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Racional identities in the Catalan triangle

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Dedicated to Dr. Manuel Calvo in his 65th birthday

Abstract

In this paper we consider different racional identities in which appear the numbers $(B_{n,p})_{n,1\leq p\leq n}$ given by

$$B_{n,p} := \frac{p}{n}\binom{2n}{n-p}, \; n,p\in\mathbb{N}, \; p\leq n.$$

The set of numbers $(B_{n,p})_{n,1\leq p\leq n}$ is known as the Catalan triangle due to the Catalan numbers $(C_n)_{n\in\mathbb{N}}$,

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \qquad n\in\mathbb{N},$$

appear in the first column. These identities have been recently proved and some of them are connected with the dynamic behavior of certain iterative methods applied to quadratic polynomials. In the last section we conjeture some new identities which involve this family of numbers $(B_{n,p})_{n,1\leq p\leq n}$.

1 Introduction

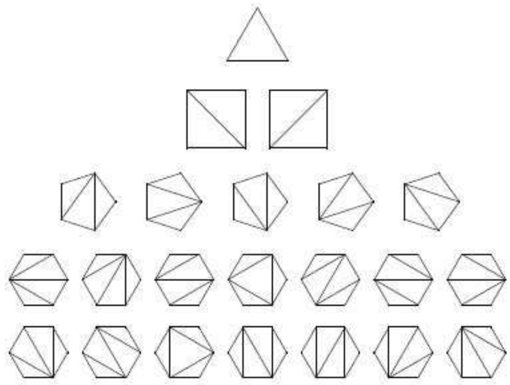
The Catalan number C_n is defined by the expression

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \qquad n\in\mathbb{N}.$$

The first ten values of C_n are $1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796$. Note that Catalan numbers have more than 165 different combinatorial interpretations, see for example [15, p. 219] and

http://www-math.mit.edu/∼rstan/ec/catadd.pdf

In particular, the number C_n is the solution to the Euler problem: how many different ways you can divide a convex polygon of $n+2$ sides in triangles using diagonals ([3]),



They also gives the number of binary bracketings of n letters (Catalan's problem) or the solution to the ballot problem [6].

In 1976, L. W. Shapiro introduced in [11], the following triangle of numbers

$n \setminus p$	1	2	3	4	5	6	...
1	1						
2	2	1					
3	5	4	1				
4	14	14	6	1			
5	42	48	27	8	1		
6	132	165	110	44	10	1	
...	...	...	...	...	...	...	...

(1)

which entries are given by

$$B_{n,p} := \frac{p}{n}\binom{2n}{n-p}, \quad n, p \in \mathbb{N}, \;\; p \leq n.$$

These numbers $(B_{n,p})_{1 \leq n, n \in \mathbb{N}}$ also satisfy a recurrence relation,

$$B_{n,p} = B_{n-1,p-1} + 2B_{n-1,p} + B_{n-1,p+1}, \qquad p \geq 2.$$

Note that $B_{n,1} = C_n$ for $n \geq 1$.

Although the numbers $B_{n,k}$ are not as famous as Catalan numbers, they have also several applications (see [2, 11, 13] for more details). As a sample, we cite some of them:

- $B_{n,p}$ is the number of leaves at level $p+1$ in all ordered trees with $n+1$ edges
- $B_{n,p}$ is the number of walks of n steps, each in direction N, S, W or E, starting at the origin, remaining in the upper half-plane and ending at height p.
- $B_{n,p}$ denote the number of pairs of non-intersecting paths of length n and distance p (see the definitions in [11, p.84]).

In this short note, we present some of the main results given in [5, 8, 9] which involve $(B_{n,p})_{1\le p\le n, n\in\mathbb{N}}$. For example, we give the explicit expressions of the moments $(\Omega_m)_{m\ge 0}$,

$$\Omega_m(n) := \sum_{p=1}^{n} p^m B_{n,p}^2, \qquad n \in \mathbb{N}$$

for $1 \le m \le 7$ and general expressions for arbitrary m. Other formulae which appear in the dynamical study of certain iterative problem are also given. This collection of results have been considered and studied by other mathematicians, [1, 4, 12]. In the last section, we present two conjectures about new identities in which appear the numbers $(B_{n,p})_{1\le p\le n, n\in\mathbb{N}}$; the second one is connected with the values of some determinants associated to the triangle (1).

