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Generalized van der Waals 4-D oscillator. Invariant tori and relative equilibria in $\Xi = L = 0$ surface

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Abstract

An uniparametric 4-DOF Hamiltonian family of perturbed oscillators in 1:1:1:1 resonance is studied. The model includes some classical cases, in particular Zeeman and the van der Waals systems. First several invariant manifolds are identified. Normalization by Lie-transforms (only first order is considered here) as well as geometric reduction related to the invariants associated to the symmetries is used, based on previous work of the authors. More precisely we find that crossing two of the integrable cases, $\beta = 1/2$ and 1, the family undergoes degenerate Hopf bifurcations, which at first order shows up as a center-cusp bifurcation. Higher order normalization and singularity analysis is needed, in order to fully describe the dynamics around those integrable cases.

1 Introduction

Continuing previous work [7, 8],[6] and [9] on perturbed isotropic oscillators in four dimensions (harmonic oscillators in 1:1:1:1 resonance), we consider in $\mathbb{R}^8$, the uniparametric family of Hamiltonian systems defined by

$$\mathcal{H}_\beta(Q, q) = \mathcal{H}_2 + \varepsilon\, \mathcal{H}_6 \tag{1}$$

where

$$\mathcal{H}_2(Q, q) = \frac{1}{2}(Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2) + \frac{1}{2}\omega^2(q_1^2 + q_2^2 + q_3^2 + q_4^2) \tag{2}$$

is the 4-D isotropic oscillator,

$$\mathcal{H}_6(-,q;\beta) = \left(q_1^2+q_2^2+q_3^2+q_4^2\right)\left(\beta^2\left(q_1^2+q_2^2-q_3^2-q_4^2\right)^2+4\left(q_1^2+q_2^2\right)\left(q_3^2+q_4^2\right)\right)$$

and ε is an small parameter $\varepsilon \ll 1$; in what follows we take $\omega = 1$.

The system defined by Hamiltonian function Eq. (1) has two first integrals in involution given by

$$I_1 = q_1Q_2 - Q_1q_2 + q_3Q_4 - Q_3q_4, \quad I_2 = q_3Q_4 - Q_3q_4 - q_1Q_2 + Q_1q_2, \tag{3}$$

associated to which we have 'rotational symmetries'. In order to refer to recent previous papers, we maintain the names $I_1 = \Xi$ and $I_2 = L_1$ given to the integrals.

Particular cases connected with problems of physical interest ($\Xi = 0$) are $\beta = 0$, the Zeeman effect, and $\beta = \sqrt{2}$, which corresponds to the Van der Waals effect (see Elipe and Ferrer [11] and references therein). For this reason we propose to name our system as *generalized Van der Waals 4-D oscillator*.

Notable values of the external parameter β are those cases of the family that define an *integrable system*. More precisely, in 3-D it is known that $\beta = \frac{1}{2}, 1, 2$ are those values (see Farrelly and Howard [13]; Ferrer and Mondéjar [16]); in fact one of them is even maximally superintegrable. In four dimensions, preliminary computations indicate that for each of those cases there is the corresponding extended integral [10].

This paper will concentrate in some aspects, within the long list of issues which lie behind the differential system proposed. In Section 2 we refer to some relevant invariant manifolds of our system. In Section 3 we give a quick look to the 3-torus reduction induced by the two symmetries and thr normalization. In Section 4 we collect the main features of the dynamics of the normalized system. For more details of the general case see Diaz *et al.* [6], in particular the Hopf bifurcations of relative equilibria. Note that only first order normalization has been used for our study. Although some results are valid for any value of ε, we will be more interested in the case when $\varepsilon << 1$, in order to compare with the analysis done with perturbation theory.

2 Some invariant manifolds

• We begin trying to identify *invariant sets* of our system. First, the origin is an *isolated equilibrium*. Moreover, a search for 'rectilinear solutions through the origin': $q(t) = \gamma(t)\,(a_1, a_2, a_3, a_4)$, ($a_i$ constants, $\sum a_i^2 = 1$) as solutions of the differential system defined by Eq. (1), leads to the conclusion that these rectilinear solutions in configuration space have to satisfy

$$\ddot{\gamma} + \gamma = -6\,\varepsilon\beta^2\,\gamma^5, \qquad a_3 = a_4 = 0. \tag{4}$$

and

$$\ddot{\gamma} + \gamma = -6\,\varepsilon\beta^2\,\gamma^5, \qquad a_1 = a_2 = 0. \tag{5}$$

i.e. rectilinear solutions in any direction in the planes Oq_1q_2 and Oq_3q_4. It is not difficult to check that there are other rectilinear solutions for particular values of β, related to the integrable cases. We will see later that among the solutions given by Eqs. (4)-(5) some correspond to the singular points of the thrice reduced orbit space. Thus, the Hopf bifurcations and their stability that we identify related to those points, are precisely connected with the evolution of these solutions.

• By direct computations from the differential Hamiltonian system defined by (1), we identify that the phase subspaces $\mathcal{W}_1 = \{(q,Q)\,|\,q_1 = q_2 = Q_1 = Q_2 = 0\}$ and $\mathcal{W}_2 = \{(q,Q)\,|\,q_3 = q_4 = Q_3 = Q_4 = 0\}$ are invariant manifolds. In fact, those subsystems are integrable.

• We have still another path to take (R. Cushman permitting!, [4]) which is to switch, only for a short while, to polar coordinates; in fact to the canonical extension of the *double polar* transformation $(\rho_1, \alpha_1, \rho_2, \alpha_2) \to (q_1, q_2, q_3, q_4)$. Explicitly, the transformation is

$$\begin{aligned} q_{2i-1} &= \rho_i \cos\alpha_i, \qquad & Q_{2i-1} &= P_i\,\cos\alpha_i - \frac{A_i}{\rho_i}\sin\alpha_i,\\ q_{2i} &= \rho_i \sin\alpha_i, \qquad & Q_{2i} &= P_i\,\sin\alpha_i + \frac{A_i}{\rho_i}\cos\alpha_i \end{aligned}$$

with $i = 1, 2$, where we have assumed $(q_1, q_2) \neq (0, 0)$ and $(q_3, q_4) \neq (0, 0)$. The Hamiltonian function (1) in those variables reads

$$\mathcal{H}_\beta = \frac{1}{2}\big(P_1^2 + P_2^2 + \frac{A_1^2}{\rho_1^2} + \frac{A_2^2}{\rho_2^2}\big) + \frac{1}{2}(\rho_1^2 + \rho_2^2) + \varepsilon\,(\rho_1^2 + \rho_2^2)[\,\beta^2(\rho_1^2 - \rho_2^2)^2 + 4\rho_1^2\rho_2^2\,]. \tag{6}$$

We identify immediately that two of the variables α_1, α_2 are ignorable, with A_1 and A_2 the corresponding first integrals; they are linear combination of the integrals Eqs. (3) mentioned above.

