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Qualocation for periodic pseudodifferential operators: additional order convergence, an overview

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Abstract

The aim of this note is to report on some new additional order convergence results for the qualocation method applied to periodic pseudodifferential operators using splines from $S_h^{r,M}$ as trial and from $S_h^{r',M}$ as test space. Here $S_h^{r,M}$ denotes the space of 1-periodic splines of order r and knot multiplicity $M \leq r$ on an equidistant mesh with mesh-size h.

1 Introduction

Additional order convergence (in brief: AOC) in discretization methods has attracted the interest of many mathematicians over decades. The AOC has the character of a gift. With a few changes in the original method which do not increase the overall numerical costs significantly a more accurate approximation is obtained. To be a little bit more detailed, in most cases the accuracy of a method will be described by a discretization parameter, say h, tending to zero with increasing approximation quality. The error between the exact solution u and the numerical solution u_h is measured, apart from a h-independent constant, by some power of h as $h \to 0$. One speaks of an AOC if another approximation $\tilde{u}_h$ can be calculated with comparably the same complexity or if by measuring the error in more subtile norms a higher convergence order is exhibited.

Perhaps the oldest and structural simplest AOC is provided by Richardson extrapolation. But as the mother of all AOC results the DeBoor & Swartz paper [1] on collocation at Gaussian points can be considered. And, naturally, a wide variety of AOC results can be found in the context of finite element methods (see for example the Lecture Note [20] and the overview article [10]).

In this paper we concentrate on AOC obtained for spline qualocation methods applied to elliptic periodic pseudodifferential operators (in brief: ψdos). Qualocation, introduced by Sloan [15], is a Petrov-Galerkin method with quadrature as a compromise between the (full) Petrov-Galerkin and the collocation method: on the one hand it discretizes the inner product in the Petrov-Galerkin method making the numerical implementation easier, on the other hand it can use more mesh-points than in the collocation method thereby stabilizing it. In accordance, the word "qualocation" means quadrature modified collocation.

From the different principles invented for obtaining AOC we use here the principles of parameter selection for cancelling the leading error term and duality combined with negative norms.

Already when introducing the qualocation method in [15] and shortly after when analyzing it in greater generality in [3] AOC was in the center of interest. The analysis of qualocation was firstly developed for constant coefficient operators and smoothest splines in [15], [3], [18], where straight Fourier analysis could be applied, and later generalized to variable coefficient operators and multiple knot splines in [11], [19], [8] requiring a more sophisticated analysis.

In this paper we report on some recent AOC results obtained by the author and relate them to corresponding earlier results. In the first part the qualocation method is introduced and the principle convergence result is given. The approximation power of multiple knot splines in the Sobolev spaces H^s for $s \in \mathbb{R}$, which serve as trial and test spaces, are reviewed. In the last section it is shown that the conditions for additional order convergence from the third section hold true if the basic quadrature rule is symmetric and satisfies certain exactness properties thereby extending conditions given in [19].

2 The given problem

The given problem is of the form $Lu = f$, where L is a periodic ψdo and u and f are functions in certain Sobolev spaces. In this section we provide the definition of the function spaces and of ψdos together with some of their properties.

2.1 Periodic ψdos in the spaces $H^s(\mathbb{T})$

Denote by

$$\hat{v}(n) = \int_{\mathbb{T}} v(x) e^{-i2\pi nx}\, dx \quad \text{for } n \in \mathbb{Z}$$

the n-th complex Fourier coefficient of a 1-periodic distribution v and by $\mathbb{T} := \mathbb{R}\backslash\mathbb{Z}$ the one-dimensional torus of length 1. Then the ψdo L is defined by

$$L = L_0 + L_1, \tag{1}$$

where

$$(2) \qquad L_0 v(x) := \sum_{n=-\infty}^{\infty} \sigma_0(x,n)\hat{v}(n)e^{i2\pi nx} \quad \text{for } x \in \mathbb{T}.$$

The symbol σ_0 has the form

$$(3) \qquad \sigma_0(x,\xi) := a^+(x)|\xi|^\beta + a^-(x)\operatorname{sign}(\xi)|\xi|^\beta \quad \text{for } x \in \mathbb{T} \text{ and } 0 \neq \xi \in \mathbb{R}$$

with coefficients a^+ and a^- in $C^\infty(\mathbb{T})$, where $\beta \in \mathbb{R}$ is the order of L_0. We assume σ_0 to be normalised by $\sigma_0(x,0) = 1$ for $x \in \mathbb{T}$. If a^- or a^+ vanishes the symbol σ_0 and the operator L_0 are said to be even or odd, respectively. L is assumed to be elliptic, i.e. $\sigma_0(x,\xi) \neq 0$ for $x \in \mathbb{T}$ and $|\xi| = 1$, and to have index $\kappa = 0$, where

$$\kappa := \frac{1}{2\pi}\left[\arg\frac{a^+(x)+a^-(x)}{a^+(x)-a^-(x)}\right]_0^1$$

is the winding number of the closed curve $(a^+ + a^-)/(a^+ - a^-)$ in the complex plane. It is known that then $L_0 : H^s \to H^{s-\beta}$ is a Fredholm operator with index 0 for all $s \in \mathbb{R}$, where $H^s \equiv H^s(\mathbb{T})$ is the usual Sobolev space of periodic distributions v equipped with the norm

$$(4) \qquad \|v\|_s := \left(\sum_{n=-\infty}^{\infty} \langle n\rangle^{2s} |\hat{v}(n)|^2\right)^{1/2} \quad \text{with } \langle n\rangle := \begin{cases} 1 & \text{if } n = 0, \\ |n| & \text{if } n \neq 0. \end{cases}$$

It is at least assumed that L_1 maps $H^s \to H^{s-\beta+\delta}$ for some $\delta > 0$ and all $s \in \mathbb{R}$ and hence L is also Fredholm with index 0.

2.2 Examples for ψdos

Since the topic of the proceedings are mainly differential equations and not primarily ψdos it may be convenient for the readers to see two standard examples for the latter which we borrow from [16].