2 Main results

Different techniques are used in the proof of the following results: Chu-Vandermonde convolution formula; W-Z theory and Newton interpolation formula. Details of the power W-Z theory may be found in the monographic [10] and in [16].

Theorem 2.1 *[8] Let $n \in \mathbb{N}$. Then*

(i) $$\Omega_0(n) := \sum_{p=1}^{n} (B_{n,p})^2 = C_{2n-1}.$$

(ii) $$\Omega_2(n) := \sum_{p=1}^{n} p^2 (B_{n,p})^2 = \frac{(3n-2)n}{4n-3} C_{2n-1}.$$

(iii) $$\Omega_4(n) := \sum_{p=1}^{n} p^4 (B_{n,p})^2 = \frac{(15n^3-30n^2+16n-2)n}{(4n-3)(4n-5)} C_{2n-1}.$$

(iv) $$\Omega_6(n) := \sum_{p=1}^{n} p^6 (B_{n,p})^2 = \frac{(105n^5-420n^4+588n^3-356n^2+96n-10)n}{(4n-3)(4n-5)(4n-7)} C_{2n-1}.$$

Theorem 2.2 *[8] Let $n \in \mathbb{N}$. Then*

(i) $\Omega_1(n) := \sum_{p=1}^{n} p\,(B_{n,p})^2 = (n+1)C_n(2n-3)C_{n-2}.$

(ii) $\Omega_3(n) := \sum_{p=1}^{n} p^3(B_{n,p})^2 = (n+1)C_n n(2n-3)C_{n-2}.$

(iii) $\Omega_5(n) := \sum_{p=1}^{n} p^5(B_{n,p})^2 = (n+1)C_n n(3n^2-5n+1)C_{n-2}.$

(iv) $\Omega_7(n) := \sum_{p=1}^{n} p^7(B_{n,p})^2 = (n+1)C_n n(6n(n-1)^2-1)C_{n-2}.$

Remarks. Note that the polynomials which appear in the Theorem 2.1 and 2.2 do not belong to any known classical family. In the following theorem we give the moments of arbitrary order although a explicit expression is unknown.

Theorem 2.3 *[9] Let $n \in \mathbb{N}$. Then there exist P_{3m+1}, Q_{2m+2}, R_{3m-1} polynomials of integer coefficients and degree at least $3m+1$, $2m+2$ and $3m-1$ respectively such that*

$$\begin{aligned}
\Omega_{2m}(n) &= \frac{P_{3m+1}(n)}{\prod_{l=1}^{m}(4n-(2l+1))}C_{2n-1}, \qquad m \geq 0,\\
\Omega_{2m+1}(n) &= Q_{2m+2}(n+1)C_nC_{n-2}, \qquad m \leq 3,\\
\Omega_{2m+1}(n) &= \frac{R_{3m-1}(n)}{\prod_{l=1}^{m-3}(2n-(2l+3))}(n+1)C_nC_{n-2}, \qquad m \geq 4.
\end{aligned}$$

Theorem 2.4 *[5, 8] Let $n \in \mathbb{N}$, and $1 \leq i \leq n$. Then*

(i) $\sum_{p=1}^{i} B_{n,p}B_{n,n+p-i}(n+2p-i) = (n+1)C_n\binom{2(n-1)}{i-1}.$

(ii) $\sum_{p=1}^{i} B_{n,p}B_{n,n+p-i}(n+2p-i)^3 = (n+1)C_n\binom{2(n-1)}{i-1}\left(n^2+4n-2ni+i^2\right).$

3 An application to Newton-like iterative methods

The application of some iterative methods for solving nonlinear equations to a polynomial equation could give raise to rational iteration functions which dynamics are not well-known.

We present in the complex plane a study of the dynamical behavior of the following Newton-like methods

$$\begin{cases} z_{m+1} = R_n(z_m) = z_m - H_n(L_f(z_m))\dfrac{f(z_m)}{f'(z_m)}, \quad m \geq 0, \\ H_n(z) = \displaystyle\sum_{j=0}^{n} \frac{1}{2^j} C_j z^j, \quad n \geq 0, \quad L_f(z) = \dfrac{f(z)f''(z)}{f'(z)^2}, \end{cases} \tag{2}$$

which are written in terms of the Catalan numbers.

