Monografías de la Real Academia de Ciencias de Zaragoza **32**, 1–17, (2009).

Survey on central configurations related with regular polyhedra

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Abstract

This paper is a survey on the known results about central configurations related with regular polyhedra.

Key words and expressions: n–body problem, spatial central configurations, regular polyhedra.

AMS (MOS) Subject Classification: 70F10, 70F15.

1 Introduction

Let $\mathbf{q}_1, \ldots, \mathbf{q}_N \in \mathbb{R}^\ell$ denote the position of the N punctual masses $m_1, \ldots, m_N$ respectively. The motion of these masses is governed by the equations

$$m_i\, \ddot{\mathbf{q}}_i = -\sum_{j=1,\, j\neq i}^{N} G\, m_i\, m_j\, \frac{\mathbf{q}_i - \mathbf{q}_j}{|\mathbf{q}_i - \mathbf{q}_j|^3}\ , \qquad i = 1, \ldots, N,$$

where G is the gravitational constant which can be taken equal to one by choosing conveniently the unit of time. The *configuration space* of the N–body problem in $\mathbb{R}^\ell$ with the center of mass at the origin is

$$\mathcal{E} = \{(\mathbf{q}_1, \ldots, \mathbf{q}_N) \in \mathbb{R}^{\ell N}\ :\ \sum_{i=1}^{N} m_i\, \mathbf{q}_i = 0,\ \mathbf{q}_i \neq \mathbf{q}_j,\ \text{for } i \neq j\}.$$

Given positive masses $m_1, \ldots, m_N$ a configuration $(\mathbf{q}_1, \ldots, \mathbf{q}_N) \in \mathcal{E}$ is *central* if there exists a positive constant λ such that

$$\ddot{\mathbf{q}}_i = -\lambda\, \mathbf{q}_i, \qquad i = 1, \ldots, N.$$

That is, in a central configuration the acceleration $\ddot{\mathbf{q}}_i$ of each point mass m_i is proportional (with the same constant of proportionality) to its position $\mathbf{q}_i$ relative to the center of mass of the system and is directed towards the center of mass. Thus a central configuration $(\mathbf{q}_1, \ldots, \mathbf{q}_N) \in \mathcal{E}$ of the N–body problem with positive masses $m_1, \ldots, m_N$ is a solution of the system of equations

$$\sum_{j=1,\, j\neq i}^{N} m_j \frac{\mathbf{q}_i - \mathbf{q}_j}{|\mathbf{q}_i - \mathbf{q}_j|^{3/2}} = \lambda\, \mathbf{q}_i \ , \qquad i = 1, \ldots, N,$$

with $\lambda > 0$.

A central configuration of $\mathbb{R}^\ell$ is called *planar* if the configuration of the N bodies is contained in a plane, and it is called *spatial* if does not exist a plane containing the configuration of the N bodies.

Two central configurations in $\mathbb{R}^\ell$ are in the same *class* if there exists a rotation of $SO(\ell)$ and a homothecy of $\mathbb{R}^\ell$ which transform one into the other.

The simplest known planar central configuration of the N–body problem for $N \geqslant 2$ is obtained by taking N equal masses at the vertices of a regular N–gon. We cannot find in the literature who was the first in knowing such planar central configurations. It is also known the existence of planar central configurations for the $2N$–body problem where the masses are at the vertices of two nested regular N–gons with a common center. In such configurations all the masses on the same N–gon are equal but masses on different N–gons could be different. It seems that the first in studying these nested planar central configurations was Longley [11] in 1907, later on in 1927 and 1929 Bilimovithc (see [6]) and in 1967 Klemplerer [7] also studied them. More recently they have been also studied in [12, 13]. This result can be extended to planar central configurations of the pN–body problem where the masses are at the vertices of p nested regular N–gons with a common center (see [5]) for all $p \geqslant 2$. As above, in such configurations the masses on the same N–gon are equal but masses on different N–gons could be different. It is also known the existence of planar central configurations for the 9–body problem where the masses are at the vertices of three nested equilateral triangles with a common center, and the intermediate triangle is rotated an angle of $\pi/3$ in relation to the other two ones (see [10]).

In [1] the authors provide classes of non–planar symmetric central configurations. In particular they prove that the configuration formed by N equal masses at the vertices of a regular polyhedron of N vertices is central for the N–body problem.

If we put an additional mass $m_0 = \mu$ at the center of mass of the polyhedra, then for all $\mu > 0$ the resulting configuration is still central.

In this paper we give a survey on the known results of spatial central configurations related with nested regular polyhedra. As we will see, the previous results for nested regular N-gons can be extended to nested regular polyhedra.

Tetrahedron	$\mathbf{a}_1^{(4)} = (-1, -1/\sqrt{3}, -1/\sqrt{6})$ $\mathbf{a}_2^{(4)} = (1, -1/\sqrt{3}, -1/\sqrt{6})$ $\mathbf{a}_3^{(4)} = (0, 2/\sqrt{3}, -1/\sqrt{6})$ $\mathbf{a}_4^{(4)} = (0, 0, \sqrt{3/2})$
Octahedron	$\mathbf{a}_1^{(6)} = (1,0,0)$ $\mathbf{a}_2^{(6)} = (-1,0,0)$ $\mathbf{a}_3^{(6)} = (0,1,0)$ $\mathbf{a}_4^{(6)} = (0,-1,0)$ $\mathbf{a}_5^{(6)} = (0,0,1)$ $\mathbf{a}_6^{(6)} = (0,0,-1)$
Cube	$\mathbf{a}_1^{(8)} = (1,1,1)$ $\mathbf{a}_2^{(8)} = (1,1,-1)$ $\mathbf{a}_3^{(8)} = (1,-1,1)$ $\mathbf{a}_4^{(8)} = (-1,1,1)$ $\mathbf{a}_5^{(8)} = (1,-1,-1)$ $\mathbf{a}_6^{(8)} = (-1,1,-1)$ $\mathbf{a}_7^{(8)} = (-1,-1,1)$ $\mathbf{a}_8^{(8)} = (-1,-1,-1)$
Icosahedron	$\mathbf{a}_1^{(12)} = (0,1,\phi)$ $\mathbf{a}_2^{(12)} = (0,1,-\phi)$ $\mathbf{a}_3^{(12)} = (0,-1,\phi)$ $\mathbf{a}_4^{(12)} = (0,-1,-\phi)$ $\mathbf{a}_5^{(12)} = (1,\phi,0)$ $\mathbf{a}_6^{(12)} = (1,-\phi,0)$ $\mathbf{a}_7^{(12)} = (-1,\phi,0)$ $\mathbf{a}_8^{(12)} = (-1,-\phi,0)$ $\mathbf{a}_9^{(12)} = (\phi,0,1)$ $\mathbf{a}_{10}^{(12)} = (\phi,0,-1)$ $\mathbf{a}_{11}^{(12)} = (-\phi,0,1)$ $\mathbf{a}_{12}^{(12)} = (-\phi,0,-1)$
Dodecahedron	$\mathbf{a}_1^{(20)} = (1,1,1)$ $\mathbf{a}_2^{(20)} = (-1,1,1)$ $\mathbf{a}_3^{(20)} = (1,-1,1)$ $\mathbf{a}_4^{(20)} = (1,1,-1)$ $\mathbf{a}_5^{(20)} = (-1,-1,1)$ $\mathbf{a}_6^{(20)} = (-1,1,-1)$ $\mathbf{a}_7^{(20)} = (1,-1,-1)$ $\mathbf{a}_8^{(20)} = (-1,-1,-1)$ $\mathbf{a}_9^{(20)} = (0,1/\phi,\phi)$ $\mathbf{a}_{10}^{(20)} = (0,-1/\phi,\phi)$ $\mathbf{a}_{11}^{(20)} = (0,1/\phi,-\phi)$ $\mathbf{a}_{12}^{(20)} = (0,-1/\phi,-\phi)$ $\mathbf{a}_{13}^{(20)} = (1/\phi,\phi,0)$ $\mathbf{a}_{14}^{(20)} = (-1/\phi,\phi,0)$ $\mathbf{a}_{15}^{(20)} = (1/\phi,-\phi,0)$ $\mathbf{a}_{16}^{(20)} = (-1/\phi,-\phi,0)$ $\mathbf{a}_{17}^{(20)} = (\phi,0,\ 1/\phi)$ $\mathbf{a}_{18}^{(20)} = (-\phi,0,1/\phi)$ $\mathbf{a}_{19}^{(20)} = (\phi,0,-1/\phi)$ $\mathbf{a}_{20}^{(20)} = (-\phi,0,-1/\phi)$