As first example, consider the boundary value problem

$$U_{XX} + U_{YY} = 0 \text{ in } \Omega \subset \mathbb{R}^2,\ \Omega \text{ bounded}, \quad U = F \text{ on } \Gamma := \partial\Omega,$$

where Γ is a smooth curve. One method to solve this problem is to express U as a single-layer potential with unknown (charge-)density W:

$$U(X) = \mathbb{V}W(X) := \frac{1}{\pi}\int_\Gamma \log\frac{1}{|X-Y|} W(Y)\,dY \quad \text{for } X \in \overline{\Omega}.$$

Parametrize Γ in the form $X = \gamma(x)$ for $x \in [0,1]$ to obtain

$$(\mathbb{V}W)(\gamma(x)) = \frac{1}{\pi}\int_0^1 \log\frac{1}{|\gamma(x)-\gamma(y)|} W(\gamma(y))|\gamma'(y)|\,dy$$

$$
\begin{aligned}
&= 2\int_0^1 \log\frac{1}{|2r\sin\pi(x-y)|}\frac{1}{2\pi}W(\gamma(y))|\gamma'(y)|\,dy \\
&\quad + 2\int_0^1 \log\frac{|2r\sin\pi(x-y)|}{|\gamma(x)-\gamma(y)|}\frac{1}{2\pi}W(\gamma(y))|\gamma'(y)|\,dy \\
&=: V_0u(x) + Ku(x),
\end{aligned}
$$

where

$$
u(x) := \frac{1}{2\pi}W(\gamma(x))|\gamma'(x)|. \tag{5}
$$

V_0u is the single-layer potential for a circle of radius r. For $\Phi_n := \exp(i2\pi nx)$ it can explicitely calculated that

$$
V_0\Phi_n = \begin{cases} \frac{1}{|n|}\Phi_n & \text{for } n \neq 0, \\ 1 & \text{for } n = 0. \end{cases}
$$

Thus, applying V_0 to a 1-periodic distribution

$$
u = \sum_{n\in\mathbb{Z}} \hat{u}(n)\Phi_n \in H^s
$$

yields the representation of V_0 as a ψdo,

$$
V_0u = \hat{u}(0) + \sum_{n\neq 0}\frac{1}{|n|}\hat{u}(n)\Phi_n.
$$

The symbol of V_0 is even and given by

$$
\sigma_0(x,\xi) = \frac{1}{|\xi|} \quad \text{for } \xi \neq 0,
$$

the order is $\beta = -1$. Evidently, V_0 maps $H^s \to H^{s+1} \equiv H^{s-\beta}$ boundedly for $s \in \mathbb{R}$.

More briefly, as second example the Hilbert transform is presented:

$$
\mathbb{S}U(X) := \frac{1}{i\pi}\int_\Gamma \frac{1}{Y-X}U(Y)\,dY,
$$

which can be transformed as in the first example to coordinates (x,y) yielding the principal part

$$
S_0u(x) := 2\int_0^1 \frac{\exp(i2\pi y)}{\exp(i2\pi y)-\exp(i2\pi x)}u(y)\,dy.
$$

The order of S_0 is $\beta = 0$, the symbol is odd and given by

$$
\sigma_0(x,\xi) = \operatorname{sign}(\xi) \quad \text{for } \xi \neq 0.
$$

3 The qualocation method

We consider the discretisation of the given problem by qualocation using splines with multiple knots on equidistant meshes as test and trial spaces. Let r, M, N with $1 \le M \le r$ be positive integers. We define the set of knots

$$\pi_h := \{x_j = jh, j = 0, \dots, N-1\}, \quad h \in \mathcal{H} := \{1/N, N \in \mathbb{N}\},$$

and denote by $S_h^{r,M}$ the space of 1-periodic splines of order r with M-fold breakpoints in π_h. $S_h^{r,M}$ is a subspace of C^{r-M-1} of dimension MN, where $C^k = C^k(\mathbb{T})$ is the space of 1-periodic k times continuously differentiable functions (with C^{-1} meaning piecewise continuity with jumps only at the knots in π_h). By $\mathcal{H}_1$ we denote a final section of the null-sequence of stepsizes $\mathcal{H}$, not necessarily the same at different occurencies.

Qualocation is based on a composite quadrature rule

$$Q_N f = h \sum_{k=0}^{N-1} \sum_{j=1}^{J} \omega_j f(x_{k,j}), \quad x_{k,j} := x_k + h\xi_j,$$

derived from the basic quadrature formula

$$Qf = \sum_{j=1}^{J} \omega_j f(\xi_j),$$

where the quadrature points $\{\xi_j\}$ and weights $\{\omega_j\}$ satisfy

$$0 \le \xi_1 < \xi_2 < \cdots < \xi_J < 1, \quad J \ge M, \quad \sum_{j=1}^{J} \omega_j = 1, \quad \omega_j > 0. \tag{6}$$

Associated with the quadrature rule we define an inner product

$$(v_h, w_h)_h := Q_N(v_h \overline{w}_h) \tag{7}$$

on the linear space W_h of mesh-functions v_h and w_h, which are functions on the set of mesh-points

$$\pi_h' := \{x_{k,j},\ k = 0, \dots, N-1,\ j = 1, \dots, J\}.$$

The inner product in (7) can be thought of as an approximation to

$$(v, w)_0 := \int_0^1 v(x)\overline{w}(x)\, dx \quad \text{for } v, w \in L^2(\mathbb{T}).$$

In the next section we give conditions for $(\cdot,\cdot)_h$ to be an inner product on $S_h^{r,M}$.

We choose splines of order r as trial space and splines of a possibly different order r' as test space. The qualocation method for approximately solving the equation $Lu = f$ is to find $u_h \in S_h^{r,M}$ such that

$$(Lu_h, z_h)_h = (f, z_h)_h \quad \text{for all } z_h \in S_h^{r',M}. \tag{8}$$

4 The spline space $S_h^{r,M}$ in qualocation

For the operator formulation of the qualocation equations the so-called qualocation projection $R_h : W_h \to S_h^{r,M}$ is needed which is defined by

$$(R_h v_h, \psi_h)_h = (v_h, \psi_h)_h \quad \text{for } \psi_h \in S_h^{r,M}.$$

It is not trivial that R_h is well-defined, i.e. that $(\cdot,\cdot)_h$ is an inner product on $S_h^{r,M}$, and in the following subsection we give criteria for this to be the case. Then in the next subsection we focus on the approximation power of R_h.

