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Lagrangians of a non-mechanical type for second order Riccati and Abel equations

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Abstract

The Helmholtz approach to the inverse problem of the Lagrangian dynamics is studied first in the particular case of the second-order Riccati equation and then in the case of the second-order Abel equation. The existence of two alternative Lagrangian formulations is proved, both Lagrangians being of a non-natural class (neither potential nor kinetic term). These second-order Riccati and Abel equations are studied by means of their Darboux polynomials and Jacobi last multipliers. The existence of a family of constants of the motion is also discussed.

Keywords: Helmholtz conditions. Jacobi last multipliers. Second order Riccati and Abel equations. Alternative Lagrangians

AMS classification: 34A34; 34A26; 34C14; 37J05; 70H03

On the beginning of his scientific career Prof. Calvo was teaching Lagrangian mechanics for several years and beyond doubt he spent many hours thinking on the Inverse problem of mechanics. We report here several recent results on alternative Lagrangians for two interesting equations, the second order Riccati and Abel equations.

1 Introduction

In mathematical terms, the Newtonian approach to classical mechanics, constructed on the use of the second Newton Law, states that the behaviour of a mechanical system is governed by second-order differential equations. On the other side, the Lagrangian approach makes use of a variational formulation associated to the Hamilton's principle:

the motions of the system are those making extremal the action integral defined by the Lagrange function L of the system; therefore such trajectories are solutions of the corresponding set of Euler-Lagrange equations.

The inverse problem of Lagrangian dynamics is to obtain necessary and sufficient conditions for a system of second-order differential equations

$$\ddot{q}^i = F^i(q,\dot{q}), \quad i = 1,\dots,n, \tag{1}$$

to be equivalent to the set of Euler-Lagrange equations of some regular Lagrangian function L. In other words, this amounts to look for a Lagrangian L such that

$$W_{ij}(\ddot{q}^j - F^j(q,\dot{q})) = W_{ij}\ddot{q}^j - \frac{\partial L}{\partial q^i} + \frac{\partial^2 L}{\partial q^j\,\partial \dot{q}^i}\dot{q}^j, \qquad i,j = 1,\dots,n.$$

where the summation convention is used and W is the Hessian matrix with elements defined by

$$W_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i\,\partial \dot{q}^j}, \quad i,j = 1,\dots,n. \tag{2}$$

Such a function L only exists if some compatibility conditions hold; moreover, in some particular cases it can even exist several different (and non gauge equivalent) solutions. In these cases the alternative Lagrangians can be used to construct constants of the motion as was proved in [1] for the one-dimensional case and generalised in [2] for the multidimensional case (see also [3] for a geometric approach).

The aim of the paper is to show that these properties are related with a rather old result by Jacobi [4] which is called the theory of Jacobi Last Multiplier and to illustrate the theory with the determination of Lagrangians of non-mechanical type for second-order Riccati and Abel equations.

The organization of the paper is as follows: in section 2 we give a concise survey of the theory of the Inverse problem in mechanics and the Helmholtz conditions necessary for the existence of a Lagrangian. In section 3 we recall several notions of Darboux polynomials for polynomial vector fields and the relation with the theory of Jacobi Last Multiplier. The relevance of such multipliers in the search for Lagrangians for one-dimensional systems is shown in section 4 and the theory is illustrated with some particular examples. The method of Jacobi last ultipliers is used to determine alternative Lagrangians for the second order Riccati equation and the second order Abel equation.

2 The inverse problem and the Helmholtz conditions

From the geometric viewpoint, the system of equations (1) determines a $2n$-dimensional vector field Γ on the velocity phase space $\mathbb{R}^{2n} = \{(q^i, v^i) \mid i = 1,\dots,n\}$ with $(v^i, F^i(q,v))$ as components, i.e.

$$\Gamma = v^i \frac{\partial}{\partial q^i} + F^i(q,v)\frac{\partial}{\partial v^i}, \tag{3}$$

the solutions of the system being the integral curves of Γ (the physical time t coincides with the parameter of the curves).