We may ask also for solutions of the differential system defined by (6)

$$\begin{aligned} \dot{\rho}_1 &= P_1, \qquad \dot{\alpha}_1 = \frac{A_1}{\rho_1^2},\\ \dot{\rho}_2 &= P_2, \qquad \dot{\alpha}_2 = \frac{A_2}{\rho_2^2},\\ \dot{P}_1 &= \frac{1}{\rho_1^3}\big(A_1^2 - \rho_1^4(1 + 2\varepsilon\,[3\beta^2\rho_1^4 + 2(4-\beta^2)\rho_1^2\rho_2^2 + (4-\beta^2)\rho_2^4])\big),\\ \dot{P}_2 &= \frac{1}{\rho_2^3}\big(A_2^2 - \rho_2^4(1 + 2\varepsilon\,[3\beta^2\rho_2^4 + 2(4-\beta^2)\rho_1^2\rho_2^2 + (4-\beta^2)\rho_1^4])\big), \end{aligned} \tag{7}$$

in the manifold satisfying $P_1 = P_2 = 0$, $\rho_1 = \rho_1^0$, $\rho_2 = \rho_2^0$ (functions of A_1, A_2 and β), *i.e. invariant 2-tori* made of quasi-periodic orbits (Note that those solutions project into 'circles' in the principal planes Oq_1q_2 and Oq_3q_4). Under these assumptions, the differential system leads to the following equations

$$A_1^2 - \rho_1^4(1 + 2\varepsilon\,[3\beta^2\rho_1^4 + 2(4-\beta^2)\rho_1^2\rho_2^2 + (4-\beta^2)\rho_2^4]) = 0, \tag{8}$$

$$A_2^2 - \rho_2^4(1 + 2\varepsilon\,[3\beta^2\rho_2^4 + 2(4-\beta^2)\rho_1^2\rho_2^2 + (4-\beta^2)\rho_1^4]) = 0. \tag{9}$$

Combining them we obtain another alternative relation

$$A_1^2\rho_2^4 - A_2^2\rho_1^4 = 8\varepsilon(\beta^2-1)(\rho_2^4-\rho_1^4)\rho_1^4\rho_2^4, \tag{10}$$

for the computation of the invariant tori. For example, taking Eqs. (8) and (10) we may obtain a polynomial in one of the variables, and then discuss the conditions for positive roots. In fact, we may take a short way because in this paper we are interested only in the case $\varepsilon \ll 1$. Indeed, by sucesive approximations, we obtain the following expressions for ρ_1^0 and ρ_2^0:

$$(\rho_1^0)^4 = A_1^2 - 2\varepsilon A_1^2[3\beta^2A_1^2 + 2(4-\beta^2)A_1A_2 + (4-\beta^2)A_2^2] \;+\; \mathcal{O}(\varepsilon^2),$$

$$(\rho_2^0)^4 = A_2^2 - 2\varepsilon A_2^2[3\beta^2A_2^2 + 2(4-\beta^2)A_1A_2 + (4-\beta^2)A_1^2] \;+\; \mathcal{O}(\varepsilon^2).$$

Then, replacing in the third integral, the Hamiltonian function (6), we will obtain in the energy-momentum space $(\mathcal{H}, A_1, A_2)$ the bifurcation surface:

$$\Phi(\mathcal{H}, A_1, A_2; \beta, \varepsilon) = 0,$$

parametrized by β and ε. Fixing a value of ε, it remains to study how this tori bifurcation surface evolves with the physical parameter β. Moreover, note that periodic solutions within these tori are the ones satisfying

$$m_1\,\frac{A_1}{(\rho_1^0)^2} + m_2\,\frac{A_2}{(\rho_2^0)^2} = 0, \qquad m_1, m_2 \in \mathbb{Z}.$$

My guess is that they will bifurcate from the special case $\rho_1 = \rho_2 = \rho_0(A_i, \beta)$. How these solutions relate to the *relative equilibria*, that to identify below, associated to the normalized system, is part of the study of the present paper. A full analysis is given in Díaz *et al.* [6].

• Another possible invariant set is the one defined by the intersection of $\mathbb{S}^3_{\sqrt{k}}$ and the integrals. Indeed, let us consider a solution such that $\rho_1^2 + \rho_2^2 = k^2 =$ constant. If this is the case, with the help of the Hamiltonian function, we find that the following relation ought to be satisfied

$$\rho_1^2 + \rho_2^2 = h - 4\varepsilon\,(\rho_1^2+\rho_2^2)[\,4\rho_1^2\rho_2^2 + \beta^2(\rho_1^2-\rho_2^2)^2\,].$$

Then, we conclude that this is only possible when $\beta^2 = 1$, with the relation $k^2 = h - 4\varepsilon k^6$. Something that we already knew because this corresponds to the integrable case of the central potential.

3 Symmetries, normalization and the reduced orbit spaces

We have already explained [7],[8] the reduction process related to this family of Hamiltonian systems. Here, in order to make the paper self contained, we gather the basic steps leading to the thrice reduced orbit space where we plan to do our bifurcation analysis.

Note that one can combine the three successive reductions into one. The composition of the three orbit maps gives an orbit map from $\mathbb{R}^8 \to \mathbb{R}^8$, which is an orbit map for the three-torus action generated by the rotational flows of the three integrals $\mathcal{H}_2^2$, Ξ, L_1, which are independent and commute. Consequently the generic relative equilibrium (i.e. stationary point on the reduced phase space) will correspond to a T^3. Due to the shape of the reduced phase spaces the intersection of the Hamiltonian and these spaces will be circles in general. Thus the generic fibre of the energy momentum map will be a T^4. There will of course also be fibres that are a point (the origin which is a stationary point of the original system and a fixed point for all circle symmetries), a circle (two of the circle symmetries will have a fixed point) or a T^2 (one of the circle symmetries will have a fixed point). The rank of the energy momentum map $\mathbb{R}^8 \to (\mathcal{H}, \mathcal{H}_2, \Xi, L_1)$ will correspond to the dimension of the fibre.