4.1 The qualocation projection R_h

The analysis of the approximation power and, more general, of the whole topic depends on Fourier techniques. An important role in this analysis plays a suitable spline basis. In the case of $S_h^{r,1}$, the space of smoothest splines of order r, a basis was found by Chandler & Sloan [3]:

$$\psi_\mu(x) := \sum_{j=1}^{N} \exp(i2\pi\mu x) b_j(x) \quad \text{for } \mu \in \Lambda_h := \Big(-\frac{N}{2}, \frac{N}{2} \Big] \cap \mathbb{Z},$$

where $\{b_j\}$ is the B-spline basis in $S_h^{r,1}$. A nice thing about the basis is that the qualocation equations for the principal part L_0 become diagonal if L_0 has constant coefficients.

The situation with M-fold knots, $M > 1$, is more delicate. In their collocation analysis McLean & Prößdorf [11] used the following characterization of splines.

Lemma 1 *$v \in S_h^{r,M}$ iff there exist trigonometric polynomials a_j such that*

$$m^r \hat{v}(m) = \sum_{j=0}^{M-1} m^j a_j(mh) \quad \textit{for } m \in \mathbb{Z}.$$

Working with this characterization makes the analysis uncomfortable. It was a step forward when a basis in $S_h^{r,M}$ was found in [7] which extends the one in [3]. Define

$$\begin{aligned}
\tilde{\Delta}_k(\xi, y) &:= \sum_{\ell \neq 0} \frac{\ell^{k-1}}{(y+\ell)^r} \Phi_\ell(\xi) \quad \text{for } |y| \le \frac{1}{2} \text{ and } \xi \in \mathbb{R},\\
\Phi_\ell(\xi) &:= \exp(i2\pi\ell\xi) \quad \text{for } \ell \in \mathbb{Z} \text{ and } \xi \in \mathbb{R},\\
\Delta_1(\xi, y) &:= 1 + y^r \tilde{\Delta}_1(\xi, y), \quad \Delta_k(\xi, y) := \tilde{\Delta}_k(\xi, y) \quad \text{for } k = 2, \dots, M,\\
\psi_{k,\mu}(x) &:= \Phi_\mu(x) \Delta_k\Big(Nx, \frac{\mu}{N}\Big) \quad \text{for } k = 1, \dots, M \text{ and } \mu \in \Lambda_h.
\end{aligned}$$

Then $\{\psi_{k,\mu}\}$ is a basis in $S_h^{r,M}$. The use of this basis makes the qualocation equations for the principal part L_0 block diagonal with blocks of size M if the coefficients are constant.

With the aid of $\{\psi_{k,\mu}\}$ it can be characterized whether $(\cdot,\cdot)_h$ is definite on $S_h^{r,M}$.

Proposition 1 *$(\cdot,\cdot)_h$ defines an inner product on $S_h^{r,M}$ iff the functions $\{\Delta_k(\cdot, y), k = 1,\ldots,M\}$ restricted to the quadrature points $\{\xi_j, k = 1,\ldots,J\}$ are linearly independent for $y = \mu/N$ and $\mu \in \Lambda_h$.*

We say that Condition (R) is satisfied if the condition in Proposition 1 holds for all $|y \leq 1/2|$. It is known that Condition (R) holds in the following cases.

- [3]: $M = 1$

  (R) holds unless $J = 1$ and $\begin{cases} \xi_1 = \frac{1}{2} & \text{if } r \text{ is even,} \\ \xi_1 = 0 & \text{if } r \text{ is odd.} \end{cases}$

- [11]: $J = M = 2$

  $\xi_1 = 0, \xi_2 = \frac{1}{2}$: (R) holds iff r is odd,

  $\xi_1 = \varepsilon, \xi_2 = 1 - \varepsilon$ with $\varepsilon \in (0, \frac{1}{2})$: (R) fails if r is odd.

- [7], [12]: $J = M = 2$

  $\xi_1 = \varepsilon, \xi_2 = 1 - \varepsilon$ with $\varepsilon \in (0, \frac{1}{2})$: (R) holds iff r is even.

- [6]: $J = M = 3$

  $\xi_1 = 0, \xi_2 = \varepsilon, \xi_3 = 1 - \varepsilon$ with $\varepsilon \in (0, \frac{1}{2})$: (R) holds iff r is even,

  $\xi_1 = \varepsilon, \xi_2 = \frac{1}{2}, \xi_3 = 1 - \varepsilon$ with $\varepsilon \in (0, \frac{1}{2})$: (R) holds iff r is odd,

- [6]: for all J, M:

  (R) holds if $J > M$.

In the remaining part of the paper it is assumed that Condition (R) and Condition (R$'$) (this is Condition (R) with r replaced by r') hold.

4.2 Approximation power of R_h

The approximation power of the qualocation projection R_h proved in [7] is the content of the next proposition.

Proposition 2 *Let $0 \leq s < r - M + \frac{1}{2}, s \leq t \leq r, t > \frac{1}{2}$. Then*

$$\|R_h v - v\|_s \leq C h^{t-s} \|v\|_t \quad \text{for } v \in H^t.$$

An explanation for the given range of indices may be helpful.

- $s < r - M + \frac{1}{2}$: the limited smoothness of the spline $\psi \in S_h^{r,M}$ implies $\psi \in H^s$ for $s < r - M + \frac{1}{2}$ only,

- $t \leq r$: the maximal t allowed, giving the highest error order, is determined by the order of the splines,

- $t > \frac{1}{2}$: the definition of $R_h v$ requires pointwise evaluation of v which is not well-defined for $t \leq \frac{1}{2}$ since then $H^t \not\hookrightarrow C(\mathbb{T})$.

