant vanishes. However, a vanishing discriminant is not enough to ensures that the multiple root lies inside the interval $[|H|, L]$. Thus, the second item must be followed by a numerical process in order to discard those situations accounting for multiple roots outside the interval $[|H|, L]$.

By considering all the above, we arrive to six new bifurcation surfaces. Four of then obtained by application of the first item

$$
\begin{aligned}
&L_1 : \mathcal{P}_+(|H|) = 0 \longrightarrow 8\,H^4\,L^2 + J_2\,(7\,H^2 - 12\,L\,|H| - 31\,L^2) = 0, \\
&L_2 : \mathcal{P}_+(L) = 0 \longrightarrow 16\,L^8 - 80\,H^2\,L^6 + J_2\,(425\,H^4 - 146\,H^2\,L^2 + 9\,L^4) = 0, \\
&L_3 : \mathcal{P}_-(H) = 0 \longrightarrow 2\,H^4\,L - 3\,J_2\,(|H| + 2\,L) = 0, \\
&L_4 : \mathcal{P}_-(L) = 0 \longrightarrow 16\,L^8 - 80\,H^2\,L^6 + J_2\,(365\,H^4 - 82\,H^2\,L^2 + 5\,L^4) = 0,
\end{aligned}
\tag{7}
$$

and two more derived from the second one

$$
\begin{aligned}
&L_5 : \operatorname{disc}(\mathcal{P}_+) = 0, \\
&L_6 : \operatorname{disc}(\mathcal{P}_-) = 0.
\end{aligned}
\tag{8}
$$

The expression of the equations defining L_5 and L_6 are too involved and they are given in Appendix 1.

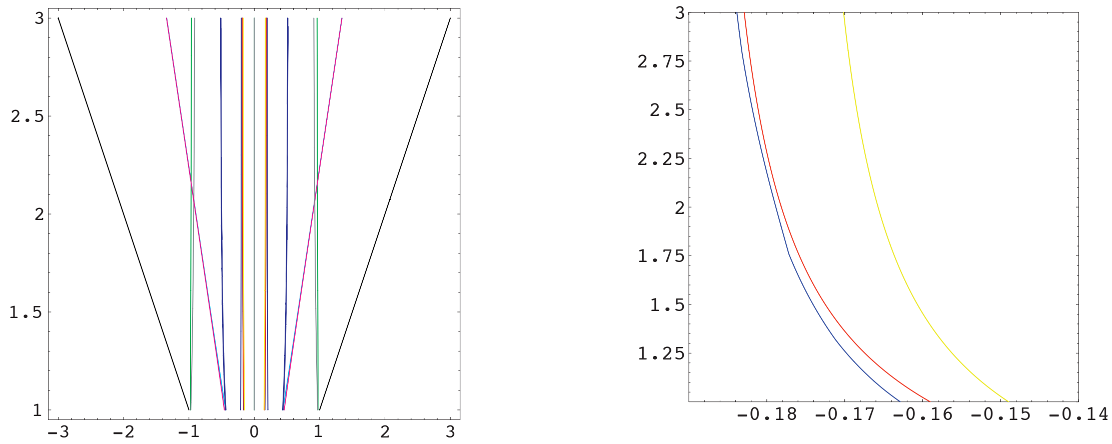

Figure 1.— The plane (H, L) divided in different regions where the number and stability of frozen orbits change, for $J_2 = -0.2$. The right figure gives an enlarged view of the lines when H is close to 0.

Now, we can depict the bifurcation surfaces in the parameter space (H, L, J_2) once we have discarded those branches of the surfaces L_5 and L_6 related with multiple roots outside the interval $[|H|, L]$. However, for the sake of simplicity, we will plot on the plane (H, L) a contour level of the surfaces of bifurcation for a fixed value of J_2. In this way, we choose $J_2 = -0.2$, a large negative value of the oblateness coefficient, in order highlight the many different regions where families of frozen orbits live, as Figure 1 shows.

Once the regions are established, we can determine the frozen orbits by solving the corresponding equations in (3). Nevertheless, they can also be visualized by depicting the phase flow on the reduced space, after normalisation. This is what we do by the technique of paint by number on the spheres defined in [5], although these spheres do not constitute the fully reduced space (see for instance [7]). As an example, we show in Figura 2 the phase flow in the regions enlarged in Figure 1, where frozen orbits correspond to the equilibrium points. It can be seen how the number of frozen orbits changes as the bifurcation lines are crossed. Indeed, if one of the lines is crossed the number of frozen orbits changes in two, and the region with the largest number of frozen orbits (14 in total) is that in between the red and yellow lines.

4 Conclusions

We have presented an analytical study of the families of frozen orbits in the main problem of artificial satellite for a prolate body. Despite the general treatment, an example is considered to see how the families of frozen orbits can be localized in the phase space. A more detailed study, by considering bifurcations surfaces in the general parameter space (H, L, J_2) is left for further development, as well as the evolution of the families in terms of physical parameters, inclination and eccentricity.

