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Multivariate polynomial interpolation: some new trends

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Dedicated to Manuel Calvo with esteem and friendship in occasion of his 65th birthday

Abstract

In this paper we comment on some recent research in the field of multivariate polynomial interpolation with special emphasis in the influence of the relative position of the interpolation nodes to extend certain univariate techniques like simple Lagrange formulae, Aitken–Neville formulae and Lebesgue constants.

Keywords: Multivariate polynomial interpolation

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1 Introduction

Univariate polynomial interpolation is a classical subject, with a long history and a rather complete theory. Its multivariate counterpart is much more complicated. Only very isolated papers, although due to important mathematicians as Kronecker or Jacobi, can be found before the beginning of the 20th Century.

Except for tensor product problems, which are obvious extensions of univariate problems, multivariate techniques have only been systematically considered in the second half of the 20th century due to the development of computers. Another reason for the interest of multivariate problems was the emergence of new mathematical methods, as finite element methods for solving partial differential equations, cubature formulae, etc. Several surveys on the history of the subject and its development have been written at the end of the last century (cf. [24, 25]). The purpose of this paper is to point out some advances in the field and show some new trends which have appeared in the last decade.

2 Constructing sets of nodes suitable for interpolation problems

The usual interpolating space in one variable is $P_n(\mathbb{R}) := \{p \in P(\mathbb{R}) : \deg p \leq n\}$, the space of polynomials of degree not greater than n. In contrast, there exist many different choices for subspaces of $P(\mathbb{R}^d)$, the space of polynomials in d variables, for solving interpolation problems in the case $d > 1$, depending on the number and distribution of the interpolation points (also called nodes). In fact, it is necessary for the existence and uniqueness of the interpolant that the number of nodes equals the dimension of the interpolating space. The most common interpolating space is the space

$$P_n(\mathbb{R}^d) := \{p \in P(\mathbb{R}^d) : \deg p \leq n\}$$

of polynomials of total degree less than or equal to n. Another subspace of polynomials, specially used for rectangular grids, is the space

$$P_{n_1,\dots,n_d}(\mathbb{R}^d) := \{p \in P(\mathbb{R}^d) : \deg_i p \leq n_i,\ i = 0, \dots, d\}$$

where $\deg_i$ denotes the partial degree, that is, the degree with respect to the ith variable x_i. Generalizations of these approaches introducing polynomials whose directional degree, that is, degree along certain directions, is prescribed have also been used [12].

A set of nodes X is said to be correct for an interpolating space S, if the Lagrange interpolation problem on X has always a unique solution in S. An interpolating space S, $\dim S = N$, satisfies the Haar condition on a given domain D if any set of N nodes in D is correct for S. There are many spaces which satisfy this condition in one variable, in particular that of polynomials of degree not greater than N on any subinterval of the real line. However, except for the trivial case of problems with only one interpolation point, there exist no spaces in more than one variable satisfying the Haar condition on domains D containing an open set. Therefore the fact that a set of nodes is correct depends on the geometric distribution of the nodes. This is a remarkable difference with the univariate case and provides one of the main research subjects in multivariate interpolation.

Chung and Yao [21] identified the $P_n(\mathbb{R}^d)$-correct sets of nodes whose Lagrange polynomials can be factored as products of first degree polynomials. This geometric condition (usually called for brevity GC) describes distributions of nodes leading to simple Lagrange formulae. A GC_n-set X is a set with $\dim P_n(\mathbb{R}^d)$ nodes such that for each $x \in X$, there exist n hyperplanes containing all nodes but x. Chung and Yao provided two important examples of distributions of nodes satisfying their geometric condition: principal lattices and natural lattices. The geometric characterization can be easily used to check whether a given set is a GC_n set but it provides no suggestion about how to construct such sets. One of the research lines recently developed has focused on describing examples which generalize those provided by Chung and Yao. For the sake of simplicity we restrict ourselves to the bivariate case.