5 Principle error estimate

The application of L_0 to the spline basis leads to the functions (see [8])

$$\begin{aligned}
\tilde{\Omega}_k(\xi, y; x) &:= \sum_{\ell \neq 0} \sigma_0(x, y+\ell) \frac{\ell^{k-1}}{(y+\ell)^r} \Phi_\ell(\xi) \quad \text{for } k = 1, \ldots, M, \\
\Omega_1(\xi, y; x) &:= 1 + (\sigma_0(x, y))^{-1} y^r \tilde{\Omega}_1(\xi, y; x) \quad \text{for } y \neq 0, \\
\Omega_k(\xi, y; x) &:= \tilde{\Omega}_k(\xi, y; x) \quad \text{for } k = 2, \ldots, M.
\end{aligned}$$

We omit the argument x if L_0 has constant coefficients. The stability of the qualocation method is connected with the ellipticity of the numerical symbol $D(y; x)$, which is a $M \times M$-matrix with elements

$$[D(y; x)]_{k,\ell} := Q\big(\Omega_\ell(\cdot, y; x), \Delta'_k(\cdot, y)\big), \quad Q(v, w) := \sum_{j=1}^{J} \omega_j (v\overline{w})(\xi_j).$$

The numerical symbol is encountered as the coefficient matrix in the linear system of qualocation equations if L_0 has constant coefficients. Ellipticity of D means that $D(y; x)$ is invertible for $|y| \leq 1/2$ and $x \in \mathbb{T}$.

Theorem 1 *Let L be elliptic and injective. Assume that D is elliptic and that*

$$\beta + M < r, \; s < r - M + \frac{1}{2}, \; \beta + \frac{1}{2} < t, \; \beta \leq s \leq t \leq r.$$

Then the qualocation equations have a unique solution u_h for $h \in \mathcal{H}_1$ satisfying

$$\|u - u_h\|_s \leq C h^{t-s} \|u\|_t \quad \text{if } u \in H^t.$$

The condition $\beta + M < r$ ensures the absolute convergence of the series defining $\tilde{\Omega}_k$.

The theorem has been proved under varying assumptions.

- [15]: constant coefficients, $L \equiv L_0$, even symbol, $M = 1$,
- [3]: constant coefficients, even or odd symbol, $M = 1$,
- [18]: constant coefficients, symbol that may be neither even nor odd, $M = 1$,
- [11]: variable coefficients, collocation, multiple knots,
- [19]: variable coefficients, strongly and oddly elliptic L, $M = 1$,
- [8]: variable coefficients, multiple knots.

The proof for variable coefficient operators uses a localization technique. Such techniques are known from PDEs but are considerably more intrigued to apply for integral operators which themselves are non-local. The underlying abstract result is due to Prößdorf [13]. Basic tools for applying Prößdorf's result are the following superapproximation from [4] and commutator property from [8]. Both provide an AOC with order 1.

Proposition 3 *Let $g \in C^r(\mathbb{T})$ and $M < r, 0 \le s < r - M + \frac{1}{2}, t \le r - M$. Then*

$$\|(I - R_h)(gv_h)\|_s \le Ch^{1+t-s}\|g'\|_{r-1,\infty}\|v_h\|_t \quad \text{for } v_h \in S_h^{r,M}.$$

Proposition 4 *Let $g \in C^r(\mathbb{T})$ and $M < r$, $0 \le s < r - M + \frac{1}{2}, \frac{1}{2} < t \le r$. Then*

$$\|R_h g(I - R_h)v\|_s \le Ch^{1+t-s}\|g'\|_{r-1,\infty}\|v\|_t \quad \text{for } v \in H^t.$$

6 Additional order of convergence

The highest error order in the principle convergence theorem is

$$\|u - u_h\|_\beta \le Ch^{r-\beta}\|u\|_r \quad \text{if } u \in H^r.$$

For example, if L_0 is the single-layer potential, where $\beta = -1$, and choosing continuous linear splines as trial space, this means order $r = 2$ and $M = 1$, then

$$\|u - u_h\|_{-1} \le Ch^3\|u\|_2 \quad \text{if } u \in H^2.$$

This convergence order is disappointing when compared to the highest order negative norm error bound for the Galerkin solution u_h^G (see [9]),

$$\|u - u_h^G\|_{-3} \le Ch^5\|u\|_2 \quad \text{if } u \in H^2.$$

For the qualocation method progress to catch up with the order 5 was made by Sloan [15] who showed that with specially designed quadrature rules one can obtain the same error order,

$$\|u - u_h\|_{-3} \le Ch^5\|u\|_4 \quad \text{if } u \in H^4. \tag{9}$$

The estimate requires the higher regularity $u \in H^4$ compared to $u \in H^2$ for the Galerkin method. This disadvantage was overcome in [17] with the tolerant version of qualocation, where the inner product on the right-hand side is evaluated exactly.

6.1 Constant coefficient L_0

The AOC result (9) is proved by the principle of parameter selection for cancelling the leading error term. A key idea of the proof is the following. In [3] the asymptotic error expansion for the operator $L = L_\beta^+$ or $L = L_\beta^-$ is obtained by Fourier transform in the form

$$\hat{u}(\mu) - \hat{u}_h(\mu) = D\Big(\frac{\mu}{N}\Big)^{-1} E\Big(\frac{\mu}{N}\Big)\hat{u}(\mu) + \text{higher order terms in } \Big(\frac{\mu}{N}\Big)$$

for $\mu \in \Lambda_h \backslash \{0\}$, where the symbol of L_α^+ and L_α^- is $\sigma_0 = |\xi|^\alpha$ and $\sigma_0 = \operatorname{sign} \xi \, |\xi|^\alpha$, respectively, and

$$E(y) := \sum_{j=1}^{J} \omega_j \big(\Omega_1(\xi_j, y) - 1\big) \overline{\Delta_1'(\xi_j, y)} \quad \text{for } |y| \le \frac{1}{2}.$$

Note that due to the numerical ellipticity $\left|D(\frac{\mu}{N})^{-1}\right| \le C$. For any choice of the quadrature rule the function E behaves like

$$E(y) = \mathcal{O}(|y|^{r-\beta}) \quad \text{as } y \to 0.$$

The qualocation method (in the case $M = 1$ considered here) is said to have additional order $b > 0$ (see [15]) if

$$E(y) = \mathcal{O}(|y|^{r-\beta+b}) \quad \text{as } y \to 0. \tag{10}$$

The functions Ω_1 and Δ_1' are given if L_0 and the spline spaces are fixed. One can try to obtain an additional order by selecting the quadrature rule appropriately. Sloan showed that the choice

$$\xi_1 = 0,\ \xi_2 = \frac{1}{2},\ \omega_1 = \frac{3}{7},\ \omega_1 = \frac{4}{7}$$

combined with linear continuous splines as test and trial space gives $b = 2$ for the single-layer equation. Note that the quadrature points are from the trapezoidal rule but not the weights.