The solution of the Inverse Problem is given by a family of functions $g_{ij}(q,\dot{q})$ such that the equations

$$g_{ij}\,(\ddot{q}^j - F^j(q,\dot{q})) = 0, \qquad i,j = 1,\ldots,n,$$

become the Euler-Lagrange equations of some function L. The problem was studied long time ago by Helmholtz [5], who established the so-called Helmholtz conditions (see e.g. [6, 7, 8, 9]). In terms of the n functions F^i, these conditions can be presented as follows: there should exist a non-degenerate symmetric matrix valued function $g = [g_{ij}]$, i.e. a family of $n(n+1)/2$ functions $g_{ij} = g_{ij}(q,v)$, such that

i) $\det[g_{ij}] \neq 0$

ii) $\dfrac{\partial g_{ij}}{\partial v^k} = \dfrac{\partial g_{ik}}{\partial v^j}$

iii) $\Gamma(g_{ij}) + \dfrac{1}{2} g_{kj} \dfrac{\partial F^k}{\partial v^i} + \dfrac{1}{2} g_{ik} \dfrac{\partial F^k}{\partial v^j} = 0$

iv) $g_{ik} \left[\dfrac{\partial F^k}{\partial q^j} + \dfrac{1}{4} \dfrac{\partial F^k}{\partial v^l} \dfrac{\partial F^l}{\partial v^j} - \dfrac{1}{2} \Gamma\left(\dfrac{\partial F^k}{\partial v^j}\right)\right] = g_{jk} \left[\dfrac{\partial F^k}{\partial q^i} + \dfrac{1}{4} \dfrac{\partial F^k}{\partial v^l} \dfrac{\partial F^l}{\partial v^i} - \dfrac{1}{2} \Gamma\left(\dfrac{\partial F^k}{\partial v^i}\right)\right].$

These properties lead to the existence of a function L such that the g_{ij} take the form

$$g_{ij} \;=\; \frac{\partial^2 L}{\partial v^i\,\partial v^j}\,.$$

The regularity of L is a consequence of condition i). Although we are not concerned in this article with the abstract geometric formalism, at this point we make the observation that these conditions are related to the existence of a symplectic structure in the phase space.

These equations involve not only the known functions $F^i(q,v)$ but also the other functions $g_{ij}(q,v)$ whose form is completely unknown. Usually the problem is solved by using an ansatz on the components of the matrix $[g_{ij}]$. Moreover, these equations do not guarantee uniqueness: every set of $n(n+1)/2$ functions $g_{ij}(q,v)$ satisfying Eqs. i)-iv) leads to a particular Lagrangian. Thus, different sets of such functions $g_{ij}^{(a)}(q,v)$, $a = 1,\ldots,A$, give rise to different alternative Lagrangians $L^{(a)}$.

Hojman and Harleston proved [2] that if a system admits two alternative regular Lagrangians $L^{(a)}$, $a = 1,2$, then the traces of the powers of the product matrix $W_{21} = W_2^{-1}W_1$ are constants of motion which are obtained by a non-Noether procedure [3].

Consequently, the existence of alternative Lagrangians for a certain Lagrangian system also means the existence of a certain set of integrals of motion.

Two situations are particularly interesting. First, in the case of velocity-independent forces, that is $F^i = F^i(q)$, the last three Helmholtz conditions reduce to

$$\begin{array}{rl} \text{ii b)} & \dfrac{\partial g_{ij}}{\partial v^k} = \dfrac{\partial g_{ik}}{\partial v^j} \\ \text{iii b)} & \Gamma(g_{ij}) = 0 \\ \text{iv b)} & g_{ik}\left(\dfrac{\partial F^k}{\partial q^j}\right) = g_{jk}\left(\dfrac{\partial F^k}{\partial q^i}\right). \end{array}$$

We note that the equation iii b) means that the functions g_{ij} must be constants of the motion for the dynamics.