Table 1.— The coordinates, modulus homothecies and rotations, of the vertices of the five regular polyhedra centered at the origin. Here $\phi = (1+\sqrt{5})/2$ is the golden ratio.

There are five convex regular polyhedra: the tetrahedron, the octahedron, the cube, the icosahedron and the dodecahedron with 4, 6, 8, 12 and 20 vertices, respectively. The vertices of the regular polyhedra centered at the origin, modulus homothecies and a rotations, are the ones given in Table 1.

We define the *two nested homothethic polyhedra configuration* as the configuration formed by two regular polyhedra with the same number of vertices N, the same center, and where one of the polyhedra is a homothecy of the other one; that is, the positions of the vertices of the inner polyhedron $\mathbf{r}_i^{in}$ and the ones of the outer polyhedron $\mathbf{r}_i^{ou}$ satisfy the relation $\mathbf{r}_i^{ou} = \rho\,\mathbf{r}_i^{in}$ for some *scale factor* $\rho > 1$ and for all $i = 1, \ldots, N$. In a similar way, the *three nested homothetic polyhedra configuration* is the configuration formed by three regular polyhedra with the same number of vertices N, the same center and such that the positions of the vertices of the inner polyhedron $\mathbf{r}_i^{in}$, the ones of the medium polyhedron $\mathbf{r}_i^{m}$ and the ones of the outer polyhedron $\mathbf{r}_i^{ou}$ satisfy the relation $\mathbf{r}_i^{m} = \rho\,\mathbf{r}_i^{in}$ and $\mathbf{r}_i^{ou} = R\,\mathbf{r}_i^{in}$ for some *scale factors* $R > \rho > 1$ and for all $i = 1, \ldots, N$.

We define the *two nested rotated polyhedra configuration* as the configuration formed by two regular polyhedra with the same number of vertices N, the same center, and where one of the polyhedra is a homothecy plus a rotation of the other one; that is, the positions of the vertices of the inner polyhedron $\mathbf{r}_i^{in}$ and the ones of the outer polyhedron $\mathbf{r}_i^{ou}$ satisfy the relation $\mathbf{r}_i^{ou} = \rho\,\mathcal{R}\mathbf{r}_i^{in}$ for some *scale factor* $\rho \geqslant 1$, for some rotation matrix of $SO(3)$:

$$\mathcal{R} = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with Euler angles $\alpha \in [0, 2\pi)$, $\beta \in [0, \pi]$ and $\gamma \in [0, 2\pi)$, and for all $i = 1, \ldots, N$. In a similar way, the *three nested rotated polyhedra configuration* is the configuration formed by three regular polyhedra with the same number of vertices N, the same center, and where one of the polyedra is a homothecy of the other one, and the third one is a homothecy followed by a rotation of the other two.

In Section 2 we provide results related with two and three nested homothetic polyhedra central configurations. In Section 3 we provide results on two and three nested rotated polyhedra. In particular we provide central configurations of two and three nested rotated tetrahedra because we cannot guarantee the existence of such configurations for the other type of polyhedra. We have some evidence that the unique central configuration of two nested octahedra and two nested cube is the homothetic one. Unfortunately, at this moment, we have not a rigorous prove of it, we left this subject for future works.

2 Nested homothethic regular polyhedra

Without loss of generality we take the unit of mass in such a way that the masses of the inner polyhedron are equal to one. Recall that the set of central configurations is invariant under homothecies and rotations, so by choosing conveniently the unit of length we can assume that the positions of the vertices of the inner polyhedron are the ones of Table 1.

2.1 Two nested homothethic polyhedra

Theorem 1 *Consider N equal masses $m_i = 1$ for $i = 1, \ldots, N$ located at the vertices of a regular polyhedron of N vertices having positions $\mathbf{q}_i = \mathbf{a}_i^{(N)}$ for $i = 1, \ldots, N$. Consider N additional equal masses $m_{i+N} = m$ for $i = 1, \ldots, N$ at the vertices of a second nested regular polyhedron of N vertices having positions $\mathbf{q}_{i+N} = \rho\,\mathbf{a}_i^{(N)}$ for $i = 1, \ldots, N$ and scale factor $\rho > 1$ (see Fig. 1). Then the following statements hold.*

(a) For $N = 4$ (i.e. two nested homothetic tetrahedra) such configuration is central for

the spatial 8*–body problem when*

$$m = f_8(\rho) = \frac{\dfrac{(2/3)^{3/2}}{(\rho-1)^2} - \dfrac{\rho}{2} + \dfrac{2\sqrt{2}\,(3\rho+1)}{(3\rho^2+2\rho+3)^{3/2}}}{-\dfrac{1/2}{\rho^2} - \dfrac{(2/3)^{3/2}\rho}{(\rho-1)^2} + \dfrac{2\sqrt{2}\,\rho\,(\rho+3)}{(3\rho^2+2\rho+3)^{3/2}}},$$

and $\rho > \alpha^{(4)} = 1.8899915758\ldots$, *where* $\alpha^{(4)}$ *is the unique real solution of* $f_8(\rho) = 0$ *with* $\rho > 1$.

(b) *For* $N = 6$ *(i.e. two nested homothetic octahedra) such configuration is central for the spatial* 12*–body problem when*

$$m = f_{12}(\rho) = \frac{\dfrac{4\rho}{(\rho^2+1)^{3/2}} - \dfrac{\left(1+4\sqrt{2}\right)\rho}{4} + \dfrac{2\,(\rho^2+1)}{(\rho^2-1)^2}}{-\dfrac{4\rho^2}{(\rho^2-1)^2} + \dfrac{4\rho}{(\rho^2+1)^{3/2}} - \dfrac{1+4\sqrt{2}}{4\rho^2}},$$

and $\rho > \alpha^{(6)} = 1.7298565115\ldots$, *where* $\alpha^{(6)}$ *is the unique real solution of* $f_{12}(\rho) = 0$ *with* $\rho > 1$.