In the case of multiple knot splines the condition for additional order $b > 0$ from [5] is

$$\sum_{k=1}^{M} D(y)^{-1}_{1,k} Q\big(\tilde{\Omega}_1(\cdot, y), \Delta_k'(\cdot, y)\big) = \mathcal{O}(|y|^b) \quad \text{as } y \to 0, \tag{11}$$

which is in the case $M = 1$ and $b \le r'$ equivalent to Condition (10) and in the case $J = M$ to Condition (2.12) in [11].

The general AOC result for constant coefficient L_0 is stated in the following theorem.

Theorem 2 *Let $L = L_0 + K : H^\beta \to H^0$ be elliptic and injective, where L_0 has constant coefficients and K maps $H^q \to H^{q-\beta+b}$ boundedly for $q \in \mathbb{R}$. Assume that D is elliptic, that Condition (11) holds and that*

$$\beta + M < r,\ M \le r',\ s < r - M + \frac{1}{2},\ s \le t \le r,\ \beta - b \le s \le \beta < t - \frac{1}{2}.$$

Then the qualocation equations have a unique solution u_h for $h \in \mathcal{H}_1$ satisfying

$$\|u - u_h\|_s \le C h^{t-s} \|u\|_{t-s+\beta} \quad \text{if } u \in H^{t-s+\beta}.$$

The theorem has been proved in varying settings.

- [15]: even symbol, $L \equiv L_0$, $M = 1$, quadrature rule of Simpson type,
- [3]: even or odd symbol, $M = 1$,
- [11]: collocation, i.e. $M = J$, multiple knots,
- [18]: qualocation, symbol may be neither even nor odd, $M = 1$,
- [5]: qualocation, multiple knots.

6.2 *Variable coefficients*

The analysis for variable coefficient operators L_0 is technically considerably more involved than for constant coefficients. In the case of variable coefficients AOC has been proved for smoothest splines in [19]. The authors assume that $L = L_0 + L_1 + K$, where

$$(12) \qquad L_1 = \sum_{i=1}^{b-1} \Big(a_i^+(x) L_{\beta-i}^+ + a_i^-(x) L_{\beta-i}^- \Big), \quad K : H^q \to H^{q-\beta+b+\nu} \text{ boundedly}$$

for $q \in \mathbb{R}$ and some $\nu > 1/2$. The basic assumptions in [19, Th. 4 and Th. 5] are conditions for the quadrature rule, which is supposed to be symmetric and to integrate certain functions exactly (see Lemma 4). In [5] multiple knot splines are considered for the same class (12) of ψdos and the following conditions for AOC in the spirit of [3] are given:

$$(13) \qquad \big| Q\big(\tilde{\Omega}_k(\cdot, y), 1\big) \big| \le C|y|^b \;\text{ as }\; y \to 0 \;\text{ for } k = 1, \dots, M$$

with $b \in \mathbb{N}$ satisfying $\beta - s \le b \le \min(r', r - \beta)$, where $\tilde{\Omega}_k$ has to be taken for L_β^+ and L_β^-; additionally,

$$(14) \qquad \big| Q\big(1, \tilde{\Delta}_1'(\cdot, y)\big) \big| \le C|y|^{r-\beta+b-r'} \;\text{ as }\; y \to 0,$$

$$(15) \qquad \big| Q\big(1, \tilde{\Delta}_k'(\cdot, y)\big) \big| \le C|y|^{r-\beta} \;\text{ as }\; y \to 0 \;\text{ for } k = 2, \dots, M.$$

It may be helpful for interpreting the conditions in (13) - (15) to hint to the fact that $Q\big(\tilde{\Omega}_k(\cdot, y), 1\big)$ can be considered as the result of the quadrature rule Q applied to the integral $(\tilde{\Omega}_k(\cdot, y), 1)_0$, which vanishes as is immediately seen from the definition of $\tilde{\Omega}_k$. Also $(1, \tilde{\Delta}_k'(\cdot, y))_0 = 0$.

Theorem 3 *Let $L = L_0 + L_1 + K$ satisfy (12) with $\nu = 0$ and be elliptic and injective. Assume also that D is elliptic, that Conditions (13) - (15) hold and that*

$$\beta + M < r,\; M \le r',\; s < r - M + \frac{1}{2},\; s \le t \le r,\; s \le \beta < t - \frac{1}{2},$$

$$\beta - s \le b \le \min(r', r - \beta).$$

Then the qualocation equations have a unique solution u_h for $h \in \mathcal{H}_1$ satisfying

$$\|u - u_h\|_s \le C h^{t-s} \|u\|_{t-s+\beta} \quad \text{if } u \in H^{t-s+\beta}.$$

The operator L_1 in (12) is said to be even (odd) if $a_i^- = 0$ ($a_i^+ = 0$) for $i = 1, \dots, b$.

Remark 1 *If L_0 and L_1 are both even or odd then it is sufficient that the qualocation method has strong additional order b of convergence to require (13) for L_β^+ or L_β^- only, respectively.*

6.3 Negative norm estimates are useful

A word on the significance of the negative norm estimates in the context of boundary integral equations may be in order (see [16]). As described in Subsection 2.2, for a given point $X_0 \in \Omega$ the solution of the boundary value problem

$$U_{XX} + U_{YY} = 0 \text{ in } \Omega \subset \mathbb{R}^2, \quad U = F \text{ on } \Gamma,$$

can be written in the form

$$U(X_0) = \frac{1}{\pi} \int_0^1 \frac{1}{\log|X_0 - \gamma(y)|} u(y)\, dy,$$

where u is from (5). The approximation

$$U_h(X_0) = \frac{1}{\pi} \int_0^1 \frac{1}{\log|X_0 - \gamma(y)|} u_h(y)\, dy$$

satisfies the error bound

$$\begin{aligned} |U(X_0) - U_h(X_0)| &= \frac{1}{\pi} (\log|X_0 - \gamma|, u - u_h)_0 \\ &\le C \|\log|X_0 - \gamma|\|_t \|u - u_h\|_{-t} \quad \text{for } t \in \mathbb{R}. \end{aligned}$$

Thus, the higher the order of $\|u - u_h\|_{-t}$ the higher the error order $(U - U_h)(X_0)$ since $\log|X_0 - \gamma| \in H^t$ for all $t \in \mathbb{R}$.

