

Is there chaos in Copenhagen problem?

Roberto Barrio*, Fernando Blesa† and Sergio Serrano‡

GME, Universidad de Zaragoza

Abstract

We present a preliminary study of the Copenhagen problem providing new techniques to approach this classical problem. In particular, the OFLI2 Chaos Indicator and the Crash Test. With the OFLI2 indicator we have studied the chaoticity of the orbits and with the Crash Test we have classified the orbits as bounded, escape or collisions.

1 Introduction

The three body problem is one of the oldest problems in dynamical systems. The restricted three-body problem (RTBP) supposes that the mass of one of the three bodies is negligible. It was considered by Euler (1772) and Jacobi (1836), and it can serve as an example of classical chaos. For the remaining two bodies, the case of equal masses was first investigated by Strömberg and his colleagues of the Copenhagen group. Its name is derived from the series of papers published by them starting in 1913.

Defining the distances to the respective primaries as:

$$\begin{aligned}r_1^2 &= (x + \mu)^2 + y^2 \\r_2^2 &= (x - (1 - \mu))^2 + y^2,\end{aligned}$$

where $\mu = m_1/(m_1 + m_2)$ with m_1 and m_2 the masses of the two bodies, the equations of motion of the Restricted Three Body Problem are

$$\begin{aligned}\ddot{x} - 2\dot{y} &= x - (1 - \mu)\frac{x + \mu}{r_1^3} - \mu\frac{x - 1 + \mu}{r_2^3} \\ \ddot{y} + 2\dot{x} &= y - (1 - \mu)\frac{y}{r_1^3} - \mu\frac{y}{r_2^3}\end{aligned}$$

The only known integral of motion of this problem is the Jacobian constant

*Depto. Matemática Aplicada

†Depto. Física Aplicada

‡Depto. Informática e Ingeniería de Sistemas

$$C = -(\dot{x}^2 + \dot{y}^2) + 2\frac{1-\mu}{r_1} + 2\frac{\mu}{r_2} + x^2 + y^2$$

what can be used to define the effective energy as $E_J = -C/2$. Since the Copenhagen problem is the particular case of $m_1 = m_2$ we have to take the value $\mu = 1/2$.

2 The numerical techniques

In this paper we focus our attention in the use of state-of-the-art numerical techniques for the analysis of the qualitative properties of this problem. The first technique is OFLI2 [1], which is a *fast chaos indicator*, that is, a fast numerical technique to detect chaos. The method is based on the use of the second order variational equations and as numerical ODE integrator we use a specially developed Taylor method [2, 3] that gives a fast and accurate numerical integration. The OFLI2 looks for detecting the set of initial conditions where we may expect sensitive dependence on initial conditions. Note that this is one of the classical requirements for chaos (Devaney's definition of chaos [4] requires transitivity, density of periodic orbits and sensitive dependence on initial conditions). On the figures with the OFLI2 results we have used the red colour to point the chaotic regions and blue for the most regular ones, being the intermediate colours the transition for one to the another situation.

In the numerical study of a dynamical system a common tool is the Poincaré surfaces of section [5] (from now on PSS). The basic idea is to select a two dimensional manifold perpendicular to most of the trajectories of the system and to study their cuts. Quite recently it has been developed an extension of such a technique quite useful for systems with several bodies. This extension is denoted as *the crash test* [8]. The crash test (in the following CT) is based on classifying the orbits in a PSS depending on several parameters. By instance, we may study if an orbit is direct or retrograde, bounded or an escape orbit, if the orbit crashes to another body or not, and so on. To each kind of orbits is assigned a different colour and so the picture gives much more information than just the classical PSS. Another extension [7, 8] is to make an analysis of a mesh of initial conditions and to assign them a colour depending on their behaviour. The difference of this second extension is that now we do not look for cuts with the manifold, being just a manifold of initial conditions. Note that Nagler [7, 8] denominated Orbit Type Diagrams to these methods. We have preferred to call them as Crash Tests because we do not intend to classify any orbit, just to distinguish among some basic behaviors:

- *Collision with the central body.* We consider that a collision with a body occurs when the distance of the orbit to the body is lower than $r_c = 10^{-4}$. We give it a blue or cyan colour depending on the collision body.