Secondly, the case of only one equation corresponding to a one degree of freedom system. In this case the matrix reduces to a function g and only one condition remains:

$$\Gamma(g) + g\,\frac{\partial F}{\partial v} = 0, \tag{4}$$

which will be shown to be that g is a Jacobi last multiplier, a concept we recall next.

3 Jacobi Last Multipliers

Given a vector field X in an oriented manifold (M, Ω), a function R such that $R\, i(X)\Omega$ is closed is said to be a Jacobi last multiplier (JLM) for X. Recall that the divergence of the vector field X (with respect to the volume form Ω) is defined by the relation

$$\mathcal{L}_X\Omega = (\operatorname{div} X)\,\Omega\,.$$

This means that R is a multiplier if and only if $R\,X$ is a divergence-less vector field and then

$$\mathcal{L}_{RX}\Omega = (\operatorname{div} RX)\,\Omega = [X(R) + R\,\operatorname{div}X]\,\Omega = 0\,,$$

and therefore we see that R is a last multiplier for X if and only if

$$X(R) + R\,\operatorname{div}X = 0\,. \tag{5}$$

Moreover, $f\,R$ is also a JLM iff f is a constant of the motion for X, because, for any function f,

$$X(f\,R) + f\,R\operatorname{div} X = (Xf)\,R + f(X(R) + R\operatorname{div} X).$$

In the particular case of X being the vector field Γ given by (3) in the velocity phase space corresponding to the second order differential equation (1) with $n = 1$, as $\operatorname{div}\Gamma = \partial F/\partial v$, (5) becomes

$$v\,\frac{\partial R}{\partial q} + F(q,v)\,\frac{\partial R}{\partial v} + R(q,v)\,\frac{\partial F}{\partial v} = 0, \tag{6}$$

that when compared with (4) means that g is a JLM for Γ.

There is a method due to Darboux for finding JLM for polynomial vector fields. We recall that a polynomial function $\mathcal{D} : U \to \mathbb{R}$ is a Darboux polynomial for a polynomial vector field X if there is a polynomial function f defined in U such that $X\mathcal{D} = f\mathcal{D}$ [10]. The function f is said to be the cofactor corresponding to such Darboux polynomial and the pair $(f, \mathcal{D})$ is called a Darboux pair.

The remarkable point is that if $\mathcal{D}_1, \ldots, \mathcal{D}_k$, are Darboux polynomials with corresponding cofactors f_i, $i = 1, \ldots, k$, one can look for multiplier factors of the form

$$R = \prod_{i=1}^{k} \mathcal{D}_i^{\nu_i} \tag{7}$$

and then

$$\frac{X(R)}{R} = \sum_{i=1}^{k} \nu_i \frac{X(\mathcal{D}_i)}{\mathcal{D}_i} = \sum_{i=1}^{k} \nu_i\, f_i \,,$$

and therefore, if the coefficients ν_i can be chosen such that

$$\sum_{i=1}^{k} \nu_i\, f_i = -\mathrm{div}\, X \tag{8}$$

holds, then we arrive to

$$\frac{X(R)}{R} = \sum_{i=1}^{k} \nu_i\, f_i = -\mathrm{div}\, X \,,$$

and consequently R is a Jacobi last multiplier for X.

4 Looking for Lagrangians from Jacobi last multipliers

The main point of the relation between Jacobi last multipliers and Lagrangian formulations for an autonomous second-order differential equation is given in the following theorem:

Theorem 1 *The normal form of the differential equation determining the solutions of the Euler-Lagrange equation defined by the regular Lagrangian function $L(y, v)$ admits the function*

$$R = \frac{\partial^2 L}{\partial v^2}. \tag{9}$$

as a Jacobi last multiplier. Conversely, if $R(y, v)$ is a multiplier function for a second order differential equation in normal form, then there exists a Lagrangian L for the system that is related to R by (9).