Principal lattices are distributions of points formed by the intersections of 3 pencils of equidistant parallel lines, $n+1$ lines each, in such a way that any node is the intersection of one line of each pencil. The standard example is the set of points $(i/n, j/n) : 0 \leq$

$i + j \leq n$, where $(i/n, j/n)$ is the intersection of the lines $x - i/n = 0$, $y - j/n = 0$, $x + y - (i + j)/n = 0$. Principal lattices were extended by Lee and Phillips to sets called 3-pencil lattices, allowing concurrent pencils of lines (parallel lines can be considered as a particular case of concurrence at infinity).

Jaklič et al. [30] have used a barycentric form as a useful tool to extend three-pencil lattices to triangulations covering polygonal domains. In this way, they construct continuous piecewise polynomials interpolating Lagrange data, analyzing the degrees of freedom in the selection of the nodes in each subtriangle. Multivariate extensions of these results have also been considered recently by the same authors.

In the last decade, the authors [14, 15] have extended the Lee-Phillips construction to lattices generated by cubic pencils. Cubic pencils are families of lines $ax + by + c = 0$ whose coefficients satisfy a cubic equation. An addition in the set of nonsingular lines Λ^* of a cubic pencil is introduced as a dualization of the addition of points of a cubic curve (a common tool in algebraic geometry). Three lines sum up to 0 if and only if they meet at a point which is not a vertex of the pencil. Usually the lines are parameterized in terms of an isomorphic classical group G,

$$L : t \in G \mapsto L(t) \in \Lambda^*,$$

so that $L(-t_1 - t_2)$ is the line in Λ^* concurrent with $L(t_1)$ and $L(t_2)$. For each $t_0, t_1, t_2 \in G$ with $t_0 + t_1 + t_2 = 0$ and $h \in G$, the set of points $X = \{x_{ijk} \mid i + j + k = n\}$, where

$$\{x_{ijk}\} = L(t_0 - (n - i)h) \cap L(t_1 + jh) \cap L(t_2 + kh), \quad i + j + k = n,$$

is a generalized principal lattice if the lines $L(t_r + ih)$, $i = 0, \dots, n$, $r = 0, 1, 2$, are all distinct. This construction generalizes 3-pencil lattices. In fact, the product of the three linear pencils arising in the Lee-Phillips construction form a cubic pencil of lines

As an example, we might consider the cubic pencil formed by all lines

$$L(t) \equiv y = \tan(t/2)x - \sin(t), \quad t \in \mathbf{R}/2\pi\mathbf{Z}$$

tangent to a deltoid

$$x(t) = \cos t(\cos t + 1), \quad y(t) = \sin t(\cos t - 1).$$

Here the parameter group is $G = \mathbf{R}/2\pi\mathbf{Z}$. Figure 1 below illustrates an example of this construction. We observe that each of the three families $L(t_r + ih)$, $i = 0, \dots, n$, do not belong to the same linear pencil, that is, they do not meet at a vertex.

An apparently more general situation was described in [14, 15] defining generalized principal lattices as distributions of points obtained from the intersections of three families of lines L_i^r, $i = 0, \dots, n$, $r = 0, 1, 2$, not necessarily related to a cubic pencil. Later on, it