- *Escape orbits.* Since we are dealing with a non-integrable problem, it is not easy to prove if the motion is bounded or not. After several numerical tests we have decided to perform a numerical integration up to a final time $t_{\text{final}} = 20000$ and to check if, along the computed trajectory, the orbit reaches a value of $R = \|(x, y)\| > R_{\text{max}} = 10$, then we suppose the orbit is far enough of the two bodies, and we stop the integration. We assign it a yellow (t_{escape} small) or red colour (t_{escape} large). We have fixed the value of t_{final} and R_{max}
- *Bounded motion.* In other case we classify the orbit as bounded orbit and we give it a green colour (in the online version).

3 Numerical results

In Figure 1 we show the evolution with the energy of the CT (C1) and the OFLI2 (O1) studying the variation in the x -axis by fixing $y = \dot{x} = 0$ and depending on the energy E_J . We note as these pictures give us a clear idea of the evolution of the system. When E_J is low the system has large forbidden regions and the motion is highly regular. Increasing E_J the system is more and more complex. The OFLI2 plot shows that the chaotic behavior appears mainly in the range $E_J \in [-1.75, 0]$. When E_J is very high the behavior is again more regular. The CT plot tells us that the chaotic regions are associated with regions where the probability of a collision is quite high. The bounded regions are inside and around the collision orbits and several regions of escape orbits appear, specially far from the bodies.

In Figure 2 we show some pictures for several values of the energy E_J in the (x, y) plane. For each value we present two pictures, one for the CT and another one for OFLI2. On white we plot the forbidden regions. The initial conditions are $x(0)$ and $y(0)$ given by the point of the (x, y) plane, $\dot{x}(0) = 0$ and $\dot{y}(0)$ determined by the energy E_J . Each figure consist on a regular grid of 600x600 points. The cases C2 and O2 are obtained for $E_J = -1.73$, cases C3 and O3 for $E_J = -1.5$ and cases C4 and O4 for $E_J = -0.75$. Note how the correspondence between the CT and the OFLI2 plots is high. Pictures C2 and O2 have a forbidden region which divides the motion in two separate zones. Inside near the bodies the motion is bounded or finishes with a collision. Outside there is a small bounded motion on the left (which corresponds to an elliptical orbit). Surrounding it there is a slow escape zone and the remaining orbits escape quickly. The escape and bounded regions have a more regular behavior and the collision regions have a more chaotic behavior.

In Figure 3 we show some pictures for several values of the energy E_J in the (x, \dot{x}) plane. For each value we present two pictures, one for the CT and another one for OFLI2. On white we plot again the forbidden regions. The initial conditions are $x(0)$ and $\dot{x}(0)$ given by the point of the (x, \dot{x}) plane, $x(0) = 0$ and $\dot{y}(0)$ obtained by the energy E_J . Each