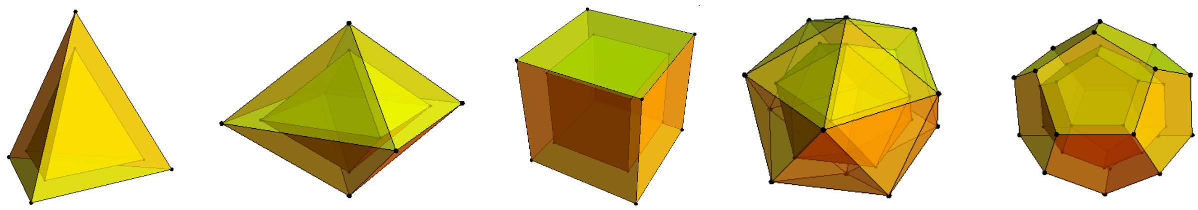
(c) *For* $N = 8$ *(i.e. two nested homothetic cube) such configuration is central for the spatial* 16*–body problem when* $m = b(\rho)/f(\rho)$ *where*

$$\begin{aligned} b(\rho) &= -\frac{1}{72}\left(18+9\sqrt{2}+2\sqrt{3}\right)\rho + \frac{2\,(\rho^2+1)}{3\sqrt{3}\,(\rho^2-1)^2} + \\ &\quad \frac{3\rho-1}{(3\rho^2-2\rho+3)^{3/2}} + \frac{3\rho+1}{(3\rho^2+2\rho+3)^{3/2}}, \\ f(\rho) &= -\frac{18+9\sqrt{2}+2\sqrt{3}}{72\rho^2} - \frac{4\rho^2}{3\sqrt{3}\,(\rho^2-1)^2} - \\ &\quad \frac{(\rho-3)\rho}{(3\rho^2-2\rho+3)^{3/2}} + \frac{(\rho+3)\rho}{(3\rho^2+2\rho+3)^{3/2}}, \end{aligned}$$

and $\rho > \alpha^{(8)} = 1.6436467629\ldots$ *where* $\alpha^{(8)}$ *is the unique real solution of* $b(\rho) = 0$ *with* $\rho > 1$.

(d) *For* $N = 12$ *(i.e. two nested homothetic icosahedra) such configuration is central for the spatial* 24*–body problem when* $m = b(\rho)/f(\rho)$ *where*

$$\begin{aligned} b(\rho) &= \frac{2\sqrt{5-2\sqrt{5}}\,(\rho^2+1)}{5\,(\rho^2-1)^2} - \frac{2\sqrt{2}\left(\sqrt{5}-5\rho\right)}{(\varphi\rho^2-4\,\phi\,\rho+\varphi)^{3/2}} + \\ &\quad \frac{2\sqrt{2}\left(5\rho+\sqrt{5}\right)}{(\varphi\rho^2+4\,\phi\,\rho+\varphi)^{3/2}} - \frac{1}{20}\left(5\sqrt{5}+\sqrt{5-2\sqrt{5}}\right)\rho, \\ f(\rho) &= -\frac{4\sqrt{5-2\sqrt{5}}\rho^2}{5\,(\rho^2-1)^2} - \frac{2\sqrt{2}\left(\sqrt{5}\rho-5\right)\rho}{(\varphi\rho^2-4\,\phi\,\rho+\varphi)^{3/2}} + \\ &\quad \frac{2\sqrt{2}\left(\sqrt{5}\rho+5\right)\rho}{(\varphi\rho^2+4\,\phi\,\rho+\varphi)^{3/2}} - \frac{5\sqrt{5}+\sqrt{5-2\sqrt{5}}}{20\rho^2}, \end{aligned}$$



(a) $N=4$ Tetrahedra (b) $N=6$ Octahedra (c) $N=8$ Cube (d) $N=12$ Icosahedra (e) $N=20$ Dodecahedra

Figure 1.— Two nested homothetic polyhedra.

$\varphi = 5+\sqrt{5}$ and $\rho > \alpha^{(12)} = 1.5493511156\ldots$ where $\alpha^{(12)}$ is the unique real solution of $b(\rho)=0$ with $\rho>1$.

(e) For $N=20$ (i.e. two nested homothetic dodecahedra) such configuration is central for the spatial 40–body problem when $m=b(\rho)/f(\rho)$ where

$$
\begin{aligned}
b(\rho) &= -\frac{1}{36}\left(18+9\sqrt{2}+\sqrt{3}+9\sqrt{5}\right)\rho+\frac{2\,(\rho^2+1)}{3\sqrt{3}\,(\rho^2-1)^2}+\frac{6\rho-2}{(3\rho^2-2\rho+3)^{3/2}}+\\
&\quad \frac{6\rho+2}{(3\rho^2+2\rho+3)^{3/2}}+\frac{6\,\phi\rho-\sqrt{5}-5}{2\,\phi\left(3\rho^2-2\sqrt{5}\rho+3\right)^{3/2}}+\frac{3\rho+\sqrt{5}}{\left(3\rho^2+2\sqrt{5}\rho+3\right)^{3/2}},\\
f(\rho) &= -\frac{4\rho^2}{3\sqrt{3}\,(\rho^2-1)^2}-\frac{2(\rho-3)\rho}{(3\rho^2-2\rho+3)^{3/2}}+\frac{2(\rho+3)\rho}{(3\rho^2+2\rho+3)^{3/2}}-\\
&\quad \frac{\left(\left(5+\sqrt{5}\right)\rho-6\,\phi\right)\rho}{2\,\phi\left(3\rho^2-2\sqrt{5}\rho+3\right)^{3/2}}+\frac{\left(\left(5+\sqrt{5}\right)\rho+6\,\phi\right)\rho}{2\,\phi\left(3\rho^2+2\sqrt{5}\rho+3\right)^{3/2}}-\\
&\quad \frac{18+9\sqrt{2}+\sqrt{3}+9\sqrt{5}}{36\rho^2},
\end{aligned}
$$

and $\rho > \alpha^{(20)} = 1.4622260542\ldots$ where $\alpha^{(20)}$ is the unique real solution of $b(\rho)=0$ with $\rho>1$.

(f) For each $N=4,6,8,12,20$ and for every fixed value of $m>0$, there exists a unique $\rho>\alpha^{(N)}$ for which the configuration of two nested homothetic polyhedra of N vertices is central for the $2N$–body problem.

Theorem 1 is proved in [2]. Statements (a) and (b) are also proved in [14] and [8], respectively in a different way. We note that in [14, 8] the authors also consider central configurations of two nested homothetic tetrahedra and octahedra with scale factor $0<\rho<1$. These central configurations can be obtained from the ones given by Theorem 1 by replacing m by $1/m$ and ρ by $1/\rho$.

If we put an additional mass $m_0=\mu>0$ at the center of mass of the two nested homothetic polyhedra, then we obtain an analogous of Theorem 1 with the difference that now m depend of μ and ρ must be greater than a value $\widetilde{\alpha}^{(N)}$ that depends also on μ.