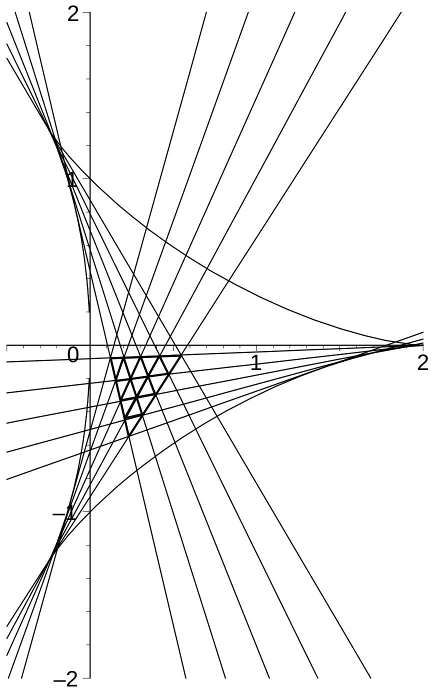

Figure 1.— A lattice generated by a cubic pencil

was proved in [20] that every bivariate generalized principal lattice can be obtained from a cubic pencil of lines. Starting with a canonical classification of cubic curves, the authors have classified in [16] all possible kinds of generalized principal lattices in two variables, up to projectivities.

An extension of these ideas to more than 2 variables was obtained in [18], showing some examples, and simultaneously pointing out the difficulties of getting a general construction.

The Aitken-Neville algorithm in one variable provides the solution of an interpolation problem of degree n on a set X of $n+1$ nodes by linear interpolation of the solutions of two subproblems of degree $n-1$ on n points of X. Multivariate extensions of the Aitken-Neville algorithm have been considered by several authors in the last half of the 20th century. In a recent paper [19] the relationship between multivariate Aitken-Neville algorithms and generalized principal lattices has been studied. Aitken-Neville sets in $\mathbb{R}^d$ were defined by Sauer and Xu [34] as distributions of points allowing a recursive interpolation formula that constructs an interpolating polynomial of degree n on $\binom{n+d}{d}$ nodes from the solutions of $d+1$ problems of degree $n-1$ with $\binom{n+d-1}{d}$ data each. The initial data provide the solutions of $\binom{n+d}{d}$ problems of degree 0 (1 data each). We construct with them the solutions of $\binom{n+d-1}{d}$ problems of degree 1 ($d+1$ data each) and so on, until getting the solution of the complete problem with $\binom{n+d}{d}$ data. The scheme extends the univariate Aitken-Neville algorithm. In [19] it has been shown that each

Aitken-Neville set satisfies the GC condition of Chung and Yao and that any generalized principal lattice is an Aitken-Neville set. As a consequence, an interpolation problem on a generalized principal lattice can be solved by an Aitken-Neville algorithm. Let us remark that interpolation problems on generalized principal lattices can also be solved by the Lagrange formula because they satisfy the GC condition.

In the plane, it has been proved in [19] that any Aitken-Neville set of degree $n > 2$ is a generalized principal lattice. However there are Aitken-Neville sets of degree 2 which are not generalized principal lattices.

Another approach to Aitken-Neville formulae for multivariate interpolation can be found in [17], where extensions to Aitken-Neville formulae (in the sense that an interpolant is obtained combining interpolants on smaller subsets of nodes) have been analyzed. The recursion formulae for Aitken-Neville sets introduced by Sauer and Xu [34, 19] are obtained as a consequence of the main result of [17], applied to the problem of obtaining an interpolant in $P_n(\mathbb{R}^d)$ in terms of subinterpolants in $P_{n-1}(\mathbb{R}^d)$.

A natural lattice of degree n in $\mathbb{R}^d$ is the set of all points obtained intersecting d hyperplanes among $n + d$ hyperplanes in general position in $\mathbb{R}^d$. The number of such intersections is $\binom{n+d}{d} = \dim P_n(\mathbb{R}^d)$. Natural lattices are adequate for total degree interpolation of degree n in $\mathbb{R}^d$ because they satisfy the GC_n condition. In the bivariate case, a natural lattice is the set of pairwise intersections among $n + 2$ lines in general position.

In one variable, Hermite problems can be seen as a limit case of Lagrange problems when nodes coalesce. Analogously, in [11], we have analyzed interpolation problems defined in the set of intersections of $n + 2$ distinct lines in any position, allowing multiple concurrences of lines at a point or parallel lines. This general situation gives rise to Hermite problems on subspaces of $P_n(\mathbb{R}^2)$ consisting of polynomials whose degree diminishes along directions corresponding to parallel or coincident lines among the lines defining the lattice.