Proof: In fact, if there exists a Lagrangian function L, the function F is given by

$$F(q, v) = \frac{1}{R}\left(\frac{\partial L}{\partial q} - v\frac{\partial^2 L}{\partial q \partial v}\right) .$$

where the function R is given by (9). We can now see that such function R satisfy condition (4). In fact, it turns out to be

$$v\frac{\partial R}{\partial q} + \frac{\partial}{\partial v}\left(\frac{\partial L}{\partial q} - v\frac{\partial^2 L}{\partial q \partial v}\right) = v\frac{\partial^3 L}{\partial v^2 \partial q} + \frac{\partial^2 L}{\partial q \partial v} - \frac{\partial^2 L}{\partial q \partial v} - v\frac{\partial^3 L}{\partial v^2 \partial q} = 0\,.$$

Conversely, let R be a function satisfying (6). Then, let L be a function such that condition (9) be satisfied, i.e. L is such that

$$\frac{\partial L}{\partial v} = \int^{v} R(q,\zeta)\, d\zeta + \phi_1(q)\,, \tag{10}$$

and then,

$$L(q,v) = \int^{v} dv' \int^{v'} R(q,\zeta)\, d\zeta + \phi_1(q)\, v + \phi_2(q)\,. \tag{11}$$

We can see that there exists a function $\phi_2(y)$ such that, for any choice of the function $\phi_1(y)$, the Euler Lagrange equation for such a Lagrangian function gives rise to the equation of motion. The function $\phi_2(y)$ is uniquely determined, while the other function is arbitrary because it corresponds to a gauge term. In fact, the Euler-Lagrange equation for (11), taking into account (10) and that

$$\frac{\partial L}{\partial q} = \int^{v} dv' \int^{v'} \left(\frac{\partial}{\partial q} R(q,\zeta)\right) d\zeta + \phi_1'(q) v + \phi_2'(q)\,,$$

turns out to be:

$$\int^{v} dv' \int^{v'} \left(\frac{\partial}{\partial q} R(q,\zeta)\right) d\zeta + \phi_1'(q) v + \phi_2'(q) = v \int^{v} \left(\frac{\partial}{\partial q} R(q,\zeta)\right) d\zeta + R(q,v) F(q,v) + \phi_1'(q) v\,.$$

Note that

$$\begin{aligned} &\frac{\partial}{\partial v}\left(v\int^{v} \left(\frac{\partial}{\partial q} R(q,\zeta)\right) d\zeta + R(q,v)F(q,v) - \int^{v} dv' \int^{v'} \left(\frac{\partial}{\partial q} R(q,\zeta)\right) d\zeta \right) \\ &\qquad = v\frac{\partial R}{\partial q}(q,v) + F(q,v)\frac{\partial R}{\partial v}(q,v) + R(q,v)\frac{\partial F}{\partial v}(q,v)\,, \end{aligned}$$

which vanishes because of the multiplier condition (5) for R and $X = \Gamma$. Then, the function $\phi_2(q)$ is uniquely determined, up to a constant, by

$$\phi_2'(q) = v \int^{v} \left(\frac{\partial}{\partial q} R(q,\zeta)\right) d\zeta + R(q,v)F(q,v) - \int^{v} dv' \int^{v'} \left(\frac{\partial}{\partial q} R(q,\zeta)\right) d\zeta. \tag{12}$$

The term $\phi_1(y)\, v$ is a gauge term which can be eliminated and then the expression (11) for the Lagrangian reduces to

$$L(y,v) = \int^{v} dv' \int^{v'} R(y,\zeta)\, d\zeta + \phi_2(y)\,. \tag{13}$$

Corollary 1 *If a system with one degree of freedom admits two different regular Lagrangians then the function f defined by*

$$f\, \frac{\partial^2 L_1}{\partial v^2} = \frac{\partial^2 L_2}{\partial v^2} \tag{14}$$

is a constant of the motion.