This value $\widetilde{\alpha}^{(N)}(\mu)$ tends to the value $\alpha^{(N)}$ given in Theorem 1 when $\mu \to 0$ and it tends to 1 when $\mu \to +\infty$. Consequently, if the ratios of the masses and the ratios of the length of the edges of the polyhedra satisfy some convenient relations, then the configuration of two nested homothetic polyhedra with a mass at their center of mass is also central for the $2N+1$–body problem.

2.2 Three nested homothetic polyhedra

Theorem 2 *Consider N equal masses $m_i = 1$ for $i = 1, \ldots, N$ at the vertices of a regular polyhedron of N vertices having positions $\mathbf{q}_i = \mathbf{a}_i^{(N)}$ for $i = 1, \ldots, N$. Consider N additional equal masses $m_{i+N} = m$ for $i = 1, \ldots, N$ at the vertices of a second nested regular polyhedron of N vertices having positions $\mathbf{q}_{i+N} = \rho\,\mathbf{a}_i^{(N)}$ for $i = 1, \ldots, N$ with scale factor $\rho > 1$, and finally we consider N additional equal masses $m_{i+2N} = M$ for $i = 1, \ldots, N$ at the vertices of a third nested regular polyhedron of N vertices having positions $\mathbf{q}_{i+2N} = R\,\mathbf{a}_i^{(N)}$ for $i = 1, \ldots, N$ with scale factor $R > \rho$ (see Fig. 2). Let*

$$
\begin{aligned}
m(R,\rho) &= \frac{1}{\det(A)} \begin{vmatrix} 1 & \beta & f(R,1) \\ \rho & g(\rho,1) & f(R,\rho) \\ R & g(R,1) & -\beta/R^2 \end{vmatrix}, \\
M(R,\rho) &= \frac{1}{\det(A)} \begin{vmatrix} 1 & f(\rho,1) & \beta \\ \rho & -\beta/\rho^2 & g(\rho,1) \\ R & -g(R,\rho) & g(R,1) \end{vmatrix},
\end{aligned} \tag{1}
$$

where $A = \begin{pmatrix} 1 & f(\rho,1) & f(R,1) \\ \rho & -\beta/\rho^2 & f(R,\rho) \\ R & -g(R,\rho) & -\beta/R^2 \end{pmatrix}$ for some functions $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $g : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ with $D = \{(x,y) \in \mathbb{R}^2 \ : \ x > y \geqslant 1\}$ and some constant $\beta > 0$ which depend on the value of N, and let $\mathcal{D}^{(N)}$ denote the set $\{(R,\rho) \in \mathbb{R}^2 \ : \ m(R,\rho) > 0,\ M(R,\rho) > 0,\ R > \rho > 1\}$. Then the following statements hold.

(a) For $N = 4$ (i.e. three nested homothetic tetrahedra) such configuration is central for the spatial 12–body problem when $m = m(R,\rho)$ and $M = M(R,\rho)$ are given by (1) with

$$
\begin{aligned}
f(x,y) &= \frac{2\sqrt{2/3}}{3(x-y)^2} - \frac{2\sqrt{2}(x+3y)}{(3x^2+2yx+3y^2)^{3/2}}, \\
g(x,y) &= \frac{2\sqrt{2/3}}{3(x-y)^2} + \frac{2\sqrt{2}(3x+y)}{(3x^2+2yx+3y^2)^{3/2}},
\end{aligned}
$$

$\beta = 1/2$ and $(R,\rho) \in \mathcal{D}^{(4)}$ (see Fig. 3(a) for the plot of $\mathcal{D}^{(4)}$).

(b) For $N = 6$ (i.e. three nested homothetic octahedra) such configuration is central for the spatial 18–body problem when $m = m(R,\rho)$ and $M = M(R,\rho)$ are given by (1)

with

$$f(x,y) = \frac{4xy}{\left(x^2-y^2\right)^2} - \frac{4y}{\left(x^2+y^2\right)^{3/2}},$$
$$g(x,y) = \frac{4x}{\left(x^2+y^2\right)^{3/2}} + \frac{2\left(x^2+y^2\right)}{\left(x^2-y^2\right)^2},$$

$\beta = \left(1+4\sqrt{2}\right)/4$ *and* $(R,\rho) \in \mathcal{D}^{(6)}$ *(see Fig. 3(b) for the plot of* $\mathcal{D}^{(6)}$*).*

(c) *For* $N = 8$ *(i.e. three nested homothetic cube) such configuration is central for the spatial* 24*–body problem when* $m = m(R,\rho)$ *and* $M = M(R,\rho)$ *are given by (1) with*

$$f(x,y) = \frac{x-3y}{\left(3x^2-2yx+3y^2\right)^{3/2}} - \frac{x+3y}{\left(3x^2+2yx+3y^2\right)^{3/2}} + \frac{4xy}{3\sqrt{3}\left(x^2-y^2\right)^2},$$
$$g(x,y) = \frac{3x-y}{\left(3x^2-2yx+3y^2\right)^{3/2}} + \frac{3x+y}{\left(3x^2+2yx+3y^2\right)^{3/2}} + \frac{2\left(x^2+y^2\right)}{3\sqrt{3}\left(x^2-y^2\right)^2},$$

$\beta = \left(18+9\sqrt{2}+2\sqrt{3}\right)/72$ *and* $(R,\rho) \in \mathcal{D}^{(8)}$ *(see Fig. 3(c) for the plot of* $\mathcal{D}^{(8)}$*).*

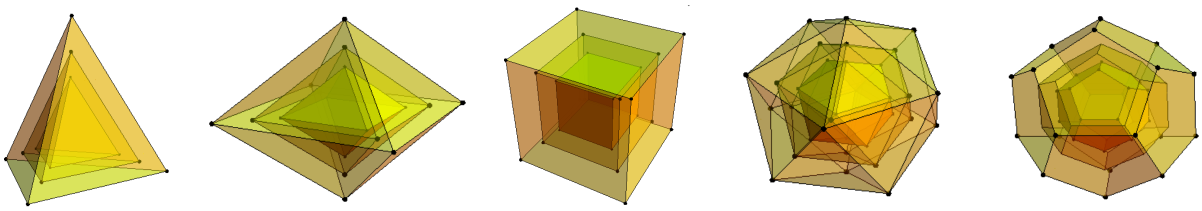
(d) *For* $N = 12$ *(i.e. three nested homothetic icosahedra) such configuration is central for the spatial* 36*–body problem when* $m = m(R,\rho)$ *and* $M = M(R,\rho)$ *are given by (1) with*

$$f(x,y) = \frac{2\sqrt{2}\left(\sqrt{5}x-5y\right)}{\left(\varphi x^2-4\phi yx+\varphi y^2\right)^{3/2}} - \frac{2\sqrt{2}\left(\sqrt{5}x+5y\right)}{\left(\varphi x^2+4\phi yx+\varphi y^2\right)^{3/2}} + \frac{4\sqrt{5-2\sqrt{5}}xy}{5\left(x^2-y^2\right)^2},$$
$$g(x,y) = \frac{2\sqrt{2}\left(5x-\sqrt{5}y\right)}{\left(\varphi x^2-4\phi yx+\varphi y^2\right)^{3/2}} + \frac{2\sqrt{2}\left(5x+\sqrt{5}y\right)}{\left(\varphi x^2+4\phi yx+\varphi y^2\right)^{3/2}} + \frac{2\sqrt{5-2\sqrt{5}}\left(x^2+y^2\right)}{5\left(x^2-y^2\right)^2},$$

$\beta = \left(5\sqrt{5}+\sqrt{5-2\sqrt{5}}\right)/20$ *and* $(R,\rho) \in \mathcal{D}^{(12)}$ *(see Fig. 3(d) for the plot of* $\mathcal{D}^{(12)}$*). Here* $\varphi = 5+\sqrt{5}$.