7 Symmetric quadrature rules

A basic quadrature rule Q satisfying the condition that if $\xi \in (0, \frac{1}{2})$ is a quadrature point then so is $(1 - \xi)$ with the same weight ω is called symmetric. In the case of smoothest splines in [18] and [19] exactness conditions for symmetric quadrature rules are given for AOC to hold. In this section we extend these conditions to multiple knot splines. We always assume that

$$\beta + M < r \quad \text{and} \quad M \le r'.$$

We need the following functions G_α for $\alpha > 0$ and $\xi \in (0,1)$ which have been studied in [2]:

$$G_\alpha(\xi) := 2\sum_{\ell=1}^{\infty} \frac{1}{\ell^\alpha}\cos 2\pi\ell\xi.$$

For symmetric quadrature rules the Conditions (13) - (15) can be further elaborated. Recall that

$$\tilde{\Omega}_k(\xi, y) = \sum_{\ell\neq 0} \sigma_0(y+\ell)\frac{\ell^{k-1}}{(y+\ell)^r}\Phi_\ell(\xi) \quad \text{for } k = 1, \dots, M,$$

where we consider these functions for the operators L_β^+ and L_β^-. As shown in [3], $\tilde{\Omega}_k(\xi,\cdot)$ has a Taylor expansion with respect to $y = 0$, where the coefficients for the real part are easily determined to be equal to

$$\begin{aligned} c_{k,m}(\xi) := \frac{1}{m!}\mathcal{R}e\frac{\partial^m\tilde{\Omega}_k(\xi,0)}{\partial y^m} &= \binom{\beta-r}{m}\sum_{\ell\neq 0}\sigma_0(\ell)\ell^{k-1-r-m}\cos 2\pi\ell\xi \\ &= \binom{\beta-r}{m}\sum_{\ell>0}\ell^{k-1-r-m}(\sigma_0(\ell)+\sigma_0(-\ell)(-1)^{k-1-r-m})\cos 2\pi\ell\xi \end{aligned}$$

for $m \in \mathbb{N}_0$ and $\sigma_0(\xi) = |\xi|^\beta$ or $\sigma_0(\xi) = \operatorname{sign}\xi\,|\xi|^\beta$. It follows that

$$c_{k,m} = \begin{cases} G_{r-\beta-k+1+m} & \text{if } \sigma_0 \text{ and } r-k+1+m \text{ have like parity,} \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

In the next two lemmas we give the Conditions (13) - (15) another form.

Lemma 2 *If L_0 is even (odd) then Condition (13) is equivalent to*

$$\sum_{j=1}^{J}\omega_j G_{r-\beta+\ell}(\xi_j) = 0 \quad \textit{for even (odd) } \ell \in [-M+1, b-1] \tag{17}$$

if σ_0 and r have like (opposite) parity. If L_0 is neither even nor odd then (13) is equivalent to the equation in (17) for all even and odd $\ell \in [-M+1, b-1]$.

P r o o f . Since $\mathcal{I}m\,\tilde{\Omega}_k(1-\xi, y) = -\mathcal{I}m\,\tilde{\Omega}_k(\xi, y)$ and $\mathcal{I}m\,\tilde{\Omega}_k(0, y) = 0$ (needed in the case $\xi_1 = 0$) it follows by virtue of the symmetry of Q that $Q(\mathcal{I}m\,\tilde{\Omega}_k(\cdot, y), 1) = 0$ and Condition (13) is equivalent to

$$Q(\mathcal{R}e\,\tilde{\Omega}_k(\cdot, y), 1) = \mathcal{O}(|y|^b) \quad \text{as } y \to 0 \text{ for } k = 1, \dots, M.$$

This relation holds iff the coefficients of y^m in the Taylor series of $Q(\mathcal{R}e\,\tilde{\Omega}_k(\cdot,y),1)$, given by $Q(c_{k,m},1)$, vanish for $m<b$. If σ_0 and r have like parity then, in view of (16), this is equivalent to $-k+1+m$ to be even and $Q(G_{r-\beta-k+1+m},1)=0$ for $m=0,\ldots,b-1$. Since $k\in[1,M]$ the equivalence of (13) with (17) is proved. If σ_0 and r have opposite parity the proof is similar. The last assertion is then implied. $\square$

Lemma 3 *Condition (14) is equivalent to*

$$(18) \qquad \sum_{j=1}^{J}\omega_j G_\ell(\xi_j)=0 \quad \textit{for even } \ell\in[r',r-\beta+b-1]$$

and Condition (15) is void if $M=1$ and if $M>1$ equivalent to

$$(19) \qquad \sum_{j=1}^{J}\omega_j G_\ell(\xi_j)=0 \quad \textit{for even } \ell\in[-M+1+r',r-\beta+r'-2].$$

P r o o f . Note that $\tilde{\Delta}'_k$ is obtained as a special case of $\tilde{\Omega}_k$ for the (even) operator L^+_β with $\beta=0$ and r replaced by r'. Taking (16) into account it is seen that (14) is equivalent to $Q(1,G_{r'+m})=0$ for even $r'+m$ and $m=0,\ldots,r-\beta+b-r'-1$. Similarly, (15) is equivalent to $Q(1,G_{r'-k+1+m})=0$ for even $r'-k+1+m$ and $m=0,\ldots,r-\beta-1$ and the equivalence of (19) follows. $\square$

Sufficient conditions for (17) - (19) can be derived by noting that G_α is for even α a multiple of the Bernoulli polynomial B_α (see [3]). From this observation the next corollary follows easily from Lemmas 2 and 3. In its formulation we use the notation of an extended symmetric quadrature formula Q. By this we mean a modification of Q, which is symmetric for periodic functions only, into a general symmetric formula $\tilde{Q}$. The modification is necessary only in the case that $\xi_1=0$. To obtain $\tilde{Q}$ the additional quadrature point $\xi_{J+1}:=1$ is introduced with weight $\omega_{J+1}:=\omega_1/2$ and the weight for $\xi_1=0$ is changed to be also equal to $\omega_1/2$.