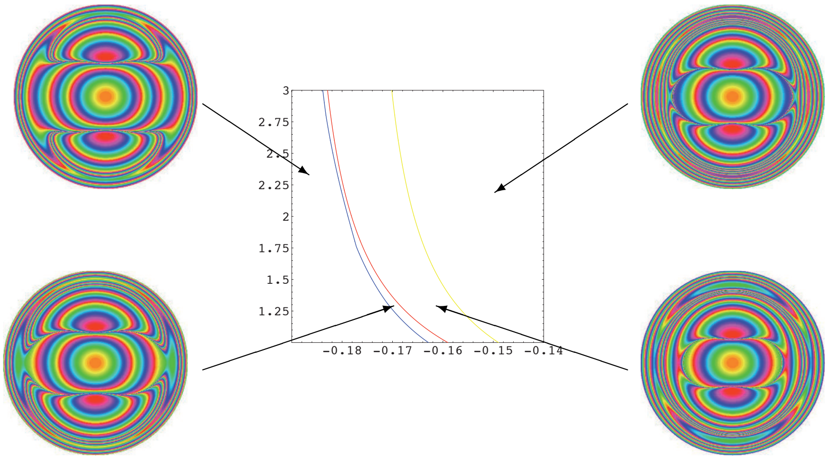

Figure 2.— Phase flow in four different regions. Frozen orbits correspond with equilibrium points.

Acknowledgments

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Appendix 1

$$L_5 :\ \mathrm{disc}(\mathcal{P}_+) = 0$$

$$\begin{aligned}
&146082502080000000000000000H^{28}J_2^2L^{12} - 151781748584760000000000000H^{26}J_2^3L^{10} - \\
&65142856800000000000000000H^{26}J_2^2L^{14} + 5530739132436480000000000000H^{26}J_2L^{18} + \\
&754300195115261737500000000H^{24}J_2^4L^8 + 7905181680943070400000000000H^{24}J_2^3L^{12} - \\
&6449256481753762560000000000H^{24}J_2^2L^{16} - 10257815540816363520000000000H^{24}J_2L^{20} + \\
&2856851368800878592000000000H^{24}L^{24} - 191164215583205221500000000H^{22}J_2^5L^6 - \\
&4464257829208807983120000000H^{22}J_2^4L^{10} + 35447650819765221392640000000H^{22}J_2^3L^{14} + \\
&14620577615920775024640000000H^{22}J_2^2L^{18} - 33575231407235737190400000000H^{22}J_2L^{22} + \\
&2432476240165935254250000H^{20}J_2^6L^4 + 1274579767733827063989600000H^{20}J_2^5L^8 - \\
&10405782778068032852706400000H^{20}J_2^4L^{12} - 96878290522803033159936000000H^{20}J_2^3L^{16} + \\
&14258568098759548962816000000H^{20}J_2^2L^{20} + 11802095650564586864640000000H^{20}J_2L^{24} - \\
&5077152410195100937500000H^{18}J_2^7L^2 - 178141991565371073324192000H^{18}J_2^6L^6 + \\
&16456310471624429441474240000H^{18}J_2^5L^{10} + 33094145013559427796913600000H^{18}J_2^4L^{14} + \\
&14728556083442997805568000000H^{18}J_2^3L^{18} - 138455914098893864140800000000H^{18}J_2^2L^{22} + \\
&29049099355419433593 75H^{16}J_2^8 + 40389870101465779393500000H^{16}J_2^7L^4 - \\
&1001899604639726674363804000H^{16}J_2^6L^8 - 592799262885792837319039360000H^{16}J_2^5L^{12} - \\
&18355123258188111927054096000 0H^{16}J_2^4L^{16} + 491205909716027481181184000000H^{16}J_2^3L^{20} + \\
&247154224515915654299648000000H^{16}J_2^2L^{24} - 22516011274717746562500H^{14}J_2^8L^2 + \\
&153698475676255849106100000H^{14}J_2^7L^6 + 36649108691334193942915161600H^{14}J_2^6L^{10} + \\
&53634282783571674772042348800H^{14}J_2^5L^{14} - 198126344363197270628966400000H^{14}J_2^4L^{18} - \\
&212498960185413325797457920000H^{14}J_2^3L^{22} - 86027207524605635062500H^{12}J_2^8L^4 - \\
&691004743635995361455220000H^{12}J_2^7L^8 - 2762359102762387862378994720H^{12}J_2^6L^{12} - \\
&248007105786811569415103347200H^{12}J_2^5L^{16} + 460667581753053391653371904000H^{12}J_2^4L^{20} + \\
&257998257349104136444968960000H^{12}J_2^3L^{24} + 34378110182393743016250 0H^{10}J_2^8L^6 + \\
&680915168260673451959052000H^{10}J_2^7L^{10} + 86392055058670807295887411 2H^{10}J_2^6L^{14} + \\
&73758450488501214428064030720H^{10}J_2^5L^{18} - 140266654976573419257357926400H^{10}J_2^4L^{22} - \\
&26503386208585901391375 0H^8J_2^8L^8 - 310585585367865288650680 80H^8J_2^7L^{12} - \\
&120425013704246647334202144H^8J_2^6L^{16} - 1369933748774047384658583552H^8J_2^5L^{20} + \\
&13577460137839808959422136 32H^8J_2^4L^{24} + 8833699994143751804250 0H^6J_2^8L^{10} + \\
&76562685935588838133000 80H^6J_2^7L^{14} + 6896525752240801270069248H^6J_2^6L^{18} + \\
&11982906748710559940463820 8H^6J_2^5L^{22} - 1416667576377440356650 0H^4J_2^8L^{12} - \\
&10046809709067354647038 56H^4J_2^7L^{16} - 6381503769278295919595 52H^4J_2^6L^{20} - \\
&16294026730226562273116 16H^4J_2^5L^{24} + 8884752806396471895 00H^2J_2^8L^{14} + \\
&54234372035912043811968H^2J_2^7L^{18} + 10021308534988322335948 8H^2J_2^6L^{22} + \\
&4935191167699167375J_2^8L^{16} + 30447329670760357555 2J_2^7L^{20} + \\
&48338741659979125555 2J_2^6L^{24} = 0.
\end{aligned}$$