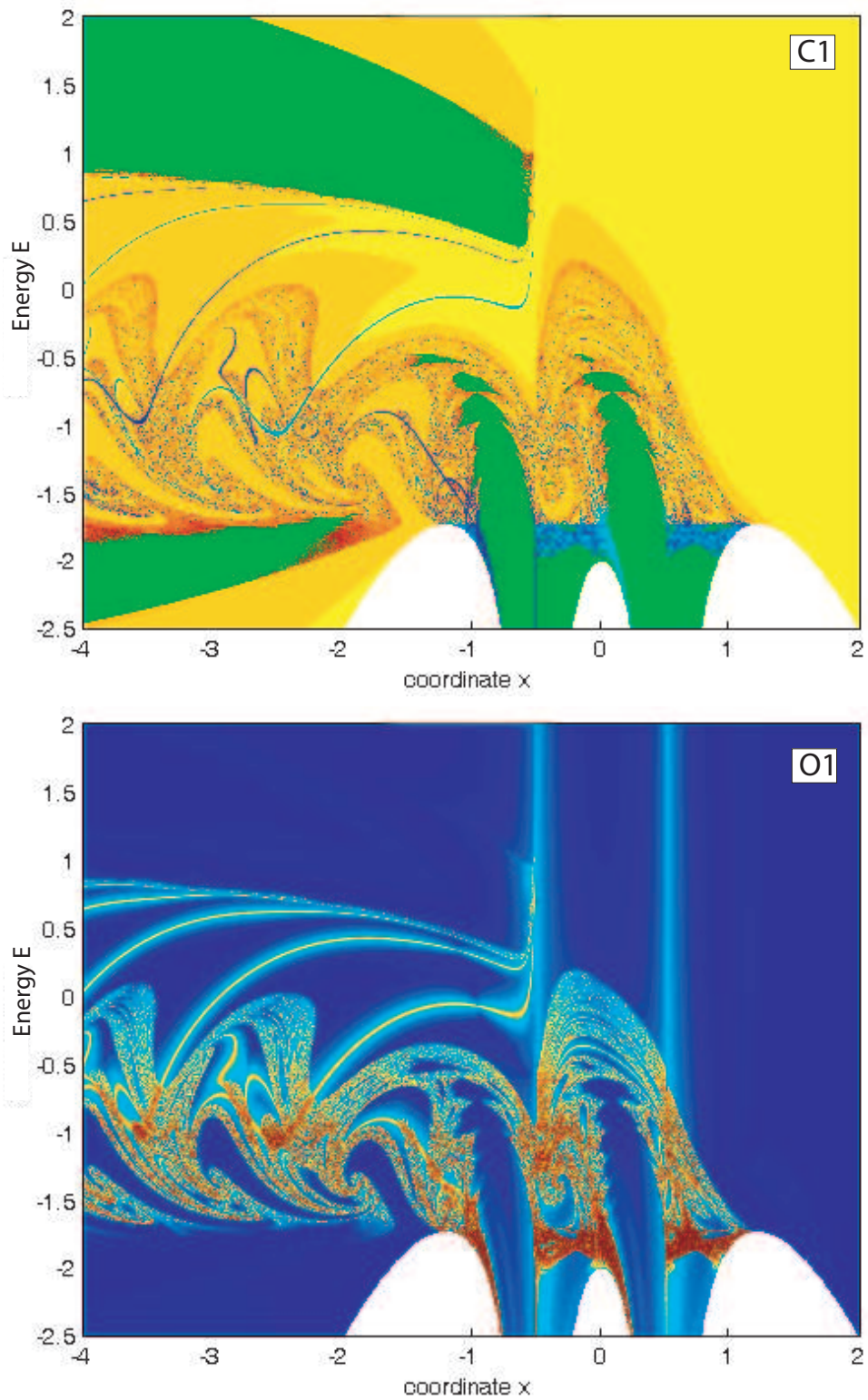


Figure 1.— CT, on the top, and OFLI2, on the bottom, plots showing the evolution with the energy (coordinate x vs. energy E_J plots).