Corollary 1 *Let σ_0 and r have like parity and $r-\beta$ be even or let σ_0 and r have opposite parity and $r-\beta$ be odd. Then the conditions (17) - (19) are satisfied if the extended symmetric quadrature rule Q has at least order $2q$ of exactness, where $q=[(r-\beta+b-1)/2]$ unless $M>1$ and $b<r'-1$, where $q=[(r-\beta+r'-2)/2]$. Here $[x]$ denotes truncation of x to the next integer not larger than x.*

In the case of a general operator L observe that by our index assumptions we have $r'\geq 1, -M+1+r'\geq 1$ and $r-\beta>0$ and the following corollary can be derived from Lemmas 2 and 3.

Corollary 2 *If the symmetric quadrature formula Q satifies*

$$\sum_{j=1}^{J} \omega_j B_\ell(\xi_j) = 0 \quad \textit{for even } \ell \in [2, r-\beta+b-1], \tag{20}$$

$$\sum_{j=1}^{J} \omega_j G_\ell(\xi_j) = 0 \quad \textit{for odd } \ell \in [r-\beta-M+1, r-\beta+b-1] \tag{21}$$

and, additionally, if $M > 1$ and $b < r' - 1$

$$\sum_{j=1}^{J} \omega_j B_\ell(\xi_j) = 0 \quad \textit{for even } \ell \in [r-\beta+b, r-\beta+r'-2] \tag{22}$$

then Conditions (17) - (19) hold true for general variable coefficient operators L.

If $M = 1$ these conditions coincide with (1.15) and (1.20) in [19].

In [18] a list of symmetric quadrature formulas with various exactness properties is provided. In the following table we collect those formulas which satisfy the conditions of Corollary 1 for certain choices of the parameters and, additionally, the formulas from Table 2. We keep the notation in [18]. A useful information for us is the first index indicating the number J.

M	$r-\beta$	b	r'	Formula
2	3	1	2	$G_{3,2,2}, L_{3,2,2}$
2	4	1	2	$G_{4,3,2}, L_{4,3,2}$
2	3	2	2,3	$G_{4,3,2}, L_{4,3,2}$
3	4	1	3	$G_{4,3,2}, L_{4,3,2}$
3	4	2	3	$G_{5,3,3}, L_{5,3,3}$

Table 1.— Quadrature formulas from [18] and Table 2 providing additional order b of convergence for general variable coefficient operators L

As an example how to determine the parameters of quadrature formulas like in Table 1 we derive Formula $G_{5,3,3}$ by an application of Corollary 2. With the parameters given in the last line in Table 1 Condition (20) is satisfied if all even polynomials of degree $\ell \in [2, 5]$ are integrated exactly (here we took into account that due to the normalization (6) constant functions are always integrated exactly). By applying the formulas to the polynomials $(\xi - 0.5)^2$ and $(\xi - 0.5)^4$ these two conditions take the form

$$\omega_1(2\xi_1 - 1)^2 + \omega_2(2\xi_2 - 1)^2 = 1/6,$$
$$\omega_1(2\xi_1 - 1)^4 + \omega_2(2\xi_2 - 1)^4 = 1/10,$$

where the symmetry relations $\omega_5 = \omega_1, \omega_4 = \omega_2, \xi_5 = 1 - \xi_1, \xi_4 = 1 - \xi_2, \xi_3 = 0.5$ were taken into account. The index ℓ in Condition (21) is odd and runs in $[2, 5]$ providing the two further equations

$$2\omega_1(G_n(\xi_1) - G_n(0.5)) + 2\omega_2(G_n(\xi_2) - G_n(0.5)) = G_n(0.5) \text{ for } n = 3, 5.$$

Condition (22) is void. Solving numerically the equations for the unknowns $\omega_1, \omega_2, \xi_1$ and ξ_2 yields the parameters for $G_{5,3,3}$ in Table 2. The parameters for $L_{5,3,3}$ are obtained similarly.

J	ξ_j	ω_j	Rule name
5	0.03675444410510	0.09796641612174	$G_{5,3,3}$
	0.20980173750308	0.24512752237399	
	0.5	0.31381212300853	
	0.79019826249692	0.24512752237399	
	0.96324555589490	0.09796641612174	
5	0.0	0.04767138349495	$L_{5,3,3}$
	0.09758560632523	0.17451387385978	
	0.34287284360121	0.30165043439274	
	0.65712715639879	0.30165043439274	
	0.90241439367477	0.17451387385978	

Table 2.— Quadrature formulas providing for $M = 3$ additional order $b = 2$ of convergence

Remark 2 *The stability of the formulas from [18] has been numerically checked there for strongly and oddly elliptic operators with integer $\beta \in [-1, 1]$. For some of the rules stability was proved analytically in [14].*

We conclude this section with some remarks concerning constant coefficient operators L_0 and the collocation method.

Lemma 4 *Let $M = 1$, $b \le r'$ and D be elliptic. Then Condition (11) to hold for both L_β^+ and L_β^- is equivalent to*

$$(23) \qquad \sum_{j=1}^{J} \omega_j G_{r-\beta+\ell}(\xi_j) = 0 \quad \textit{for } \ell = 0, \ldots, b-1,$$

and thus is identical with [18, Condition (4.13)] for AOC of order b.

The lemma follows from the observation that Condition (11) for $M = 1$ and $b \le r'$ is equivalent to

$$Q(\tilde{\Omega}_1(\cdot, y), 1) = \mathcal{O}(|y|^b) \quad \text{as } y \to 0,$$

which, as in Lemma 2, is equivalent to (23).

For collocation with double knot splines the following conditions for AOC have been given in [11].