(e) *For* $N = 20$ *(i.e. three nested homothetic dodecahedra) such configuration is central for the spatial* 60*–body problem when* $m = m(R,\rho)$ *and* $M = M(R,\rho)$ *are given by (1) with*

$$\begin{aligned} f(x,y) = {} & \frac{2(x-3y)}{\left(3x^2-2yx+3y^2\right)^{3/2}} - \frac{2(x+3y)}{\left(3x^2+2yx+3y^2\right)^{3/2}} + \\ & \frac{\left(5+\sqrt{5}\right)x-6\phi y}{2\phi\left(3x^2-2\sqrt{5}yx+3y^2\right)^{3/2}} - \frac{\left(5+\sqrt{5}\right)x+6\phi y}{2\phi\left(3x^2+2\sqrt{5}yx+3y^2\right)^{3/2}} + \\ & \frac{4xy}{3\sqrt{3}\left(x^2-y^2\right)^2}, \\ g(x,y) = {} & \frac{2(3x-y)}{\left(3x^2-2yx+3y^2\right)^{3/2}} + \frac{2(3x+y)}{\left(3x^2+2yx+3y^2\right)^{3/2}} + \\ & \frac{6\phi x-\left(5+\sqrt{5}\right)y}{2\phi\left(3x^2-2\sqrt{5}yx+3y^2\right)^{3/2}} + \frac{6\phi x+\left(5+\sqrt{5}\right)y}{2\phi\left(3x^2+2\sqrt{5}yx+3y^2\right)^{3/2}} + \\ & \frac{2\left(x^2+y^2\right)}{3\sqrt{3}\left(x^2-y^2\right)^2}, \end{aligned}$$



(a) $N = 4$ Tetrahedra | (b) $N = 6$ Octahedra | (c) $N = 8$ Cube | (d) $N = 12$ Icosahedra | (e) $N = 20$ Dodecahedra

Figure 2.— Three nested homothetic polyhedra.

$\beta = \left(18 + 9\sqrt{2} + \sqrt{3} + 9\sqrt{5}\right)/36$ *and* $(R, \rho) \in \mathcal{D}^{(20)}$ *(see Fig. 3(e) for the plot of* $\mathcal{D}^{(20)}$*).*

The proof of Theorem 2 can be found in [3].

After analyzing the level curves $m(R, \rho) = c_1$ and $M(R, \rho) = c_2$ for some $c_1, c_2 > 0$ we get numerical evidence that the following conjecture can be true (see also [3]).

Conjecture 3 *For each* $N = 4, 6, 8, 12, 20$*, fixed a pair of values* $m > 0$ *and* $M > 0$ *there exists a unique pair of values* $(R, \rho) \in \mathcal{D}^{(N)}$ *for which the configuration of three nested homothetic polyhedra of* N *vertices defined in Theorem 2 is central for the* $3N$*–body problem.*

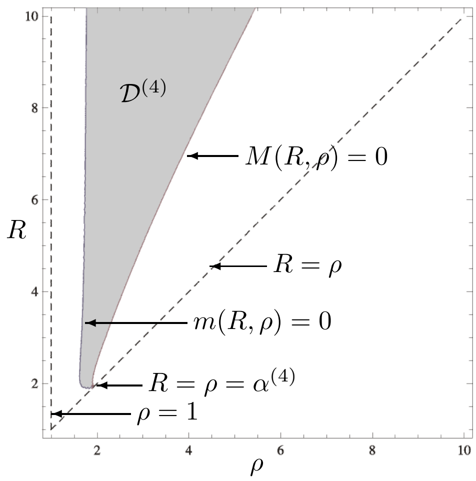
If we put an additional mass $m_0 = \mu > 0$ at the center of mass of the three nested homothetic polyhedra, then we obtain an analogous of Theorem 2 with the difference that now m, M and the region $\mathcal{D}^{(N)}$ depend of μ (see [4]). Consequently if the ratios of the masses and the ratios of the length of the edges of the polyhedra satisfy some convenient relations, then the configuration of three nested homothetic polyhedra with a mass at their center of mass is also central for the $3N + 1$–body problem.

2.3 p nested homothetic polyhedra

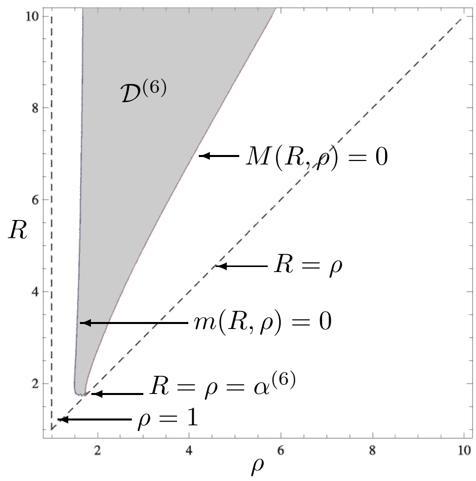
As happen with the nested N–gons, one would expect that the following conjecture can be true.

Conjecture 4 *Theorems 1 and 2 can be extended to spatial central configurations of the* pN*–body problem with* p *nested regular polyhedra of* N *vertices each one, for all* $p \geqslant 4$*.*

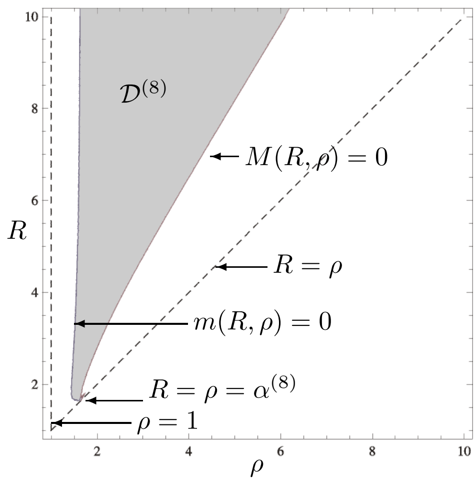
We left the proof of this conjecture for a future work.



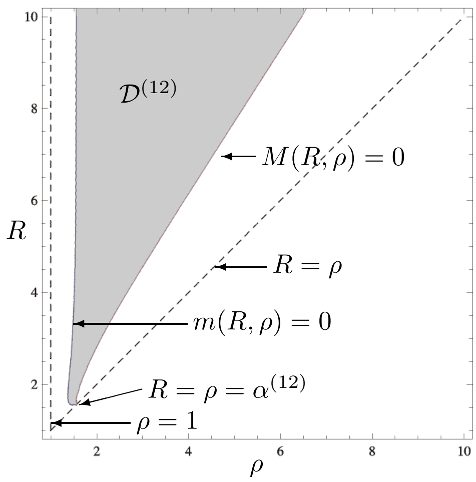
(a) $N = 4$.