Proof: The function f is the quotient of two different Jacobi last multipliers.

The last result is usually attributed to Currie and Saletan [1], but actually data back to Jacobi's time. Conversely, if L_1 is an admissible Lagrangian and f is a constant of the motion, there is an alternative Lagrangian L_2 such that relation (14) holds.

Then, the inverse problem for one-dimensional systems reduces to find the function g which is a Jacobi last multiplier and L is obtained by integrating the function g two times with respect to velocities. The function L so obtained from g is unique up to addition of a gauge term.

As an instance, Jacobi derived in [11] the multiplier $R = e^{\varphi(y)}$ for a family of differential equation of the form

$$y'' + \frac{1}{2}\frac{d\varphi}{dy}\, y'^2 + \psi(y) = 0\,,$$

which includes as a particular instance the equation studied in [12]. In fact, one can see that such a function R satisfies (6), because now $\log R = \varphi(y)$ and $F(y,v) = -\frac{1}{2}\varphi'(y)\, v^2 - \psi(y)$, and then

$$\frac{\partial F}{\partial v} = -\frac{d\varphi}{dy}\, v = -\frac{d}{dx}\log R\,.$$

The corresponding Lagrangian is given, up to a gauge term, by

$$L(y,v) = \frac{1}{2}e^{\varphi(y)}v^2 + f(y)\,,$$

where f satisfies the following equation:

$$\frac{df}{dy} + e^{\varphi(y)}\psi(y) = 0\,,$$

because using (11) in the form (13) we find that

$$L = \int_0^v v' e^{\varphi(y)}\, dv' + \phi_2(y)$$

with ϕ_2 being determined by (12), i.e.

$$L(y,v) = \frac{1}{2}e^{\varphi(y)}v^2 + f(y)\,,$$

with $f(y)$ such that the previous equation holds.

For instance, the evolution equation

$$\ddot{x} = F(x,\dot{x}) = \frac{-a + \lambda \dot{x}^2}{1 + \lambda\, x^2}\,, \quad a, \lambda \in \mathbb{R},$$

which corresponds to a nonlinear oscillator [13, 14], is a particular example for which

$$\frac{1}{2}\frac{d\varphi}{dx} = -\frac{\lambda\, x}{1 + \lambda\, x^2}\,, \qquad \psi(x) = \frac{a\, x}{1 + \lambda\, x^2}\,,$$

and therefore,

$$\varphi(x) = -\log(1 + \lambda\, x^2)\,.$$

The Jacobi last multiplier is then

$$R = e^{\varphi(x)} = \frac{1}{1+\lambda\, x^2}\,,$$

and the corresponding Lagrangian turns out to be:

$$L(x, \dot{x}) = \frac{1}{2}\,\frac{\dot{x}^2}{1+\lambda\, x^2} - \frac{1}{2}\,\frac{a\, x^2}{1+\lambda\, x^2}\,.$$

5 Alternative Lagrangians for second-order Riccati and Abel equations

5.1 Second-order Riccati equation

Riccati differential equation is a nonlinear generalization of the inhomogeneous linear equation

$$\dot{x} = c_0(t) + c_1(t)\, x + c_2(t)\, x^2\,,$$

which is just the particular case corresponding to $c_2(t) \equiv 0$. It appears in the Lie reduction process when taking into account invariance under dilations, $u \mapsto \lambda\, u$, of second-order linear equations. The infinitesimal generator of such transformations is the Liouville vector field $\Delta = u\, \partial/\partial u$. Lie's recipe for order reduction of differential equations with symmetry consists on changing the dependent variable in such a way that $\Delta = \partial/\partial w$. More specifically, $u = e^w$, and under such change of dependent variable, as