3.1 Normalization and reduction of the oscillator symmetry H_2: A choice of invariants

In order to normalize the system defined by (1) with respect to H_2, and reduce the normalized system we compute the invariants for the H_2 action. There are 16 quadratic polynomials in the variables $(\mathbf{Q}, \mathbf{q})$ that generate the space of functions invariant with respect to the action given by the flow of H_2. Explicitly they are

$$\begin{array}{llll}
\pi_1 = Q_1^2 + q_1^2 & \pi_2 = Q_2^2 + q_2^2 & \pi_3 = Q_3^2 + q_3^2 & \pi_4 = Q_4^2 + q_4^2 \\
\pi_5 = Q_1Q_2 + q_1q_2 & \pi_6 = Q_1Q_3 + q_1q_3 & \pi_7 = Q_1Q_4 + q_1q_4 & \pi_8 = Q_2Q_3 + q_2q_3 \\
\pi_9 = Q_2Q_4 + q_2q_4 & \pi_{10} = Q_3Q_4 + q_3q_4 & \pi_{11} = q_1Q_2 - q_2Q_1 & \pi_{12} = q_1Q_3 - q_3Q_1 \\
\pi_{13} = q_1Q_4 - q_4Q_1 & \pi_{14} = q_2Q_3 - q_3Q_2 & \pi_{15} = q_2Q_4 - q_4Q_2 & \pi_{16} = q_3Q_4 - q_4Q_3
\end{array} \tag{11}$$

The invariants are obtained using canonical complex variables (see [3], [7] for more details). The reduction is now performed using the orbit map

$$\rho_\pi : \mathbb{R}^8 \to \mathbb{R}^{16}; (q, Q) \to (\pi_1, \cdots, \pi_{16}) \, .$$

The image of this map is the orbit space for the H_2-action. The image of the level surfaces $H_2(q, Q) = n$ under ρ_π are the reduced phase spaces. These reduced phase spaces are isomorphic to $\mathbb{CP}^3$. The normalized Hamiltonian can be expressed in the invariants and therefore naturally lifts to a function on $\mathbb{R}^{16}$, which, on the reduced phase spaces, restricts to the reduced Hamiltonian. Expressing the H_2 normal form up to first order in ε for (1) in those invariants we have

$$\overline{\mathcal{H}} = \mathcal{H}_2 + \varepsilon \overline{\mathcal{H}}_6 \tag{12}$$

where

$$\mathcal{H}_2 = \frac{1}{2}(\pi_1 + \pi_2 + \pi_3 + \pi_4) = n \tag{13}$$

and

$$\begin{aligned}\overline{\mathcal{H}}_6 = \frac{1}{2}\big[&n\,(1-4\beta^2)(\pi_{15}^2 + \pi_{14}^2 + \pi_{13}^2 + \pi_{12}^2) \\ &+ 2(\beta^2 - 1)(\pi_{11}^2(\pi_4 + \pi_3) - \pi_{16}^2(\pi_3 + \pi_4)) \\ &+ 5n\,(1-\beta^2)(\pi_9^2 + \pi_8^2 + \pi_7^2 + \pi_6^2) + \beta^2 n\,(5n^2 - 3\pi_{11}^2) + n(\beta^2 - 4)\pi_{16}^2\big]\end{aligned}$$

However, in the following we will not use the invariants π_i as is done in [8], but instead use the (K_i, L_j, J_k) invariants as introduced in [7]. That is we replace the generating invariants π_i by the following set of invariants which is actually a linear coordinate transformation on the image of the orbit map.

$$\begin{array}{llll} H_2 = \frac{1}{2}(\pi_1 + \pi_2 + \pi_3 + \pi_4) & K_2 = \pi_8 - \pi_7 & L_2 = \pi_{12} + \pi_{15} & K_3 = -\pi_6 - \pi_9 \\ K_1 = \frac{1}{2}(-\pi_1 - \pi_2 + \pi_3 + \pi_4 & J_3 = \pi_8 + \pi_7 & J_7 = \pi_{12} - \pi_{15} & J_6 = \pi_6 - \pi_9 \\ J_1 = \frac{1}{2}(\pi_1 - \pi_2 - \pi_3 + \pi_4) & J_4 = \pi_5 + \pi_{10} & L_3 = \pi_{14} - \pi_{13} & \Xi = \pi_{16} + \pi_{11} \\ J_2 = \frac{1}{2}(\pi_1 - \pi_2 + \pi_3 - \pi_4) & J_5 = \pi_5 - \pi_{10} & J_8 = \pi_{14} + \pi_{13} & L_1 = \pi_{16} - \pi_{11} \end{array} \tag{14}$$

The normal form is in these invariants is

$$\begin{aligned}\overline{\mathcal{H}}_\Xi = \tfrac{1}{2}\big[&n\;(5\,{K_2}^2 + 5\,{K_3}^2 + 2\,{L_1}^2 + {L_2}^2 + {L_3}^2 + \beta^2\,(5\,{K_1}^2 + {L_2}^2 + {L_3}^2)\,) \\ &-((4+\beta^2)\,(K_2\,L_2 + K_3\,L_3) + (2+3\,\beta^2)K_1 L_1)\;\xi\big]\end{aligned} \tag{15}$$

The reduction of the H_2 action may now be performed through the orbit map

$$\rho_{K,L,J} : \mathbb{R}^8 \to \mathbb{R}^{16}; (q, Q) \to (H_2, \cdots, J_8)\,.$$

Note that on the orbit space we have the reduced symmetries due to the reduced actions given by the reduced flows of X_Ξ and X_{L_1}. Details on how the orbit space is defined are given in [10]. The basic relation $H_2 = n$ defines the symplectic leaves for the induced Poisson structure on this orbit space which are the reduced phase spaces. Let $B_{K,L,J}$ denote the structure matrix for induced Poisson structure$\{\ ,\ \}_{(K,L,J)}$. This matrix is given in [10]. Note that the motivation for this choice of invariants is that the reduced Ξ

invariants are the (K_i, L_j), which makes that the second reduction is easily obtained (see section 3.2).

A basic work to do at this stage is to deal with relative equilibria on $\mathbb{CP}^3$. This turns out to be not a simple task at all (see [10]). A relative equilibrium for a Hamiltonian system with respect to a symmetry group G is an orbit which is a solution of the system and simultaneously an orbit of the group. In our case the relative equilibria are therefore orbits of $X_{\bar{H}}$ as well as orbits of X_{H_2}, where $\bar{H}$ denotes the first order normal form for H. These relative equilibria correspond to stationary points of the reduced system obtained from $X_{\bar{H}}$ after reduction with respect to the X_{H_2}-action, i.e. the action of the one-parameter group given by the flow of X_{H_2}.

The reduced system on $\mathbb{R}^{16}$ is given by the differential equations

$$\frac{dz}{dt} = \{z, \bar{H}(z)\}_{(K,L,J)} =< z, B_{(K,L,J)} D\bar{H}(z) > , \qquad (16)$$

with $z = (H_2, K_1, J_1, J_1, K_2, J_3, J_4, J_5, K_3, J_6, \Xi, L_1, L_2, J_7, L_3, J_8)$, which on the reduced phase spaces restrict to a Hamiltonian system.