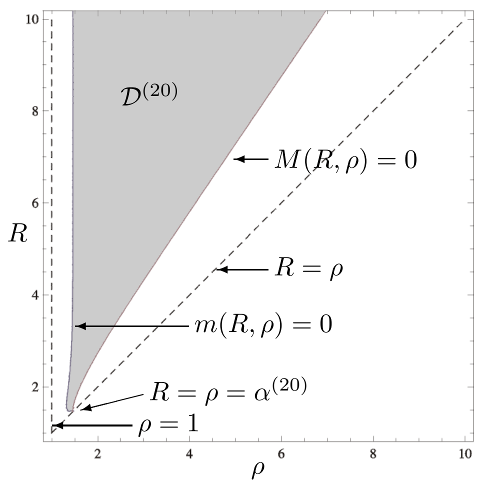
(b) $N = 6$.



(c) $N = 8$.



(d) $N = 12$.



(e) $N = 20$.

Figure 3.— The region $\mathcal{D}^{(N)}$ for $N = 4, 6, 8, 12, 20$. The numerical values of $\alpha^{(N)}$ are given in Theorem 1.

3 Nested rotated tetrahedra

3.1 Two nested rotated tetrahedra

Without loss of generality we take the unit of mass in such a way that the masses of the inner tetrahedron are equal to one. Recall that the set of central configurations is invariant under homothecies and rotations, so by choosing conveniently the unit of length we can assume that the positions of the vertices of the inner tetrahedron are the ones of Table 1.

Theorem 5 *Consider four equal masses $m_i = 1$ for $i = 1, \ldots, 4$ at the vertices of a regular tetrahedron having positions $\mathbf{q}_i = \mathbf{a}_i^{(4)}$ for $i = 1, \ldots, 4$. Consider four additional equal masses $m_{i+4} = m$ for $i = 1, \ldots, 4$ at the vertices of a second nested rotated regular tetrahedron having positions $\mathbf{q}_{i+4} = \rho\mathcal{R}\mathbf{a}_i^{(4)}$ for $i = 1, \ldots, 4$ where $\rho \geqslant 1$ and $\mathcal{R}$ is an arbitrary rotation matrix of $SO(3)$. The following statements hold.*

(a) There are two unique classes of central configurations of two nested rotated regular tetrahedra.

(a.1) The class of configurations of Type I with $\{\mathbf{q}_5, \mathbf{q}_6, \mathbf{q}_7, \mathbf{q}_8\} = \{\rho\mathbf{a}_1, \rho\mathbf{a}_2, \rho\mathbf{a}_3, \rho\mathbf{a}_4\}$ which is described in Theorem 1(a).

(a.2) The class of configurations of Type II with $\{\mathbf{q}_5, \mathbf{q}_6, \mathbf{q}_7, \mathbf{q}_8\} = \{\rho P\mathbf{a}_1, \rho P\mathbf{a}_2, \rho P\mathbf{a}_3, \rho P\mathbf{a}_4\}$ where P is the rotation matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

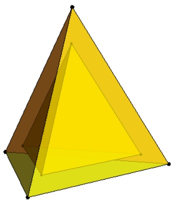
These two classes of configurations are shown in Figure 4.

(b) The configuration of Type II is central for the spatial 8–body problem when

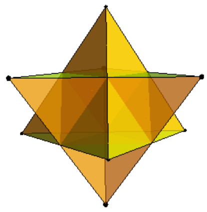
$$m = \frac{b(\rho)}{f(\rho)} = \frac{-\dfrac{\rho}{2} + \dfrac{2\sqrt{2}(3\rho-1)}{(3\rho^2-2\rho+3)^{3/2}} + \dfrac{(2/3)^{3/2}}{(\rho+1)^2}}{-\dfrac{2\sqrt{2}(\rho-3)\rho}{(3\rho^2-2\rho+3)^{3/2}} + \dfrac{(2/3)^{3/2}\rho}{(\rho+1)^2} - \dfrac{1}{2\rho^2}},$$

and $\rho \in [1, \alpha_1) \cup (\alpha_2, \infty)$ where $\alpha_1 = 1.3981650369\ldots$ is the unique real solution of $b(\rho) = 0$ with $\rho \geqslant 1$ and $\alpha_2 = 6.5360793703\ldots$ is the unique real solution of $f(\rho) = 0$ with $\rho \geqslant 1$. Moreover the following statements hold.

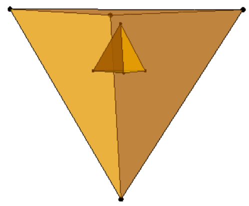
(b.1) There is a unique central configuration of Type II with $\rho \in [1, \alpha_1)$ for $m \in (0, 1]$. In this configuration the faces of both tetrahedra intersect.



(a) Conf. of Type I
$\rho > 1$



(b) Conf. of Type II
$1 \leqslant \rho \leqslant 3$



(c) Conf. of Type II
$\rho > 3$

Figure 4.— Plot of the possible classes of configurations of two nested regular tetrahedra. Notice that we have two possibilities for the configurations of Type II, in (b) the faces of both tetrahedra intersect whereas in (c) they do not intersect.

(b.2) *Let* $\alpha_3 = 8.7756058918\ldots$ *be the unique real solution of* $(n(\rho)/d(\rho))'$ *with* $\rho \geqslant 1$ *and* $m_0 = n(\alpha_3)/d(\alpha_3) = 2880.33\ldots$. *There are no central configurations for* $m \in (1, m_0)$.

(b.3) *There is a unique central configuration of Type II with* $\rho = \alpha_3$ *for* $m = m_0$. *In this configuration the faces of the tetrahedra do not intersect.*

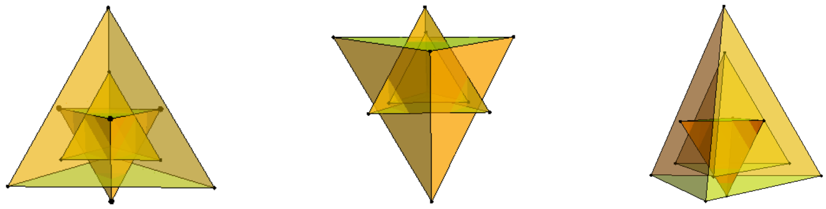
(b.4) *There are two central configurations of Type II, one with* $\rho \in (\alpha_2, \alpha_3)$ *and the other with* $\rho \in (\alpha_3, \infty)$, *for* $m \in (m_0, \infty)$. *In this configuration the faces of the tetrahedra do not intersect.*

The proof of Theorem 5 can be found in [4]. In [9] the authors prove statement (b) of Theorem 5 in a different way. We note that in [9] the authors also consider values of the scale factor $0 < \rho < 1$. The central configurations of two nested rotated tetrahedra for values of the scale factor $0 < \rho < 1$ can also be obtained from the ones given by Theorem 5 by replacing m by $1/m$ and ρ by $1/\rho$.