These methods give rise to rational functions defined in the extended complex plane, $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. In particular, we prove that these rational root-finding algorithms are generally convergent for quadratic polynomials.

The idea of general convergence of a method for polynomials of a given degree was introduced by Smale [14] and McMullen [7] and it means that the method converges to a root for almost every starting point and for almost every polynomial of a given degree.

The conjugated rational map of R_n, $S_n := MR_nM^{-1}$, via the Möbius map $M(z) = (z-a)/(z-b)$, is given by

$$S_n(z) = z^{n+2}\frac{P_n(z)}{\hat{P}_n(z)}, \tag{3}$$

where $P_n(z) = \displaystyle\sum_{p=0}^{n} B_{n+1,p+1} z^p$ and $\hat{P}_n(z) = \displaystyle\sum_{p=0}^{n} B_{n+1,n+1-p} z^p$.

A rational map R, divides $\mathbb{C}_\infty$ in two subsets, that are known as *Fatou set* and *Julia set*. The Fatou set, denoted $\mathcal{F}(R)$ is defined as the set of points $z_0 \in \mathbb{C}_\infty$ such that the family of iterates R^n is a normal family in some neighborhood U_{z_0} of z_0. That is, every infinite sequence of R^n contains a subsequence R^{n_k} that converges locally uniformly on U_{z_0} to some continuous function $f \in \mathcal{C}(\mathbb{C}_\infty)$. Recall that $R^{n_k} \to f$ locally uniformly on U_{z_0} if for all $z \in U_{z_0}$, $R^{n_k} \to f$ uniformly on some neighborhood of z. The Julia set, $\mathcal{J}(R)$, is the complement of the Fatou set, $\mathcal{J}(R) = \mathbb{C}_\infty - \mathcal{F}(R)$.

Roughly speaking, the Fatou $\mathcal{F}(R)$ set includes the points whose orbits are predictable after iteration and the Julia set includes the points whose dynamical behaviour is complicated with independency of the number of iterations.

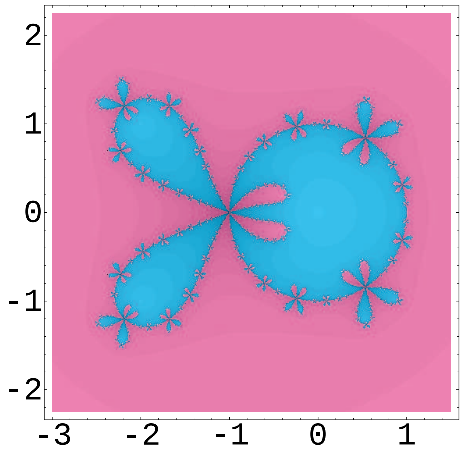
Applying item (i) of Theorem 2.4, we obtain

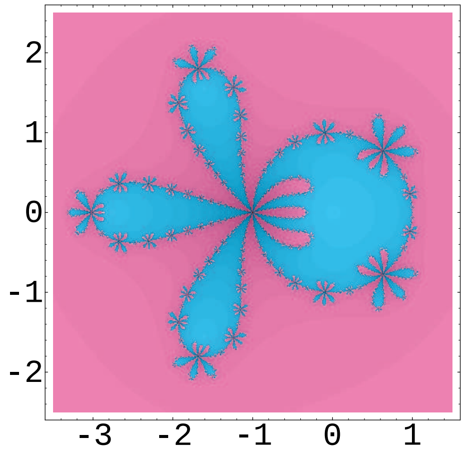
$$S_n'(z) = \frac{(n+2)C_{n+1}z^{n+1}(1+z)^{2n}}{\hat{P}_n(z)^2}.$$

Hence, we can describe the Fatou components associated to S_n, $n \geq 0$, and we can conclude that the rational map R_n is generally convergent for quadratic polynomials.

In fact, we have that the rational map $S_n(z)$, $(n \geq 0)$, defined in (3), has precisely two forward invariant Fatou components: a superattracting component where iterates converge to ∞ and a superattracting component where iterates converge to 0. The unit circle $S^1(z) = \{z \in \mathbb{C};\ |z| = 1\}$ is forward invariant and it is contained in $\mathcal{J}(S_n)$ and moreover, $\mathfrak{m}(\mathcal{J}(S_n)) = 0$, where $\mathfrak{m}$ is the Lebesgue measure on $\mathbb{C}$.