3.2 Normalization and Reduction related to the rotational symmetry Ξ

Dividing out the rotational symmetry Ξ reduces $\mathbb{CP}^3$ to a variety made of strata of dimension 4, and two stratum of dimension 2. In order to see that, we fix $\Xi = \xi$ and consider $\mathbb{CP}^3/S^1$ where S^1 is the action generated by the symmetry Ξ. We do that expressing the second reduced system in the 8 invariants defining this action

$$H_2 = \tfrac{1}{2}(\pi_1 + \pi_2 + \pi_3 + \pi_4), \qquad \Xi = \pi_{16} + \pi_{11},$$

$$K_1 = \tfrac{1}{2}(\pi_3 + \pi_4 - \pi_1 - \pi_2), \qquad L_1 = \pi_{16} - \pi_{11},$$

$$K_2 = \pi_8 - \pi_7, \qquad L_2 = \pi_{12} + \pi_{15},$$

$$K_3 = -(\pi_6 + \pi_9), \qquad L_3 = \pi_{14} - \pi_{13}.$$

This, in turn, leads us to the orbit mapping

$$\rho_2 : \mathbb{R}^{16} \to \mathbb{R}^8; \quad (\pi_1, \cdots, \pi_{16}) \to (K_1, K_2, K_3, L_1, L_2, L_3, \mathcal{H}_2, \Xi)$$

There are $2 + 2$ relations defining the second reduced space

$$K_1^2 + K_2^2 + K_3^2 + L_1^2 + L_2^2 + L_3^2 = H_2^2 + \Xi^2, \quad K_1L_1 + K_2L_2 + K_3L_3 = H_2\Xi$$

$$\Xi = \xi \qquad H_2 = n$$

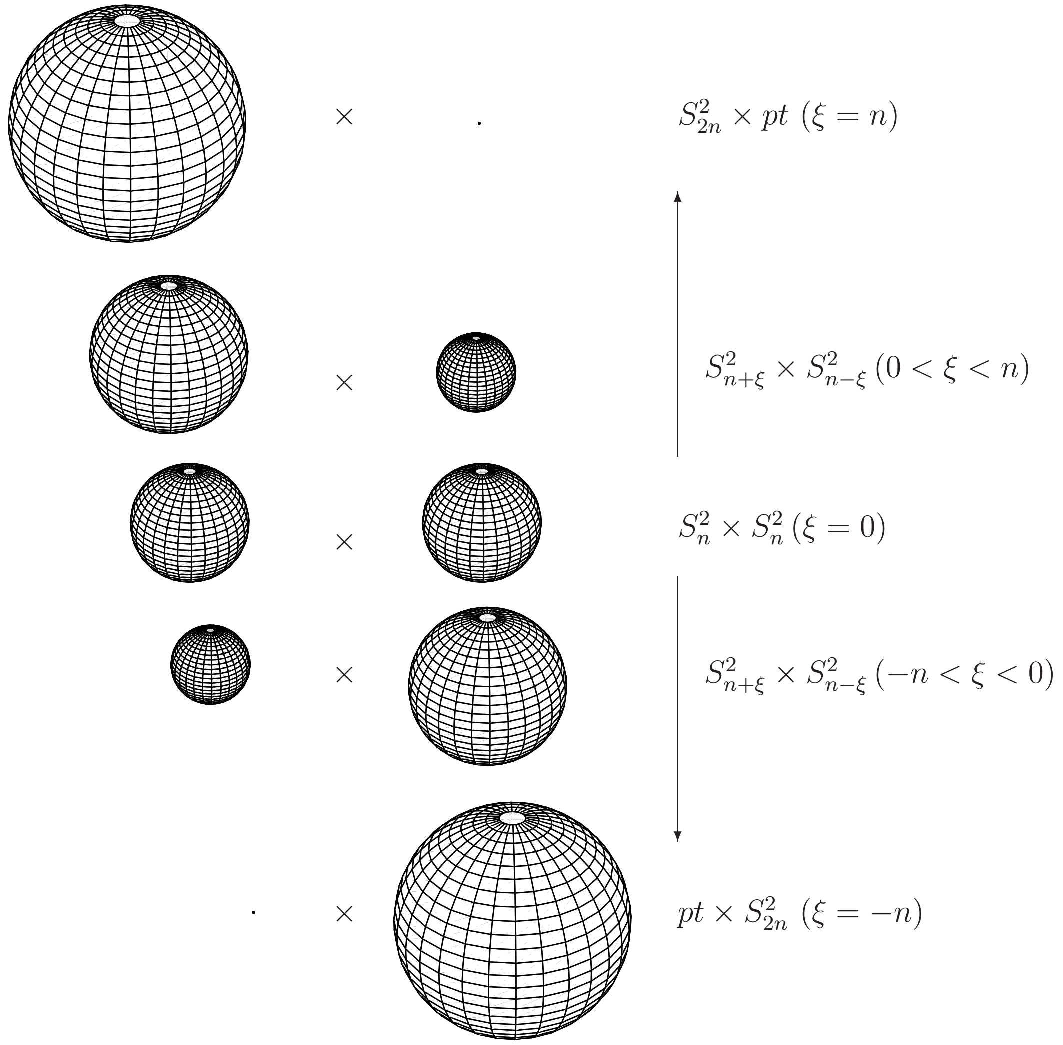

Figure 1.— Double reduced orbit space $S^2_{n+\xi} \times S^2_{n-\xi}$ for different values of the integral ξ

with $n \geq 0$. In other words, those relations give a dimension 4 space in the variables $(K_1, K_2, K_3, L_1, L_2, L_3)$

$$K_1^2 + K_2^2 + K_3^2 + L_1^2 + L_2^2 + L_3^2 = n^2 + \xi^2, \quad K_1L_1 + K_2L_2 + K_3L_3 = n\xi. \tag{17}$$

Introducing a new set of coordinates $(\sigma_1, \sigma_2, \sigma_3, \delta_1, \delta_2, \delta_3)$ by the relations $\sigma_i = K_i + L_i$ y $\delta_i = L_i - K_i$ with $i = 1, 2, 3$ they verify that

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = (n + \xi)^2 \qquad \delta_1^2 + \delta_2^2 + \delta_3^2 = (n - \xi)^2$$

Thus (17) is isomorphic to $\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}$. Note that when $\xi = 0$ the second reduced space is isomorphic to $\mathbb{S}^2_n \times \mathbb{S}^2_n$. This space, as we know, may be obtained normalizing perturbed Keplerian systems by immersion in a space of dimension 4 by means of regularization and the Kustaanheimo-Stiefel transformation [19].