If we put an additional mass $m_0 = \mu > 0$ at the center of mass of the two nested rotated tetrahedra, then we obtain an analogous of Theorem 5(b) with the difference that now m and α_1 depend of μ. The value $\widetilde{\alpha}_1(\mu)$ tends to the value α_1 given by Theorem 5(b) when $\mu \to 0$ and it tends to 1 when $\mu \to +\infty$. Consequently if the ratios of the masses and the ratios of the length of the edges of the tetrahedra satisfy some convenient relations, then the configuration of two nested rotated tetrahedra with a mass at their center of mass is also central for the 9–body problem.

3.2 Three nested rotated tetrahedra

In this section we analyze the spatial central configurations of the 12–body problem when the masses are located at the vertices of three nested regular tetrahedra with scale



(a) $1 \leqslant \rho < R$ (b) $1 < R \leqslant \rho$ (c) $0 < \rho < 1 < R$

Figure 5.— Three possible central configurations with three nested regular tetrahedra with scale factors 1, ρ and R and the one with scale factor ρ rotated with respect to the other two by a rotation of Euler angles $\alpha = 0$, $\beta = \pi$ and $\gamma = 0$. In (a) the medium tetrahedra is the rotated one, in (b) is the outermost, and in (c) is the innermost. In any of these three possibilities the tetrahedra can intersect or not.

factors 1, ρ and R and some of them rotated with respect to the others by a rotation of Euler angles $\alpha = 0$, $\beta = \pi$ and $\gamma = 0$ (i.e. by a rotation with rotation matrix P). Taking conveniently the unit of masses we can assume that all the masses of the tetrahedron with scale factor 1 are equal to one. We also choose the unit of length in such a way that the edges of the tetrahedron with scale factor 1 have length 2. Since the set of central configurations is invariant under homothecies and rotations, without loss of generality we can assume that the positions of the vertices of the tetrahera of scale factor 1 are the ones of Table 1, that there is only one rotated tetrahedra which is the tetrahedra with scale factor ρ, and that $1 \leqslant \rho < R$ when the rotated tetrahedra is the medium one, $1 < R \leqslant \rho$ when the rotated tetrahedra is the outer one, and finally $0 < \rho < 1 < R$ when the rotated tetrahedra is the inner one (see Figure 5). We define the sets

$$\begin{aligned}
\mathcal{C}_m &= \{(\rho, R) \in \mathbb{R}^2 \,:\, 1 \leqslant \rho < R\}, \\
\mathcal{C}_o &= \{(\rho, R) \in \mathbb{R}^2 \,:\, 1 < R \leqslant \rho\}, \\
\mathcal{C}_i &= \{(\rho, R) \in \mathbb{R}^2 \,:\, 0 < \rho < 1 < R\}.
\end{aligned}$$

Result 6 *Consider four equal masses $m_i = 1$ for $i = 1, \ldots, 4$ at the vertices of a regular tetrahedron having positions $\mathbf{q}_i = \mathbf{a}_i^{(4)}$ for $i = 1, \ldots, 4$. Consider four additional equal masses $m_{i+4} = m$ for $i = 1, \ldots, 4$ at the vertices of a second nested rotated regular tetrahedron having positions $\mathbf{q}_{i+4} = \rho P \mathbf{a}_i^{(4)}$ for $i = 1, \ldots, 4$ with scale factor $\rho > 0$. Finally we consider four additional equal masses $m_{i+8} = M$ for $i = 1, \ldots, 4$ at the vertices of a third nested regular tetrahedron having positions $\mathbf{q}_{i+8} = R\mathbf{a}_i^{(4)}$ for $i = 1, \ldots, 4$ with scale* *factor $R>1$. Let*

$$
\begin{aligned}
m = m(\rho,R) &= \frac{1}{\det(A)}\begin{vmatrix} 1 & 1/2 & g_1(R) \\ \rho & -f(\rho,1) & f(\rho,R) \\ R & g_2(R) & -1/(2R^2) \end{vmatrix}, \\
M = M(\rho,R) &= \frac{1}{\det(A)}\begin{vmatrix} 1 & f(1,\rho) & 1/2 \\ \rho & -1/(2\rho^2) & -f(\rho,1) \\ R & f(R,\rho) & g_2(R) \end{vmatrix},
\end{aligned} \tag{2}
$$

where $A = \begin{pmatrix} 1 & f(1,\rho) & g_1(R) \\ \rho & -1/(2\rho^2) & f(\rho,R) \\ R & f(R,\rho) & -1/(2R^2) \end{pmatrix}$ *and*

$$
\begin{aligned}
f(x,y) &= \frac{2\sqrt{2}(y-3x)}{(3x^2-2yx+3y^2)^{3/2}} - \frac{(2/3)^{3/2}}{(x+y)^2}, \\
g_1(x) &= -\frac{2\sqrt{2}(x+3)}{(3x^2+2x+3)^{3/2}} + \frac{(2/3)^{3/2}}{(x-1)^2}, \\
g_2(x) &= \frac{2\sqrt{2}(3x+1)}{(3x^2+2x+3)^{3/2}} + \frac{(2/3)^{3/2}}{(x-1)^2}.
\end{aligned}
$$

Then the following statements hold.

(a) *Such configuration is central for the spatial 12–body problem when $m = m(\rho,R)$ and $M = M(\rho,R)$ are given by the expression (2) and $(R,\rho) \in \mathcal{D} = \{(R,\rho) \in \mathbb{R}^2 : \det(A) \neq 0,\ m(R,\rho) > 0,\ M(R,\rho) > 0,\ \rho > 0,\ R > 1\}$ (see Figure 6 for the plot of $\mathcal{D}$).*

(b) *The set $\mathcal{D}$ is formed by the disjoint union of the sets $\mathcal{D}_i$ for $i = 1,\dots,6$ defined in Figure 6. Then the regions $\mathcal{D}_i$ provide central configurations of the 12–body problem with the masses located at the vertices of three nested regular tetrahedra satisfying the following.*

(b.1) *The rotated tetrahedron is the inner one and it intersects only the medium when $(\rho,R) \in \mathcal{D}_2 \cap \mathcal{C}_i$, it intersects the medium and the outer when $(\rho,R) \in \mathcal{D}_4 \cap \mathcal{C}_i$, and the three tetrahedra do not intersect when $(\rho,R) \in \mathcal{D}_1$.*

(b.2) *The rotated tetrahedron is the medium one and it intersects only the inner when $(\rho,R) \in \mathcal{D}_2 \cap \mathcal{C}_m$ and when $(\rho,R) \in \mathcal{D}_3 \cap \{\rho \leqslant 3\}$, it intersects the inner and the outer when $(\rho,R) \in \mathcal{D}_5 \cap \mathcal{C}_m \cap \{\rho \leqslant 3\}$ and when $(\rho,R) \in \mathcal{D}_4 \cap \mathcal{C}_m$, it intersects only the outer one when $(\rho,R) \in \mathcal{D}_5 \cap \mathcal{C}_m \cap \{\rho > 3\}$, and the three tetrahedra do not intersect when $(\rho,R) \in \mathcal{D}_3 \cap \{\rho > 3\}$.*

(b.3) The rotated tetrahedron is the outer one and it intersects only the medium when $(\rho, R) \in \mathcal{D}_5 \cap \mathcal{C}_o \cap \{\rho > 3\}$*, it intersects the inner and the medium when* $(\rho, R) \in \mathcal{D}_5 \cap \mathcal{C}_o \cap \{\rho \leqslant 3\}$*, and the three tetrahedra do not intersect when* $(\rho, R) \in \mathcal{D}_6$.