The search of new distributions of interpolation nodes which form a correct set on a certain space of polynomials is a natural question in multivariate polynomial interpolation. In the last years, Bojanov and Xu [3] have studied bivariate Hermite problems, where the interpolant matches prescribed data consisting of function values and consecutive normal derivatives on a set of points placed on several circles centered at the origin. Lagrange interpolation is a particular case. For a given integer n, the interpolation nodes are the intersection points of $2[\frac{n+1}{2}] + 1$ rays from the origin with a set of concentric circles (here $[m]$ means the integer part of m). The circles can be repeated and, in this case, successive radial derivatives are provided as interpolation data. In [3] the number of circles, counting multiplicities, is $[\frac{n}{2}] + 1$. So, the total number of interpolation data is the dimension of $P_n(\mathbb{R}^2)$ and, if the rays are equidistant (i.e. the nodes on each circle are equidistant), the set of nodes is correct for $P_n(\mathbb{R}^2)$. The poisedness holds if the circles freely rotate. In particular, Bojanov and Xu rediscovered in [3] a nice star-shaped example of a natural lattice, previously obtained by Hakopian [26].

Later on Bojanov and Xu [4] considered nonconcentric circles and Hakopian and Ismail [27] have extended the analysis to conic sections. Hakopian and Khalaf [28] have continued this work, proving that the poisedness of the data for the Bojanov-Xu problem [3] is equivalent to the unisolvence of certain $2[\frac{n+1}{2}] + 1$ dimensional Lagrange interpolation problems. As a consequence, they prove that the Bojanov-Xu problem is poised not only for equidistant rays, but for a wide family of sets of rays satisfying some simple conditions.

3 Some conjectures on distributions of nodes suitable for interpolation problems

Gasca and Maeztu [22] considered interpolation nodes in $\mathbb{R}^2$ defined as intersections of lines and provided a method of constructing poised Lagrange and Hermite interpolation problems on apropriate subspaces of polynomials. The novelties of their approach consisted, on the one hand, in allowing multiple concurrences of lines (giving rise to derivatives as interpolation data) and, on the other hand, in solving the problem by means of a recurrence with a simple Newton-like formula. An extension to more than two variables was also suggested in that paper.

The simplest case arises when $n + 1$ nodes lie on a line l_0, n nodes on another line l_1, none of them lying on l_0, $n - 1$ points on another line l_2, none of them lying on $l_0 \cup l_1$, and so on. Then the Lagrange interpolation problem on these nodes is poised in the space $P_n(\mathbb{R}^2)$. This distribution of points has been rediscovered several times in the literature, apparently the first times by Berzolari [2] and much later by Radon [32]. Recently it has been referred to as the Berzolari-Radon distribution of points.

The Lagrange problem on the Berzolari-Radon distribution leads to a triangular system of equations and can be solved by a Newton formula. Obviously, not any lattice of this type verifies the geometric condition (GC) of Chung and Yao. On the one hand, the Berzolari-Radon lattices are straightforward to construct, and the corresponding interpolation problem can be solved with a simple Newton formula. On the other hand, for any given set of points in the plane, the geometric characterization can be checked and leads to a simple explicit Lagrange formula as mentioned in Section 2. However, the class of GC sets is not completely known. Some particular constructions like natural lattices, principal lattices and their generalizations are often used but other instances of GC sets are not so easy to construct and describe. It is a remarkable fact that all known GC sets are particular cases of the Berzolari-Radon construction.