Brackets for the invariants $(K_1, K_2, K_3, L_1, L_2, L_3)$ defining the second reduced orbit space $\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}$ are given in the following table

$\{\cdot,\cdot\}_2$	K_1	K_2	K_3	L_1	L_2	L_3
K_1	0	$-2L_3$	$2L_2$	0	$-2K_3$	$2K_2$
K_2	$2L_3$	0	$-2L_1$	$2K_3$	0	$-2K_1$
K_3	$-2L_2$	$2L_1$	0	$-2K_2$	$2K_1$	0
L_1	0	$-2K_3$	$2K_2$	0	$-2L_3$	$2L_2$
L_2	$2K_3$	0	$-2K_1$	$2L_3$	0	$-2L_1$
L_3	$-2K_2$	$2K_1$	0	$-2L_2$	$2L_1$	0

Moreover the second reduced Hamiltonian up to first order, up to constant terms, takes the form

$$\begin{aligned}\overline{\mathcal{H}}_\Xi = \tfrac{1}{2}\,[n\ (5\,{K_2}^2 + 5\,{K_3}^2 + 2\,{L_1}^2 + {L_2}^2 + {L_3}^2 + \beta^2\,(5\,{K_1}^2 + {L_2}^2 + {L_3}^2)\,)\\ -\,((4+\beta^2)\,(K_2\,L_2 + K_3\,L_3) + (2+3\,\beta^2)K_1L_1)\ \xi]\end{aligned} \tag{18}$$

Thus $(\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}, \{\cdot,\cdot\}_2, \overline{\mathcal{H}}_\Xi)$ is a Lie-Poisson system. The dynamics in $\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}$ is given by the following set of equations

$$\frac{d\mathbf{K}}{dt} = \{\mathbf{K}, \nabla_{\mathbf{K}}\overline{\mathcal{H}}_\Xi\}_2 \tag{19}$$

with $\mathbf{K} = (K_1, K_2, K_3, L_1, L_2, L_3)$. But we do not need to give them explicitly. In what follows we take $\lambda = \beta^2$.

3.3 The rotational symmetry L_1. The third reduced system

To further reduce from $\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}$ to $V_{n\,\xi\,l}$ one divides out the S^1-action generated by $L_1 = L_1$ and fixes $L_1 = l$. The 8 invariants for the L_1 action on $\mathbb{R}^8$ are

$$\mathcal{H}_2, \qquad \Xi, \qquad L_1, \qquad K = K_1,$$

$$X = \tfrac{1}{2}\,(K_2^2 + K_3^2), \quad Y = \tfrac{1}{2}\,(L_2^2 + L_3^2), \quad Z = K_2L_2 + K_3L_3, \quad S = K_2L_3 - K_3L_2.$$

There are $3+3$ relations defining the third reduced phase space

$$\mathcal{H}_2 = n, \qquad \Xi = \xi, \qquad L_1 = l,$$

$$K^2 + 2X + L_1^2 + 2Y = \mathcal{H}_2^2 + \Xi^2, \quad KL_1 + Z = \mathcal{H}_2\Xi, \quad 4XY = Z^2 + S^2$$

However, it is more convenient to use the following invariants

$$\mathcal{H}_2, \qquad \Xi, \qquad L_1, \qquad K = K_1,$$

$$M = X + Y, \quad N = X - Y, \quad Z = K_2L_2 + K_3L_3, \quad S = K_2L_3 - K_3L_2$$

with relations

$$K^2 + L_1^2 + 2M = \mathcal{H}_2^2 + \Xi^2$$
$$Kl_1 + Z = \mathcal{H}_2\Xi$$
$$M^2 - N^2 = Z^2 + S^2$$
$$L_1 = l,\ \Xi = \xi,\ \mathcal{H}_2 = n$$

Consequently we may represent the third reduced phase space $V_{n\xi l}$ in (K, N, S)-space by the equation

$$(n^2 + \xi^2 - l^2 - K^2)^2 - 4(n\xi - lK)^2 = 4N^2 + 4S^2. \tag{20}$$

If we set

$$f(K) = (n^2 + \xi^2 - l^2 - K^2)^2 - 4(n\xi - lK)^2 = [(n+\xi)^2 - (K+l)^2][(n-\xi)^2 - (K-l)^2]$$

then our reduced phase space is a surface of revolution obtained by rotating $\phi(K) = \sqrt{f(K)}$ around the K-axis.

The reduced phase spaces as well as the Hamiltonian are invariant (see Eq. (22)) under the discrete symmetry $S \to -S$. Furthermore the reduced phase space is invariant under the discrete symmetry $N \to -N$. We choose not to further reduce our reduced phase space with respect to these discrete symmetries because the three dimensional picture makes it easy to access information about the reduced orbits and this way one does not introduce additional critical points (fixed points) which need special attention. We will make use of the fact that all the critical point will be in the plane $S = 0$.

The shape of the reduced phase space is determined by the positive part of $f(K)$. The polynomial $f(K)$ can be written as

$$f(K) = (K + n + \xi + l)(K - n - \xi + l)(K - n + \xi - l)(K + n - \xi - l),$$

thus, the four zeroes of $f(K)$ are given by

$$K_1 = -l - n - \xi\ ,\ K_2 = l + n - \xi\ ,\ K_3 = l - n + \xi\ ,\ K_4 = -l + n + \xi\ .$$

So $f(K)$ is positive (or zero) in the subsequent intervals of K:

$$\begin{array}{lll} l < \xi\,,\ -l < \xi & K_1 < K_3 < K_2 < K_4 & K \in [K_3, K_2] \\ l > \xi\,,\ -l < \xi & K_1 < K_3 < K_4 < K_2 & K \in [K_3, K_4] \\ l < \xi\,,\ -l > \xi & K_3 < K_1 < K_2 < K_4 & K \in [K_1, K_2] \\ l > \xi\,,\ -l > \xi & K_3 < K_1 < K_4 < K_2 & K \in [K_1, K_4] \end{array} \tag{21}$$

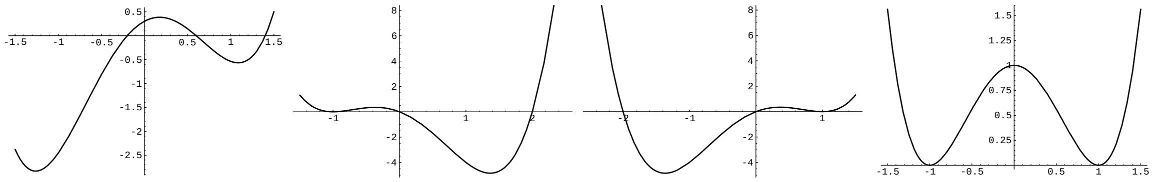

Figure 2.— Graph of $f(K)$. From left to right: $l \neq \xi \neq n$, $l = -\xi$, $l = \xi$, $l = \xi = 0$

See figure (2). When we have a simple root of $f(K)$ which belongs to one of the above intervals, we have that the intersection of the reduced phase space with the K-axis is smooth. $f(K)$ has four different roots in the following two cases: (i) $l \neq \xi$ and $\xi, l \neq 0$; (ii) $l \neq \xi$ and $\xi = 0$ or $l = 0$. In these cases the reduced phase space is diffeomorphic to a sphere. A point on this sphere corresponds to a three-torus in original phase space.