(c) Let

$$m(M, \rho, R) = \frac{\rho^2(\rho + 2f(\rho, 1))}{2\rho^3 f(1, \rho) + 1} - M\frac{2\rho^2(\rho g_1(R) - f(\rho, R))}{2\rho^3 f(1, \rho) + 1},$$

and $M_0(\rho, R) = (\rho^2(\rho + 2f(\rho, 1)))/(2\rho^2(\rho g_1(R) - f(\rho, R))$*, and let*

$$\begin{aligned} \mathbf{p} &= (p_1, p_2) = (1.5094116757\ldots, 1.9479968088\ldots), \\ \mathbf{q} &= (q_1, q_2) = (1.9926247501\ldots, 31.4606148079\ldots), \end{aligned}$$

be the solutions of system $\det(A) = \det(A_2) = \det(A_3) = 0$ *(see (1) for the definitions of* A_2 *and* A_3*). Then the following statements hold.*

(c.1) For each $M > M_0(\mathbf{p})$*, where* $M_0(\mathbf{p}) = 0.1252891302\ldots$*, and* $m = m(M, \mathbf{p}) = -0.4001317738\cdots + 3.1936671053\ldots M$ *we have a central configuration of the spatial* 12*–body problem when* $(\rho, R) = \mathbf{p}$*. In this configuration the rotated tetrahedron is the medium and it intersects with the inner and the outer.*

(c.2) For each $M > M_0(\mathbf{q})$*, where* $M_0(\mathbf{q}) = 25204.620455\ldots$*, and* $m = m(M, \mathbf{q}) = -3.5811747685\cdots + 0.0001420840585\ldots M$ *we have a central configuration of the spatial* 12*–body problem when* $(\rho, R) = \mathbf{q}$*. In this configuration the rotated tetrahedron is the medium and it intersects only the inner.*

We call the previous result as Result 6 instead of Theorem 6 because part of its proof is done numerically with the help of Mathematica. The prove of Result 6 can be found in [4].

After analyzing the plot of $m(\rho, R)$ and $M(\rho, R)$ on the regions $\mathcal{D}_i$ for $i = 1, \ldots, 6$ we observe the following (see for details [4]). On $\mathcal{D}_1$, $m(\rho, R) \in (0, 0.000347182\ldots)$ and $M(\rho, R) \in (0, \infty)$. On $\mathcal{D}_2$, $m(\rho, R) \in (0, \infty)$ and $M(\rho, R) \in (25204.62\ldots, \infty)$. On $\mathcal{D}_3$, $m(\rho, R) \in (0, \infty)$ and $M(\rho, R) \in (11625.1167\ldots, \infty)$. On $\mathcal{D}_4$, $m(\rho, R) \in (0, \infty)$ and $M(\rho, R) \in (0, \infty)$. On $\mathcal{D}_5$, $m(\rho, R) \in (0, \infty)$. and $M(\rho, R) \in (0, \infty)$. And finally, on $\mathcal{D}_6$, $m(\rho, R) \in (14997.0524\ldots, \infty)$ and $M(\rho, R) \in (0, \infty)$.

If we put an additional mass $m_0 = \mu > 0$ at the center of mass of the three nested rotated tetrahedra, then we obtain an analogous of Theorem 6 with the difference that now m, M and the region $\mathcal{D}$ depend of μ. Consequently, if the ratios of the masses and the ratios of the length of the edges of the tetrahedra satisfy some convenient relations, then the configuration of three nested rotated tetrahedra with a mass at their center of mass is also central for the 13–body problem.

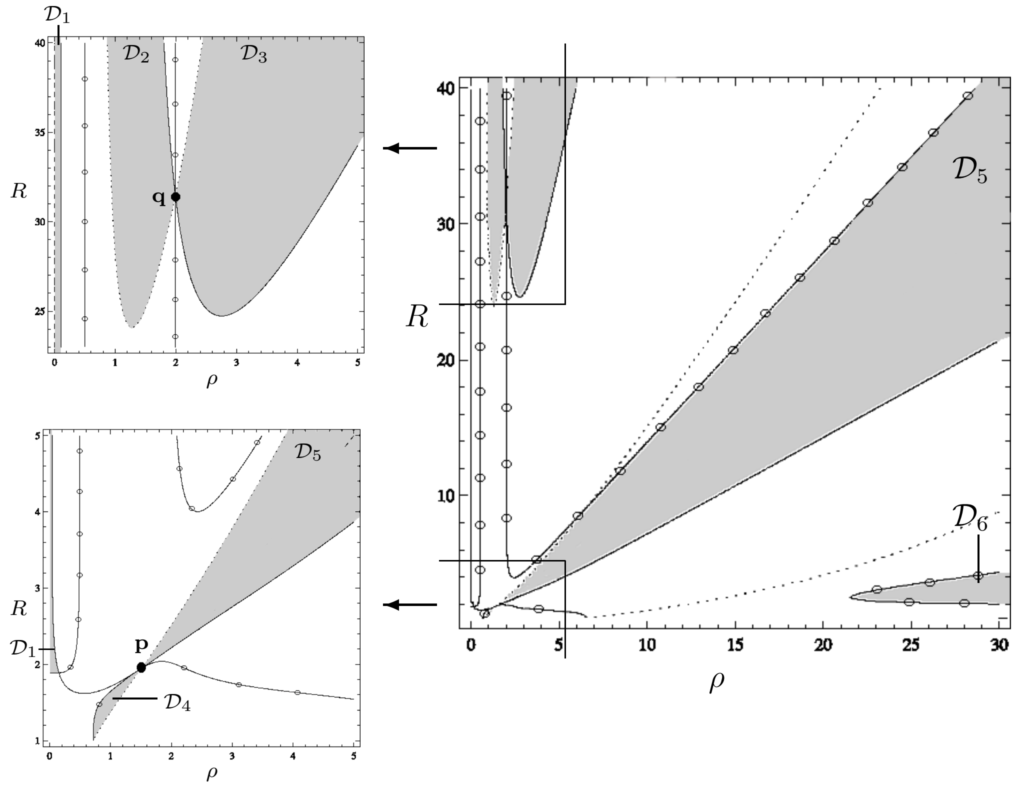


Figure 6.— The set $\mathcal{D}$. The dotted curves correspond to points where $\det(A) = 0$, the continuous curves correspond to points where $m(\rho, R) = 0$, and finally, the curves with small circles correspond to points where $M(\rho, R) = 0$.

Acknowledgements

Both authors are supported by a MEC/FEDER grant number MTM2008–03437. The second author is also supported by a CONACIT grant number 2005SGR 00550.

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