$$L_6 : \ \mathrm{disc}(\mathcal{P}_-) = 0$$

$$
\begin{aligned}
&-769433134694400000000000\, H^{24} L^{24} - 3816840127463424000000000000\, H^{26} J_2 L^{18} + \\
&887487457362739200000000000\, H^{24} J_2 L^{20} - 4886175680467107840000000000\, H^{22} J_2 L^{22} + \\
&2706094167966351360000000000\, H^{20} J_2 L^{24} + 6128017770792960000000000\, H^{28} {J_2}^2 L^{12} - \\
&182075567772672000000000000\, H^{26} {J_2}^2 L^{14} - 573979526215764480000000000\, H^{24} {J_2}^2 L^{16} + \\
&35330622559577708544000000000\, H^{22} {J_2}^2 L^{18} - 31913209726141444300800000000\, H^{20} {J_2}^2 L^{20} - \\
&12125089252785777868800000000\, H^{18} {J_2}^2 L^{22} - 2479494456282210566144000000\, H^{16} {J_2}^2 L^{24} + \\
&17018441574429254400000000000\, H^{26} {J_2}^3 L^{10} - 16096240298937617280000000000\, H^{24} {J_2}^3 L^{12} + \\
&274038014466435260160000000000\, H^{22} {J_2}^3 L^{14} + 433393829094073715942400000000\, H^{20} {J_2}^3 L^{16} - \\
&657596619027193979130880000000\, H^{18} {J_2}^3 L^{18} - 1454059311198688392949760000000\, H^{16} {J_2}^3 L^{20} + \\
&724331089819077530877952000000\, H^{14} {J_2}^3 L^{22} + 176203135034840680339865600000\, H^{12} {J_2}^3 L^{24} + \\
&18071387880653235429000000000\, H^{24} {J_2}^4 L^{8} - 28344237384836614802400000000\, H^{22} {J_2}^4 L^{10} + \\
&106217747013076277062960000000\, H^{20} {J_2}^4 L^{12} + 1786911713278039812546880000000\, H^{18} {J_2}^4 L^{14} - \\
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&33767390266573536332768870400\, H^{8} {J_2}^4 L^{24} + 891456806222842860000000000\, H^{22} {J_2}^5 L^{6} - \\
&19214082548856331995600000000\, H^{20} {J_2}^5 L^{8} + 77532144263789517818880000000\, H^{18} {J_2}^5 L^{10} - \\
&147380295287094731220800000000\, H^{16} {J_2}^5 L^{12} - 162711765209304920829171200000\, H^{14} {J_2}^5 L^{14} + \\
&505969041951600945496774400000\, H^{12} {J_2}^5 L^{16} + 92527753655803895623219200000\, H^{10} {J_2}^5 L^{18} - \\
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&212208772210727783203125\, H^{16} {J_2}^8 - 659890079247464296875000\, H^{14} {J_2}^8 L^{2} + \\
&2663403731428196804687500\, H^{12} {J_2}^8 L^{4} - 3817681335804557626125000\, H^{10} {J_2}^8 L^{6} + \\
&1093109177792065584018750\, H^{8} {J_2}^8 L^{8} - 148700459548578550445000\, H^{6} {J_2}^8 L^{10} - \\
&24596916523981525888500\, H^{4} {J_2}^8 L^{12} + 946071284037834731700\, H^{2} {J_2}^8 L^{14} - \\
&594632672565549492875\, {J_2}^8 L^{16} = 0.
\end{aligned}
$$