To find the the double zeroes of $f(K)$ we compute the discriminant of $f(K) = 0$. It is

$$(l-n)^2(l+n)^2(l-\xi)^2(l+\xi)^2(n-\xi)^2(n+\xi)^2 \ .$$

Thus there are double zeroes at $l = \pm n$, $l = \pm\xi$ and $\xi = \pm n$. If we have just one double zero the reduced phase space is a sphere with one cone-like singularity at the intersection point given by the double root ($l = \pm\xi \neq 0$). If we have two double zeroes the reduced phase space is a sphere with two cone-like singularities at the intersection points given by the double roots ($l = \xi = 0$). In the other cases the reduced phase space is just one singular point. The singular points correspond to two-tori in original phase space.

Triple zeroes occur when $|l| = |\xi| = n$. The reduced phase space is just a point which corresponds to a circle in original phase space.

Quadruple zeroes only occur when $l = n = \xi = 0$, which corresponds to the origin in original phase space and is a stationary point. See figure 3. More details on this analysis can be found in [7].

The cone-like singularities of the reduced phase space are candidates for the occurrence of Hamiltonian Hopf bifurcations, therefore in the following we restrict ourselves to the case $a = \zeta$ in which case we have a cone-like point at $K = n$.

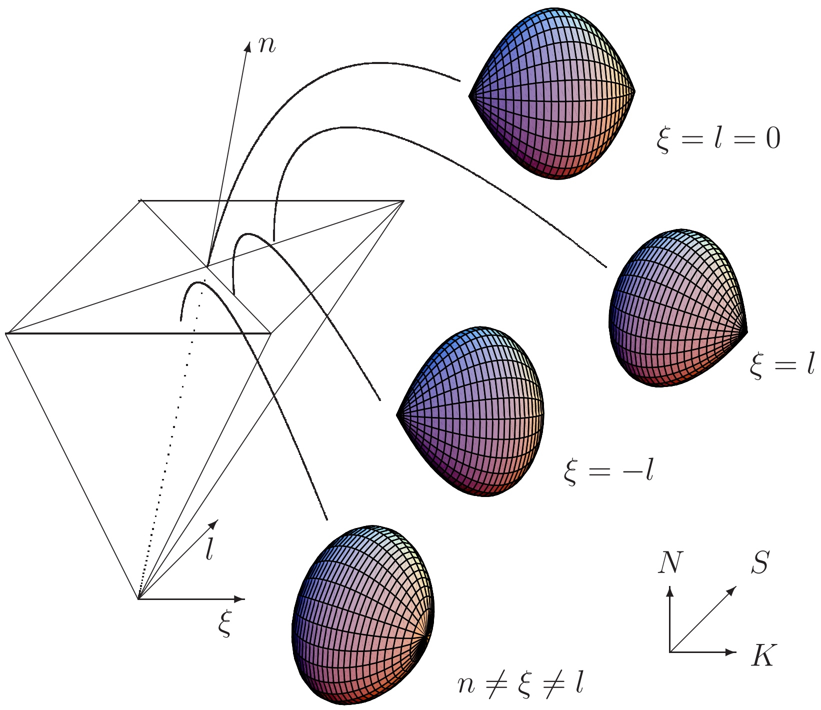

Figure 3.— The thrice reduced phase space over the parameter space. K is the symmetry axis of each surface.

3.3.1 The third reduced space $V_{n\xi l}$ is a Lie-Poisson manifold

The Poisson structure for M, N, Z, S, K, L_1 is given

$\{\cdot,\cdot\}_3$	M	N	Z	S	K	L_1
M	0	$4KS$	0	$-4KN$	0	0
N	$-4KS$	0	$-4L_1S$	$-4(KM - L_1Z)$	$4S$	0
Z	0	$4L_1S$	0	$-4L_1N$	0	0
S	$4KN$	$4(KM - L_1Z)$	$4L_1N$	0	$-4N$	0
K	0	$-4S$	0	$4N$	0	0
L_1	0	0	0	0	0	0

The Hamiltonian on the third reduced phase space is

$$\overline{\mathcal{H}}_{\Xi,L_1} = \frac{3n}{4}\,(3\lambda - 2)\,K^2 + \xi l(1-\lambda)K + \frac{n}{2}\,(4-\lambda)\,N + n^3(\frac{3}{2} + \frac{\lambda}{4}) - (l^2+\xi^2)\,(\frac{\lambda}{2}+1)\frac{n}{2} \tag{22}$$

In (K, N, S)-space the energy surfaces are parabolic cylinders. Note that for $\lambda = 2/3$, the function $\mathcal{H}$ in the variable space (K, N, S) is a plane. Likewise for $\lambda = 1$, we note that $\mathcal{H}$ is independent of ξ and l. Moreover when $\lambda = 4$, $\mathcal{H}$ is only function of K. The intersection with the reduced phase space give the trajectories of the reduced system. Tangency with the reduce phase spaces gives relative equilibria that generically will correspond to three dimensional tori in the original phase space.

Thus $(V_{n\xi l}, \{\cdot,\cdot\}_3, \overline{\mathcal{H}}_{\Xi,L_1})$ is a Lie-Poisson system. The corresponding dynamics is given by

$$
\begin{aligned}
\frac{dK}{dt} &= 2n(\lambda-4)\,S,\\
\frac{dN}{dt} &= 2[3n(3\lambda-2)K+2\xi l(1-\lambda)]\,S, \qquad (23)\\
\frac{dS}{dt} &= n(\lambda-4)(K^3-(\xi^2+l^2+n^2)K)-(3\lambda-2)[6nKN+4\xi l(\lambda-1)N+2ln^2\xi].
\end{aligned}
$$

It is easy to see that this system can be integrated by means of elliptic functions, but we do not plan o follow that path. We intend to classify the different types of flows as functions of the integrals and parameter of the system. Only then we will ready for the integration of a specific initial value problem; for details see Díaz *et al.* [6]. In the rest of the paper we focus on the dynamics in manifold $\Xi = 0$, particularly in the 'polar case'.