Lemma 5 *Assume that the symbol (3) has constant coefficients satisfying $a^+ = 0$ or $a^- = 0$ and that $M = 2$. If the quadrature points are*

$$\xi_1 = 0, \xi_2 = \frac{1}{2} \quad \text{and } \sigma_0 \text{ and } r \text{ have opposite parity,} \tag{24}$$

or

$$\xi_1 = \epsilon, \xi_2 = 1 - \epsilon \quad \text{and } \sigma_0 \text{ and } r \text{ have like parity, where } G_{r-\beta}(\epsilon) = 0, \tag{25}$$

then (11) holds with $b = 1$ or $b = 2$, respectively.

It is shown in [3] that $G_{r-\beta}$ has a unique zero in $(0, 1/2)$.

P r o o f . For $J = M$ the collocation method is a special case of the qualocation method if Condition (R′) is satisfied. With the quadrature points in Lemma 5 and the choice $r' = 3$ or $r' = 2$ in the case of (24) or (25), respectively, (R′) has been proved in [7, Prop. 5.1 and 5.2].

In both cases the quadrature rules are symmetric. The numerical symbol D is elliptic, which follows from a slight generalization of [7, Prop. 5.1 and 5.2] in combination with [8, Lemma 3.1] (in the case of opposite parity also from [11, Lemma 5.1]). The conditions

$$Q\big(\tilde{\Omega}_1(\cdot, y), 1\big) = \mathcal{O}(|y|^b) \quad \text{and} \quad D^{-1}_{1,2}(y) Q\big(\tilde{\Omega}_1(\cdot, y), \Delta'_2(\cdot, y)\big) = \mathcal{O}(|y|^b) \tag{26}$$

are seen to be sufficient for (11), where for the first condition the boundedness of D^{-1} and the form $\Delta'_1(\cdot, y) = 1 + y^{r'} \tilde{\Delta}'_1(\cdot, y)$ with $r'(\ge 2) \ge b$ was taken into account. Consider the case of like parity. With the aid of (16) we conclude for the Taylor coefficients $c_{1,m}(\xi)$ of $\mathcal{R}e\, \tilde{\Omega}_1(\xi, \cdot)$ that $c_{1,0}(1 - \epsilon) = c_{1,0}(\epsilon) = G_{r-\beta}(\epsilon) = 0$. The relation (16) yields also $c_{1,1} = 0$. Consequently, $\mathcal{R}e\, \tilde{\Omega}_1(\xi_1, y) = \mathcal{R}e\, \tilde{\Omega}_1(\xi_2, y) = \mathcal{O}(|y|^2)$ and the first relation in (26) holds with $b = 2$. In the case of opposite parity, (16) yields $c_{1,0} = 0$ and, consequently, $\mathcal{R}e\, \tilde{\Omega}_1(\xi, y) = \mathcal{O}(|y|)$. Thus the first relation in (26) holds with $b = 1$.

For the proof of the second relation in (26) first note that, due to

$$\mathcal{R}e\, \tilde{\Omega}_1(1 - \xi, \cdot) = \mathcal{R}e\, \tilde{\Omega}_1(\xi, \cdot), \quad \mathcal{I}m\, \tilde{\Omega}_1(1 - \xi, \cdot) = -\mathcal{I}m\, \tilde{\Omega}_1(\xi, \cdot), \quad \mathcal{I}m\, \tilde{\Omega}_1(0, \cdot) = 0,$$

the corresponding relations for Δ'_2 and the symmetry of the quadrature formulas, we have

$$D^{-1}_{1,2}(y) Q\big(\tilde{\Omega}_1(\cdot, y), \Delta'_2(\cdot, y)\big) = D^{-1}_{1,2}(y) Q\Big(\mathcal{R}e\, \big(\tilde{\Omega}_1(\cdot, y) \overline{\Delta'_2(\cdot, y)}\big)\Big)$$

$$= D_{1,2}^{-1}(y)Q\big(\mathcal{R}e\,\tilde\Omega_1(\cdot,y)\mathcal{R}e\,\Delta_2'(\cdot,y)\big) + D_{1,2}^{-1}(y)Q\big(\mathcal{I}m\,\tilde\Omega_1(\cdot,y)\mathcal{I}m\,\Delta_2'(\cdot,y)\big). \tag{27}$$

¿From the first part of the proof follows that the first term in (27) has the correct order since $D_{1,2}^{-1}$ is bounded. For the second term note that in the case of opposite parity $\mathcal{I}m\,\tilde\Omega_1(\xi_1,0) = \mathcal{I}m\,\tilde\Omega_1(\xi_2,0) = 0$ (see (28)), which implies order $b = 1$ of the second term in (27). Consider the case of like parity. It is not difficult to check that for $k = 1, 2$

$$\mathcal{R}e\,\tilde\Omega_k(\cdot,-y) = (-1)^{k-1}\mathcal{R}e\,\tilde\Omega_k(\cdot,y), \quad \mathcal{I}m\,\tilde\Omega_k(\cdot,-y) = (-1)^k\mathcal{I}m\,\tilde\Omega_k(\cdot,y). \tag{28}$$

The functions Δ_k' satisfy the same relations since their symbol $\sigma_0 = |\xi|^0$ and $r' = 2$ have like parity. Then calculating the matrix element $D_{1,2}^{-1}$ with Cramer's rule it is seen to be odd. It follows also from (28) that $\mathcal{I}m\,\tilde\Omega_1(\xi,0) = 0$ and that $Q\big(\mathcal{I}m\,\tilde\Omega_1(\cdot,y)\mathcal{I}m\,\Delta_2'(\cdot,y)\big)$ is odd with respect to y. Thus the second term in (27) is even and vanishes for $y = 0$ implying the required order $b = 2$. □

Remark 3 *Due to the general assumption $\beta + M < r$, which is equally made in [18] (case $M = 1$), [11] and [5] it is not allowed in the case $M = 2$ to set $r - \beta = 2$ although the quantities in Lemmas 2 and 3 are well-defined for this choice and would lead to the same additional order $b = 1$ or $b = 2$ as in Lemma 5 for even or odd variable coefficient operators L (with weights $\omega_1 = 1/3, \omega_2 = 2/3$ for the rule (24)). The reasons for this restriction may be of technical nature.*

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