A natural question arising in this context is whether or not any planar GC set is a Berzolari-Radon set. Bézout Theorem implies that no line of the plane can contain more that $n+1$ nodes of a set correct for $P_n(\mathbb{R}^2)$. A conjecture launched in [22], known presently as the GM conjecture, states that, *if X is a planar GC set of order n, then $n+1$ points of X are collinear*. It is easy to see that, if a planar GC_n set has $n+1$ collinear points, then the set obtained removing those points is a GC_{n-1} set. Hence, if the GM conjecture is true, it can be shown by induction that any planar GC set of order n is a Berzolari-Radon set of order n. This is the reason why the GM conjecture has atracted much attention in the last twenty years. In spite of the relevance of the consequences of the geometric condition, the GM conjecture has only been proved up to degree $n = 4$ (see, for instance [10, 29]). However no counterexample has been found. The conjecture was reinforced in [13], where the authors proved that, if the GM conjecture holds for any degree, then there exist at least 3 lines containing $n+1$ nodes of any planar GC_n set. The existence of at least 3 lines containing $n+1$ nodes has been considered as a new conjecture, known as the CG conjecture, equivalent to the GM conjecture for points in the plane.

A multivariate version of the GM conjecture in $\mathbb{R}^d$ (the GM_d conjecture) was stated recently by de Boor [8]: there exist always a maximal hyperplane for any GC_n set in $\mathbb{R}^d$. A maximal hyperplane for a $P_n(\mathbb{R}^d)$-correct set X is any hyperplane containing exactly $\binom{n+d-1}{d-1}$ nodes. The name maximal is based on the fact that no hyperplane can contain more than $\binom{n+d-1}{d-1}$ nodes. The same author also launched a multivariate version of the CG conjecture: there exist at least $d+1$ maximal hyperplanes for any GC_n set in $\mathbb{R}^d$. This conjecture was disproved in [8], where a GC_2 set in $\mathbb{R}^3$ with only 3 maximal hyperplanes was described. This counterexample does not disprove the GM_d conjecture nor the planar CG conjecture. The search of new approaches to the GM conjecture by Hakopian, Jetter and Zimmerman has stimulated recent research on the number of maximal hyperplanes in multivariate GC sets. In a recent paper [1], it is shown that the GM conjecture holds for trivariate GC_2 sets.

4 The search of good interpolation nodes

The Lebesgue constant

$$\Lambda_n = \max_{x\in[a,b]} \sum_{i=0}^{n} \prod_{j\neq i} \frac{|x - x_j^n|}{|x_i^n - x_j^n|}$$

is the norm of the interpolation operator $L : C[a,b] \to C[a,b]$ associated to the Lagrange interpolation problem at nodes $x_0^n < \cdots < x_n^n$ in $[a,b]$ and measures in some sense the condition and stability of the interpolation process. By the Erdös-Brutman Theorem

$$\Lambda_n > \frac{2}{\pi} \log n + 0.5212$$

for any set of nodes $x_0^n < \cdots < x_n^n$ in $[a,b]$. This implies that Λ_n must diverge as $n \to \infty$, independently of the choice of nodes. The search of points for interpolation in P_n with least possible Lebesgue constant is an interesting question without an explicit solution. The zeros of the Chebyshev polynomials are called the Chebyshev nodes and they are almost optimal in $[-1,1]$, in the sense that the Lebesgue constant has an asymptotic growth

$$\Lambda_n = \frac{2}{\pi}\log n + O(1), \quad n \to \infty.$$

The advantange of the Chebyshev nodes is that they have a simple explicit formula

$$x_i^n = -\cos\frac{(2i+1)\pi}{2(n+1)}, \quad i = 0, \ldots, n.$$

Another important choice of nodes are the Chebyshev-Lobatto nodes

$$x_k^n = -\cos\frac{kn}{n}, \quad k = 0, \ldots, n.$$

In general, the Lebesgue constant is expected to be low for all distributions of points $x_0^n < \cdots < x_n^n$ in $[-1,1]$ which tend to be uniformly distributed when $n \to \infty$ with respect to the Dubiner metric

$$d(x_1, x_2) = |\arccos x_2 - \arccos x_1|.$$

In several variables, there is no clear candidate for almost optimal points for general domains. In the last years, there have been new advances on the subject and new bivariate distributions of points have been proposed for the square and the circle.