4 Case $\Xi = 0$: Generalized van der Waals problem

The dynamics of the generalized van der Waals potential [1] corresponds to our model (1) in the manifold $\Xi = 0$, with the integral $L_1 = l$ as the axial symmetry. As it is well known, when we consider in (1) the parameter ε to be small, the reduced space for $\Xi = 0$ relates to *normalized* 3-DOF perturbed Keplerian systems.

At this point to check the classical Cushman's survey [2] (see also [21]) on this issue is due. Indeed, although he does not mention the van der Waals model, from the generic study done about the first order double reduced Hamiltonian, one notices that the generalized van der Waals family for $\Xi = 0$ falls (see below) in the class of systems studied by him, defined by Eq. (8), in Section 6 of his survey. Nevertheless, we plan on developing this Section because, in particular, what we are after is the evolution of the flow with the physical parameter λ and the connection of integrable cases to degeneracy of the normalized flow, an aspect of the problem was not tackled in the generic study done by Cushman. In fact, to our knowledge, what we report below was considered first by Elipe and Ferrer [11, 12] using symplectic Delaunay variables.

4.1 Pitchfork bifurcations in the normalized 3-D case

In this case the reduced space (20) is defined by

$$
[\,(n+l)^2-K^2\,][\,(n-l)^2-K^2\,] = 4N^2+4S^2. \qquad (24)
$$

which is diffeomorphic to an S^2 sphere if $l \neq 0$ as we have shown in Section 3. Thus, if $l > 0$, the domain is $-|n-l| \leq K \leq |n-l|$. When $l < 0$, will be $-|n+l| \leq K \leq |n+l|$. When $l = 0$ it has two singular points, and we will study it separately. The Hamiltonian function (22) reduces to

$$\overline{\mathcal{H}}_{L_1} = \frac{3n}{4}(3\lambda - 2)\, K^2 + \frac{n}{2}(4 - \lambda)\, N. \tag{25}$$

which is, in general, a parabolic cylinder.

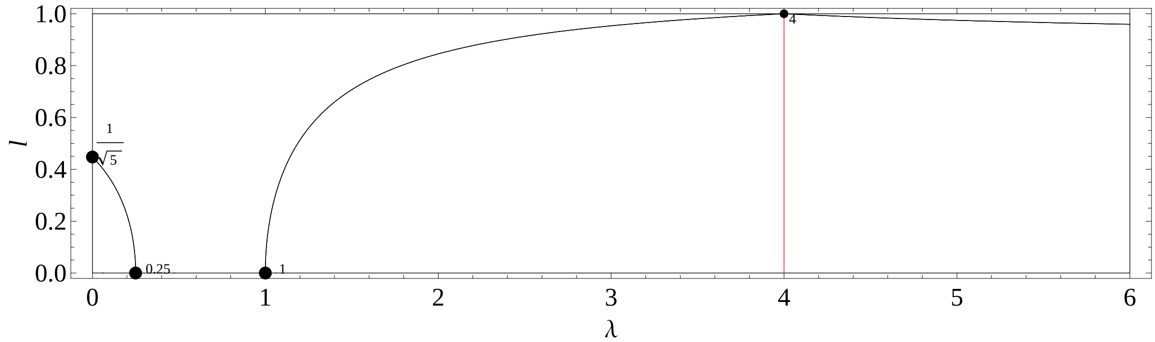

Figure 4.— Bifurcation lines of relative equilibria for the Van der Waals family. For details see [11]

From the work by Elipe and Ferrer [11], now recovered in invariants, we reproduce here Fig. 4 with the plane (λ, l) defined by the external parameter λ and the distinguished parameter l, with the pitchfork bifurcations lines related to families of circular orbits at *critical inclinations* and equatorial orbits at *critical eccentricities*.

Farrelly and collaborators [13, 14] showed that the generalized van der Waals model has three integrable cases for the values $\lambda = 1/4,\, 1,\, 4$. The third integral was given explicitly for those cases. We have found (see Egea *et al.* [10]) that they may be extended to the 4-D case. Here we are interested in the bifurcations related to those integrable cases, studying them with the invariant formalism.

4.2 Polar manifold $\Xi = L_1 = 0$. Parametric bifurcations

In this case sections of the surface defining the thrice reduced space (24) and the graph of the Hamiltonian (22) with the plane $S = 0$ are parabolas. More precisely, considering the two parabolas (the two branches) of the reduced space and the family N_h of Hamiltonian parabolas

$$N = \pm\frac{1}{2}(n^2 - K^2), \qquad N_h = \frac{3}{2}\,\frac{3\lambda - 2}{4 - \lambda}K^2 + \frac{2}{n(4 - \lambda)}\, h,$$

we compare them, distinguishing several cases. Fixing a value of λ, for each generic value of h we obtain a closed curve which can be given expliticly by elliptics functions after integrating system (23). There is a special type of solutions related to values of the energy h such that the Hamiltonian parabola contains one of the relative equilibria: 'south', singular or 'north' points respectively. They are homoclinic and heteroclinic trajectories given in Fig. 5.

Figure 5.— Three snapshots of the separatrix. Left $\lambda = 1/5$, center $\lambda = 1/2$ and right $\lambda = 2$

Finally, we identify particular values for λ such that both parabolas will coincide: this will correspond to a degeneracy. We will have then *infinity relative equilibria* at this order and higher order normalization will be needed, in order to elucidate the true nature of that flow for those values of the physical parameter.

The separatrices.- For the sake of completeness we include the explicit expressions of the separatrices:

- Homoclinic for $0 < \lambda < 1/4$

$$K = n \tanh \omega_1 t, \quad N = \frac{3(2-3\lambda)n^2}{2(\lambda-4)} \mathrm{sech}^2 \omega_1 t, \quad S = \frac{\sqrt{5(1-\lambda)(4\lambda-1)}n^2}{(\lambda-4)} \mathrm{sech}^2 \omega_1 t,$$

where $\omega_1(\lambda) = 2n^2\sqrt{5(1-\lambda)(4\lambda-1)}$.

- Heteroclinic for $1/4 < \lambda < 1$

$$K = \pm\frac{n\sqrt{4-\lambda}}{\sqrt{4\lambda-1}} \mathrm{sech}\omega_2 t, \; N = \frac{n^2}{2} + \frac{3(2-3\lambda)n^2}{2(4\lambda-1)} \mathrm{sech}^2\omega_2 t, \; S = \pm\frac{n^2\sqrt{5(\lambda-1)}}{\sqrt{4\lambda-1}} \frac{\mathrm{senh}\, \omega_2 t}{\cosh^2 \omega_2 t}$$

where $\omega_2(\lambda) = 2n^2\sqrt{5(4-\lambda)(\lambda-1)}$. Note that for $\lambda = 2/3$ the heteroclinic is a plane curve.