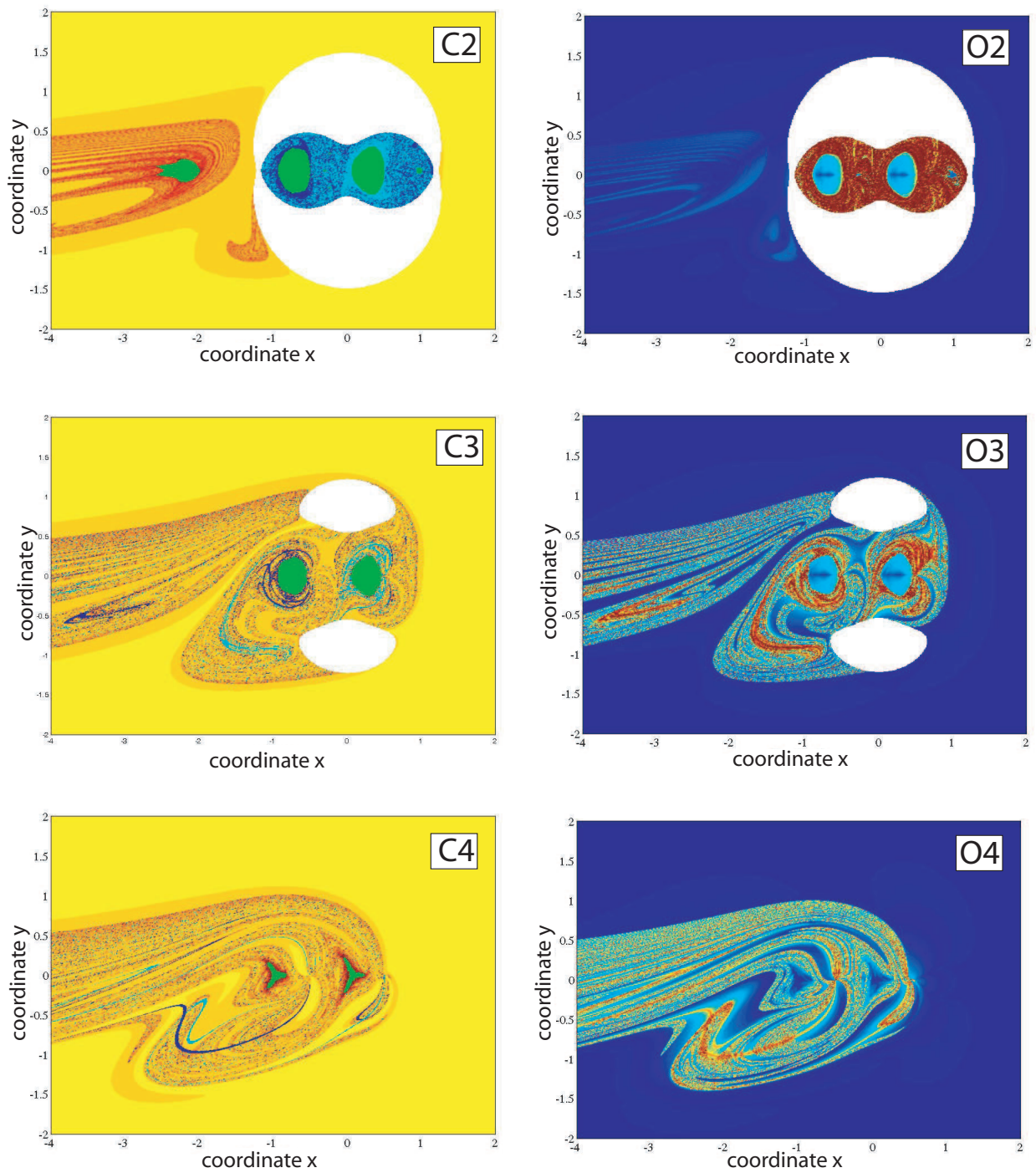


Figure 2.— CT, on the left, and OFLI2, on the right, plots (coordinate x vs. coordinate y) for three values of the Jacobi constant C .