Finally, we show the basins of attraction associated to the two roots of a quadratic polynomial $f(z) = (z-a)(a-b)$ when we apply S_2 and S_3. The basins of attraction clarify the structures of the universal Julia sets associated to the corresponding iterative methods R_2 and R_3.





Plot of the basins of attraction under S_2 and S_3 applied to quadratic polynomial $f(z) = (z-a)(z-b)$.

4 Two open problems

Now we come back to the triangle (1). Note that if we multiple the figures in the row n by the figures in the next row $n+1$, we obtain the Catalan number C_{2n}, for example

$$C_6 = 132 = 5 \cdot 14 + 4 \cdot 14 + 1 \cdot 6.$$

In fact, this result looks like true if we multiply two different rows: multiply the row n and $n+j$, we obtain the Catalan number C_{2n+j-1}. To check this conjeture, take the second and fifth rows and

$$C_6 = 132 = 42 \cdot 2 + 48 \cdot 1.$$

Then it is natural to conjeture that

$$C_{i+j-1} = \sum_{k=1}^{\min(i,j)} B_{i,k} B_{j,k}, \qquad i, j \geq 1.$$

From the triangle (1), we traslade the figures in each column p-th, $p-1$ steps to obtain the new table,

$n \setminus p$	1	2	3	4	5	6	...
1	1	1	1	1	1	1	
2	2	4	6	8	10	12	
3	5	14	27	44	65	90	
4	14	48	110	208	350	544	
5	42	165	429	910	1700	2907	
6	132	572	1638	3808	7752	14364	
...	...	...	...	...	...	...	...

(4)

We denote by $(M_n)_{n\geq 1}$ the main minors of order n in the table (4); we obtain that

n	M_n
1	$1=2^0$
2	$2=2^1$
3	$8=2^3$
4	$64=2^6$
5	$1024=2^{10}$
6	$32768=2^{15}$

(5)

Taking into account (5), it is natural to conjeture that $M_n = 2^{\frac{n(n-1)}{2}}$ for $n \geq 1$.

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References

[1] X. Chen and W. Chu: *Moments on Catalan number.* J. Math. Anal. Appl., **349**, (2) (2009), 311–316.

[2] E. Deutsch and L. Shapiro: *A survey of the Fine numbers,* Discrete Math. **241** (2001), 241–265.

[3] H. G. Forder: *Some Problems in Combinatorics,.* Math. Gaz. **41** (1961), 199–201.

[4] V.J.W. Guo and J. Zeng: *Factors of binomial sums from Catalan triangle.* J. Number Theory, **130**, (1) (2010), 172–186.

[5] J. M. Gutierrez, M.A. Hernández, P.J. Miana, and N. Romero: *New identities in the Catalan triangle.* J. Math. Anal. Appl., **341**, (1) (2008), 52–61.

[6] P. Hilton and J. Pedersen: *Catalan numbers, their generalization and their uses,* Math. Intelligencer **13** (1991), 64–75.

[7] C. McMullen, Families of rational maps and iterative root-finding algorithms. Ann. Math., **Vol. 125**, (1987) 467–493.

[8] P.J. Miana and N. Romero: *Computer proofs of new identities in the Catalan triangle.* Biblioteca de la Revista Matemática Iberoamericana. Proc. of the "Segundas Jornadas de Teoría de Números", (2007), 203–208.

[9] P.J. Miana and N. Romero: *Moments of combinatorial and Catalan numbers,* to appear in J. Number Theory.

[10] M. Petkovsek, H. S. Wilf and D. Zeilberger: $A = B$. A. K. Peters Ltd., Wellesley, 1997. http://www.cis.upenn.edu/∼wilf/AeqB.html

[11] L. W. Shapiro: *A Catalan triangle,* Discrete Math. **14** (1976), 83–90.

[12] A. Slavík: *Identities with squares of binomial coefficients,* to appear in Ars Combinatoria.

[13] N. Sloane: http://www.research.att.com/∼njas/sequences/A039598

[14] S. Smale, On the efficiency of algorithms of analysis. Bull. AMS, **Vol. 13**, (1985) 87–121.

[15] R.P. Stanley: *Enumerative Combinatorics,* vol. 2, Cambridge University Press, 1999.

[16] H. Wilf and D. Zeilberger: *Rational functions certify combinatorial identities,* J. Amer. Math. Soc. **3** (1990), 147–158.