- Homoclinic for $1 < \lambda < 4$

$$K = \frac{\pm n(\lambda-4)\mathrm{sech}\, \omega_3 t}{\sqrt{5}\sqrt{(\lambda-5)\lambda+4}}, \; N = -\frac{n^2}{2} + \frac{3(3\lambda-2)n^2}{10(\lambda-1)} \mathrm{sech}^2\omega_3 t, \; S = \frac{\pm n^2\sqrt{(4\lambda-1)}}{\sqrt{5(\lambda-1)}} \frac{\mathrm{senh}\, \omega_3 t}{\cosh^2 \omega_3 t}$$

where $\omega_3(\lambda) = 2n^2\sqrt{(\lambda-4)(4\lambda-1)}$.

What happens if $\lambda = 4$ and $\lambda > 4$? It can be seen immediately that when $\lambda = 4$ the parabola degenerates into two vertical lines and then for $\lambda > 4$ again the dynamics is similar to the first case.

In order to obtain the flow corresponding to the thrice reduced orbit space, we may proceed either by numerical computation of the diferential system for several initial values or, taking into account the geometry of the problem, forgetting about the time evolution and drawing the intersection of the reduced space with the Hamiltonian function for different values of the energy. We follow this second way, using the technique of 'painting the flow' (see [15]) gathering the different scenarios, which in increasing values of λ, come as are presented in Figs. 6 and 7.

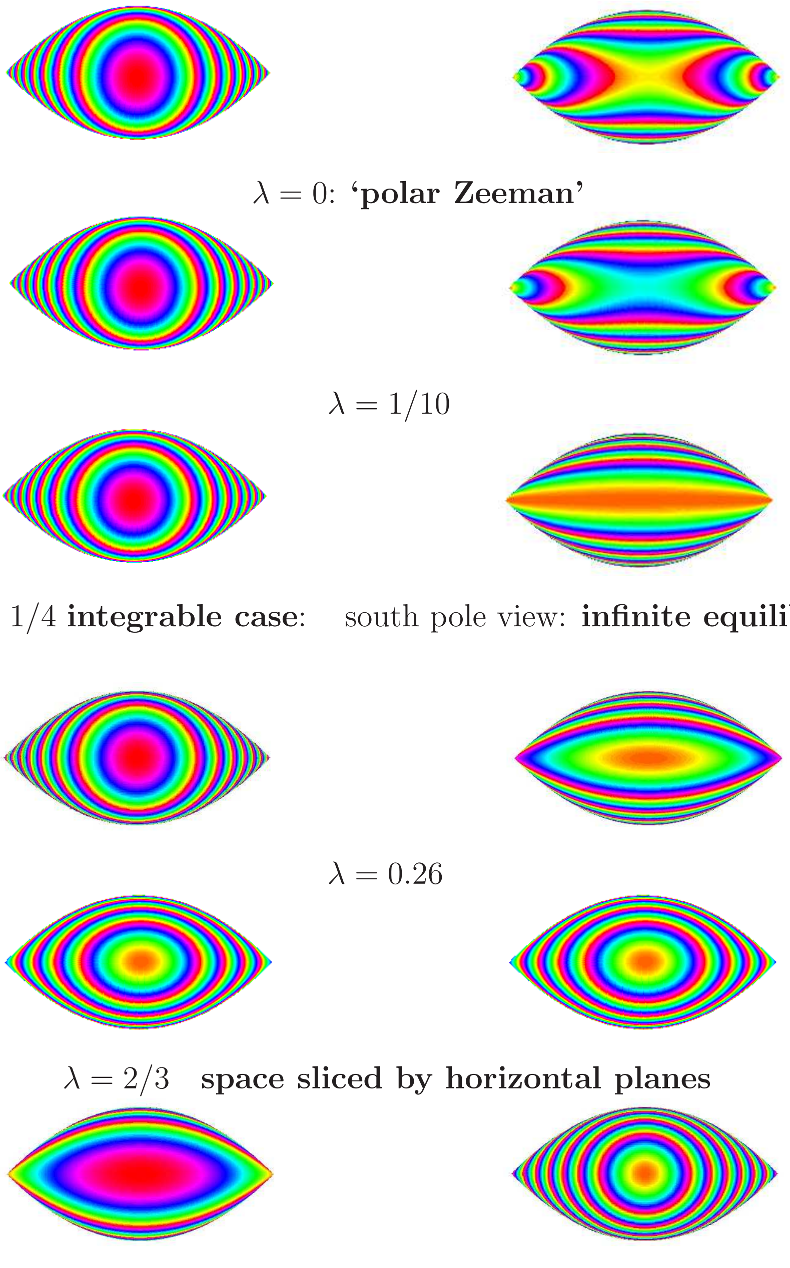

Figure 6.— Case $\Xi = L_1 = 0$: 'north pole' (left) and 'south pole' (right) views of the flow on the thrice reduced space for several values of the physical parameter λ. The integrable case $\lambda = 1/4$ is related to a Hopf bifurcation connecting the singular points

Higher order normalization of the system is needed for a investigation of the Hamiltonian Hopf Bifurcation in these values.

$\lambda = 1$, **integrable case**. north pole view: Infinite equilibria

$\lambda = 1.1$

$\lambda = 2$ **Van der Waals**

$\lambda = 3.5$

$\lambda = 4$, **integrable case**. Oyster bifurcation. Infinite equilibria in the plane $K = 0$

$\lambda = 6$: stable equilibrium is at the north; the unstable at the south.

Figure 7.— Case $\Xi = L_1 = 0$: 'north pole' (left) and 'south pole' (right) views of the flow on the thrice reduced space for several values of the physical parameter λ. From $\lambda = 1$ to $\lambda = 6$ we find two integrable cases. When $\lambda = 1$ the bifurcation associated to it involves the singular points: it is a degenerate Hopf bifurcation. When $\lambda = 4$ we identify another bifurcation called oister-bifurcations by some authors.

Conclusions

The generalized 4-D van der Waals uniparametric system model may be taken as a benchmark for dealing with symmetries, integrability and bifurcations, and the relations among them, in the perturbed isotropic oscillator in 4-D. In this paper we report on first results about those aspects (for a detailed study see [6]). In particular we focus on invariant manifolds, reduction and bifurcations. We have found that two of the integrable cases relate with Hopf bifurcations, whose analysis requires higher order normalization now in progress (see [5]).

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