In order to avoid the discussion of the different cases arising in total degree interpolation when the degree is even or odd, let us assume for the sake of simplicity that the degree $n = 2m$ is even. Let $\xi_k^n := \cos(k/n)$ denote Chebyshev-Lobatto nodes. Y. Xu [35] proposed the nodes

$$\begin{aligned}(x_{2i,2j+1}, y_{2i,2j+1}) &:= (\xi_{2i}^{2m}, \xi_{2j+1}^{2m}), \quad i = 0, \ldots, m, \quad j = 0, \ldots m-1,\\ (x_{2i+1,2j}, y_{2i,2j+1}) &:= (\xi_{2i+1}^{2m}, \xi_{2j}^{2m}), \quad i = 0, \ldots, m-1, \quad j = 0, \ldots, m,\end{aligned}$$

for interpolation on a subspace of $P_{2m}(\mathbb{R}^2)$ containing $P_{2m-1}(\mathbb{R}^2)$ on the square $[-1,1]^2$. Other authors (Caliari, de Marchi and Vianello [9]) have proposed the "Padua points" for total degree interpolation in $P_{2m}(\mathbb{R}^2)$

$$(x_i, y_j), \quad i = 0, \ldots, 2m, \quad j = 0, \ldots, m,$$

where

$$x_i = \xi_i^{2m}, \quad i = 0, \ldots, 2m,$$

and

$$y_j := \begin{cases} \xi_{2j}^{2m+1}, & \text{if } m \text{ is odd},\\ \xi_{2j+1}^{2m+1}, & \text{if } m \text{ is even},\end{cases} \qquad j = 0, \ldots, m$$

with lower Lebesgue constant than the Xu points. Other bivariate and trivariate distributions of points with low Lebesgue constants have been proposed for the square, the circle and other simple bivariate domains.

The Xu points are equally spaced in the Dubiner metric

$$d((x_1, y_1), (x_2, y_2)) = \max(|\arccos x_2 - \arccos x_1|, |\arccos y_2 - \arccos y_1|).$$

A generalization of the Dubiner metric can be defined on any compact subset of $\mathbb{R}^n$. A conjecture stated in [9] says that: *nearly optimal points for polynomial interpolation on a compact set are asymptotically equidistributed with respect to the Dubiner metric.* The research on this subject is now very active and the reader is referred to recent papers by Bos, Caliari, de Marchi, Vianello, Xu among others.

5 Multivariate divided differences

Univariate divided differences can be Êdefined in different ways: as the coefficients of the Newton interpolation formula, as a certain linear functional vanishing on the space of polynomials of a given degree and by a recurrence relation, among others. From any of these definitions the other ones can be derived and also relevant properties in applications, such as error formulae in numerical quadrature and relations with B-spline functions. The extension of the concept to the multivariate case depends on the property that we want to preserve. In other words, the way in which multivariate divided differences are defined can lead to the loss of some common properties of the univariate ones. Generalizations of the divided differences to several variables have been recently proposed by Rabut [31], de Boor [7] and Sauer [33].

A general technique for multivariate polynomial interpolation on correct sets of points is based on constructing extensions of the Newton formula. The Newton approach can be described as the problem of constructing a basis of functions in the interpolation space such that the interpolation conditions give rise to a linear system whose coefficient matrix is lower triangular or block-lower triangular. Such a basis can be called a Newton basis. The space of multivariate polynomials have a graded structure and in order to exploit it, an additional condition for Newton bases of polynomials is usually required. The degree of the polynomials of a Newton basis must be increasing (or increasing by blocks) in some sense. So, extensions to the concept of degree might be necessary for deriving Newton formulae in more general situations. Most concepts of multivariate divided differences can be interpreted in terms of the construction of a suitable generalization of the Newton formula.

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