$$\dot{u} = e^w\, \dot{w}\,, \qquad \ddot{u} = e^w(\dot{w}^2 + \ddot{w})\,.$$

Then the second-order linear differential equation $\ddot{u} + d_0\, \dot{u} + d_1\, u = 0$ becomes a Riccati differential equation for the function $\dot{w} = \dot{u}/u$:

$$\ddot{w} + \dot{w}^2 + d_0\, \dot{w} + d_1 = 0\,.$$

We call higher-order Riccati equations those appearing in a reduction process from a linear differential equations by using dilation invariance: the linear $(j+1)$-order differential equation $y^{(j+1)} = 0$ gives rise to a j-order Riccati equation. For instance the third order linear equation $y''' = 0$ defines the second order Riccati equation

$$\ddot{x} + 3\, x\, \dot{x} + x^3 = 0\,. \qquad (15)$$

More specifically, the invariance under dilations of the differential equation $y^{(n)} = 0$, according to Lie recipe, suggests to look for a new variable z such that the dilation vector field $y\, \partial/\partial y$ becomes $\partial/\partial z$. Then $y = e^z$, up to an irrelevant factor.

It has been proved in [15] that the differential equation $y^{(n)} = 0$ becomes $R^{(n-1)}(x) = 0$ with $x = \dot{z}$, where $R^{(j)}(x)$ is defined in an iterative way by

$$R^{(j)}(x, \ldots, x^{(j)}) = \mathbb{D}^j x\,, \quad j = 0, 1, \ldots,$$

with

$$\mathbb{D} = \frac{d}{dt} + x\,.$$

It has been pointed out in [16] that the second-order Riccati equation (15) admits a Lagrangian formulation with the function

$$L = \frac{1}{\dot{x} + x^2}\ .$$

as a (non-standard) Lagrangian. Our aim in this section is to rederive this Lagrangian by means of the associated Darboux polynomials. The differential equation (15) has an associated system of differential equations

$$\begin{cases} \dot{x} &= v \\ \dot{v} &= -3xv - x^3 \end{cases}$$

which determines the integral curves of the vector field

$$\Gamma^{(1)} = v\frac{\partial}{\partial x} - (3xv + x^3)\frac{\partial}{\partial v}\,.$$

We can look for a Darboux polynomial of the form

$$\mathcal{D}(x, v) = v + ax^2.$$

The condition $\Gamma^{(1)}\,\mathcal{D} = f\,\mathcal{D}$ implies first that f must be of the form $f(x, v) = (2a - 3)x$ and then that $-x = af = (2a - 3)ax$.

Therefore there are two solutions corresponding to the two roots of $2a^2 - 3a + 1 = 0$: either a must be equal to 1, and then the corresponding cofactor is $f_1 = -x$, or to $1/2$ with associated cofactor $f_{1/2} = -2x$.

In the first case, with $\mathcal{D}_1(x, v) = v + x^2$, we can choose in (8) $\nu_1 = -3$ because $\nu_1\, f_1 = 3x = -\mathrm{div}\,\Gamma^{(1)}$, and consequently we arrive to

$$R_1(x, v) = L_1^3(x, v) = \frac{1}{(v + x^2)^3}.$$

In the second case, we have $\mathcal{D}_2(x, v) = v + \frac{1}{2}x^2$, and we can choose in (8) $\nu_2 = -\frac{3}{2}$, so that we obtain

$$R_2 = \left(v + \frac{1}{2}x^2\right)^{-3/2}$$

as another Jacobi last multiplier.

¿From the first Jacobi last multiplier R_1 we obtain that the vector field $\Gamma^{(1)}$ is the Euler-Lagrange vector field of a Lagrangian L that just coincides with L_1. The alternative Lagrangian obtained from R_2 is:

$$L'(x, v) = \sqrt{v + \frac{1}{2}x^2}.$$

5.2 Second-order Abel equation

The Abel equation of first order,

$$\dot{x} = A_0(t) + A_1(t)x + A_2(t)x^2 + A_3(t)x^3,$$

is a generalisation of the Riccati equation (that appears as the particular case $A_3 = 0$).