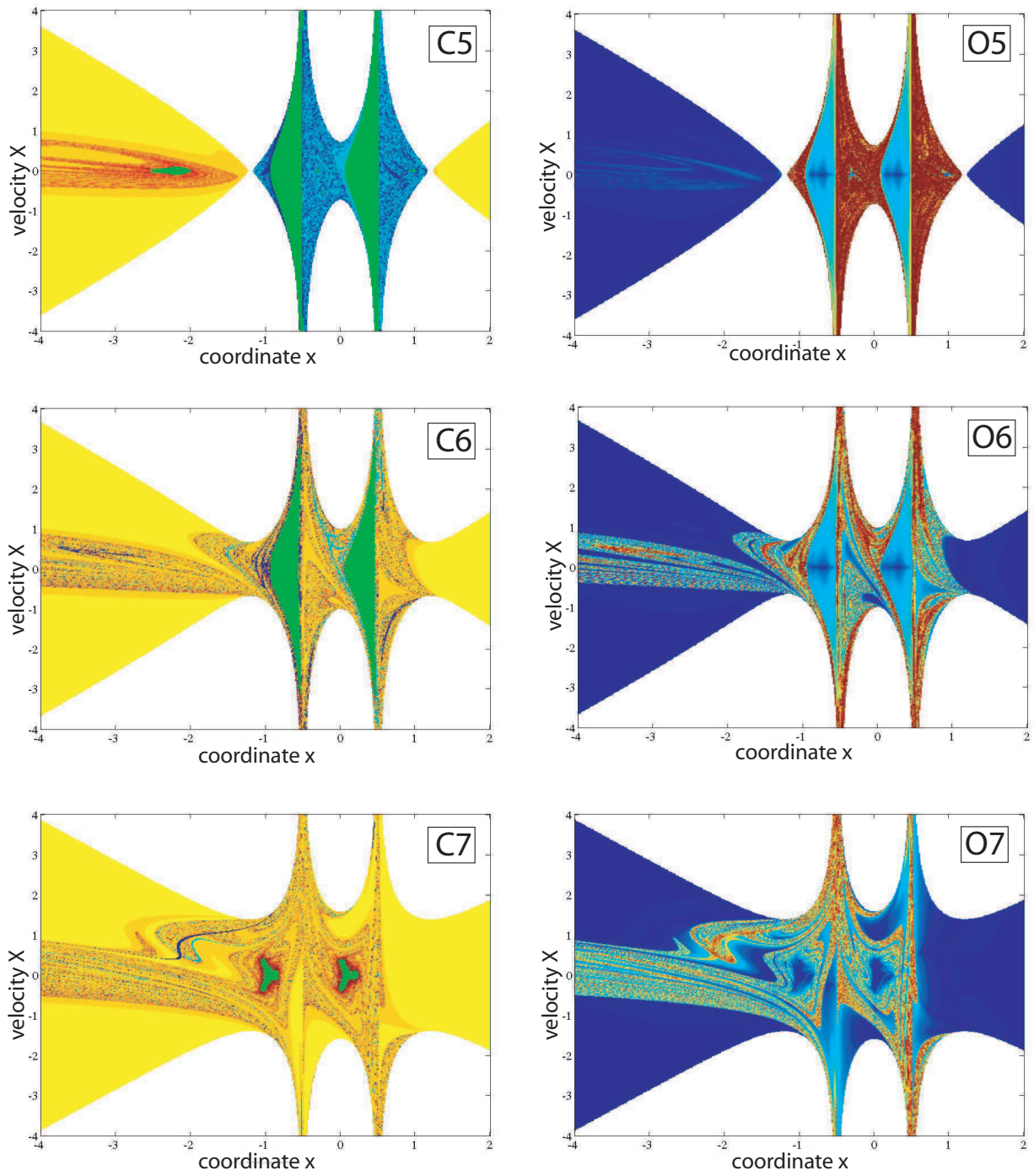


Figure 3.— CT, on the left, and OFLI2, on the right, plots (coordinate x vs. velocity \dot{x}) for three values of the Jacobi constant C .

figure consist on a regular grid of 600x600 points. The cases C5 and O5 are obtained for $E_J = -1.73$, cases C6 and O6 for $E_J = -1.5$ and cases C7 and O7 for $E_J = -0.75$.

For higher values of the energy, there are much more escape orbits, and the bounded motion regions decrease. There is a correspondence between escape and bounded motion for the CT and non chaotic regions for the OFLI2, since they are regular motions. On the other side, collision orbits are next to chaotic zones.

4 Conclusions

In this paper we have done a brief analysis of the classical Copenhagen problem, a particular case of the restricted three body problem, using some state-of-the-art numerical techniques. The OFLI2 has permitted to locate the region of initial conditions where we expect chaos and the CT to classify the kind of orbits (bounded, unbounded, collision,...). The results of the techniques give new sights on this classical problem and, moreover, both techniques complements one each other giving some confidence on the results. As a final answer to the question of the title, YES, there are chaos on the Copenhagen problem (and not in Copenhagen !!!).

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