In similarity with the Ricati case, let us define the differential operator

$$\mathbb{D}_A = \frac{d}{dt} + x^2(t)\,,$$

in such a way that iterating the action of $\mathbb{D}_A$ on x leads to the family of differential equations

$$\mathbb{D}_A^m x = 0\,,\quad m = 1, 2, 3, \ldots\,.$$

The three first equations in this hierarchy of higher-order Abel equations are given by $\mathbb{D}_A^0 x = 0$, $\mathbb{D}_A x = 0$, and $\mathbb{D}_A^2 x = 0$, with $\mathbb{D}_A^0 x$, $\mathbb{D}_A x$, and $\mathbb{D}_A^2 x$ given by

$$\begin{aligned}
\mathbb{D}_A^0 x &= x \\
\mathbb{D}_A x &= \left(\frac{d}{dt} + x^2\right) x = \dot{x} + x^3 \\
\mathbb{D}_A^2 x &= \left(\frac{d}{dt} + x^2\right)^2 x = \ddot{x} + 4x^2\dot{x} + x^5
\end{aligned}$$

The second-order Abel equation $\mathbb{D}_A^2 x = 0$ so obtained can be presented as a system of two first-order equations

$$\left\{\begin{aligned}
\frac{dx}{dt} &= v \\
\frac{dv}{dt} &= -4\,x^2 v - x^5
\end{aligned}\right. \tag{16}$$

corresponding to the following vector field on the velocity phase space $\mathbb{R}^2$

$$\Gamma^{(2)} = v\,\frac{\partial}{\partial x} - \left(4\,x^2 v + x^5\right)\,\frac{\partial}{\partial v}\,.$$

In this case the polynomial $\mathcal{D}_1$ defined by

$$\mathcal{D}_1(x, v) = v + x^3$$

is a Darboux polynomial for $\Gamma^{(2)}$ with cofactor $-x^2$ since

$$\left(v\,\frac{\partial}{\partial x} - \left(4\,x^2 v + x^5\right)\,\frac{\partial}{\partial v}\right)(v + x^3) = -x^2(v + x^3)\,.$$

The divergence of the vector field $\Gamma^{(2)}$ is $-4x^2$, and then we see that there is a Jacobi last multiplier of the form

$$R = \mathcal{D}_1^{-4}\,.$$

Consequently, the Abel equation admits a Lagrangian description by means of a function L such that

$$\frac{\partial^2 L}{\partial v^2} = \frac{1}{(v+x^3)^4}\,,$$

from where we obtain the Lagrangian $L = L_A$ given by

$$L_A = \frac{1}{(v+x^3)^2}\,. \tag{17}$$

The polynomial $\mathcal{D}_2$ defined by

$$\mathcal{D}_2(x,v) = 3v + x^3$$

is a Darboux polynomial for $\Gamma^{(2)}$ with cofactor $-3x^2$, because

$$\left(v\,\frac{\partial}{\partial x} - \left(4\,x^2 v + x^5\right)\,\frac{\partial}{\partial v}\right)(3v+x^3) = 3\,x^2 v - 3(4\,x^2 v + x^5) = -3\,x^2(3v+x^3)\,,$$

and then we can find another Jacobi last multiplier of the form $\mathcal{D}_2^{\nu_2}$ with $\nu_2 = -4/3$. Therefore the Abel equation admits a Lagrangian description by means of a second function L such that

$$\frac{\partial^2 L}{\partial v^2} = (3\,v + x^3)^{-4/3}\,, \tag{18}$$

from where we obtain the Lagrangian $L = \widetilde{L}_A$ given by

$$\widetilde{L}_A = (3v + x^3)^{2/3}\,.$$

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