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Some particularities of the collinear point L_3 in the RTBP

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Abstract

We are interested in studying the motion in a (big) neighborhood of the collinear equilibrium point L_3 of the RTBP. We consider both the planar and spatial cases. Actually different kinds of invariant objects appear: periodic orbits, invariant tori, the associated invariant manifolds, collision manifolds and homoclinic and heteroclinic phenomena among others. In this communication, we just present some particularities of L_3 and its 1-dimensional manifolds to show the difficulties that we have to cope with in order to give a global description of the motion in a global neighborhood of $_3$.

Key words and expressions: Restricted three-body problem, equilibrium point, periodic orbits, invariant tori, manifolds.

1 Introduction

Our framework is the circular RTBP, whose well known equations of motion depend on the mass parameter $\mu \in (0, 1/2]$, and in a rotating reference system are

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x, \\ \ddot{y} + 2\dot{x} &= \Omega_y, \\ \ddot{z} &= \Omega_z, \end{aligned}$$

where

$$\Omega = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1-\mu)$$

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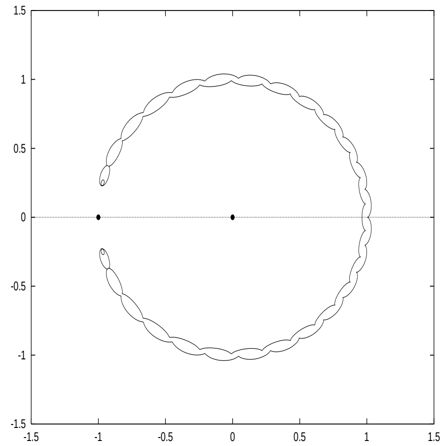

Figure 1.— Horseshoe motion

and $r_1 = \sqrt{(x-\mu)^2 + y^2 + z^2}$, $\quad r_2 = \sqrt{(x-\mu+1)^2 + y^2 + z^2}$. Such equations have 5 equilibrium points $L_1, \ldots, L_5$ (L_1, L_2 and L_3 are the collinear ones, while L_4 and L_5 are the equilateral ones), the Jacobi first integral given by

$$C = 2\Omega - \dot{x}^2 - \dot{y}^2 - \dot{z}^2.$$

and the well known symmetry

$$(x, y, z, x', y', z', t) \rightarrow (x, -y, z, -x', y', -t). \quad (1)$$

(see [13]).

In this communication we will deal with the collinear equilibrium points, with special emphasis on the L_3 point. Actually, L_3 is responsible for the **horseshoe motion** (see Figure 1). This kind of motion has been observed in some interesting situations, like the asteroid motion of Janus and Epimetheus as coorbitals of Saturn (see [11], [6], [7], [8] and [12]), and the motion of some more recent near Earth Asteriods (see [3] and [5]). Concerning horseshoe motion related to L_3, we mention [2], where a mechanism that explains horseshoe motion in the planar RTBP, for $\mu > 0$ small, is given from the $\mu = 0$ case, and also [1] where many families of horseshoe periodic orbits (both in the planar and spatial RTBP) are computed. On the other hand, we point out that the study of L_3 is also interesting from an academic point of view since it requires the development of new methodology. Actually the neighborhood of L_1, L_2 has been (and is being) systematically studied by both semi–analytical and numerical techniques (see [4] and [10]). But some of this methodology does not apply easily to the neighborhood of L_3 (see below).

We finally remark that our aim would be the description (as global as possible) of a neighborhood (as large as possible) of L_3 including homoclinic and heteroclinic phenomena, for different values of the mass parameter μ. Therefore we need to compute, in particular, periodic orbits (PO), invariant tori and the invariant manifolds of both PO

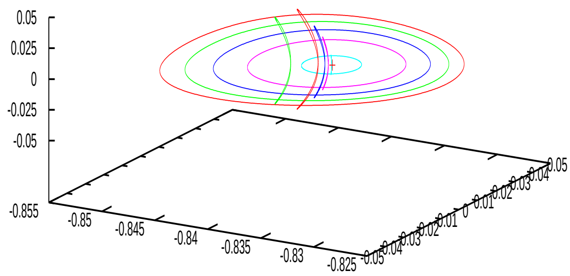

Figure 2.— Two families of Lyapunov PO.

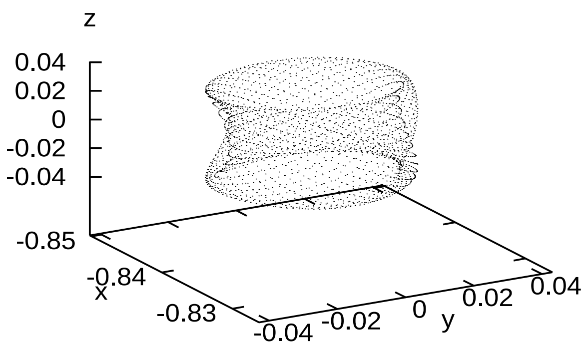

Figure 3.— A sample torus.

and tori. Along this talk, we will describe some specific particularities of L_3 and some problems that appear when studying and doing numerical explorations in a (maybe large) neighborhood of L_3.

2 Local/global behaviour around L_3

As it is well known, Spec$Df(0) = \{\pm i\omega_1, \pm i\omega_2, \pm\lambda\}$, so L_3 is a center$\times$center$\times$saddle type equilibrium point. Thus, a first insight to the local dynamics around L_3 gives two families of non-linear Lyapunov periodic orbits (PO) associated with the two centers (see Figure 2) and a 2-parametric (cantorian) family of 2-dimensional tori (see Figure 3). Of course, as far as the Jacobi constant decreases, the amplitudes of both the PO and tori increase. Therefore, in order to describe the global dynamics around L_3, it will be necessary to explore, in particular, the possible homoclinic connections to the existing invariant objects: L_3, the PO and the tori.

Therefore, first the homoclinic connections to L_3, when μ increases, is explored. Then,

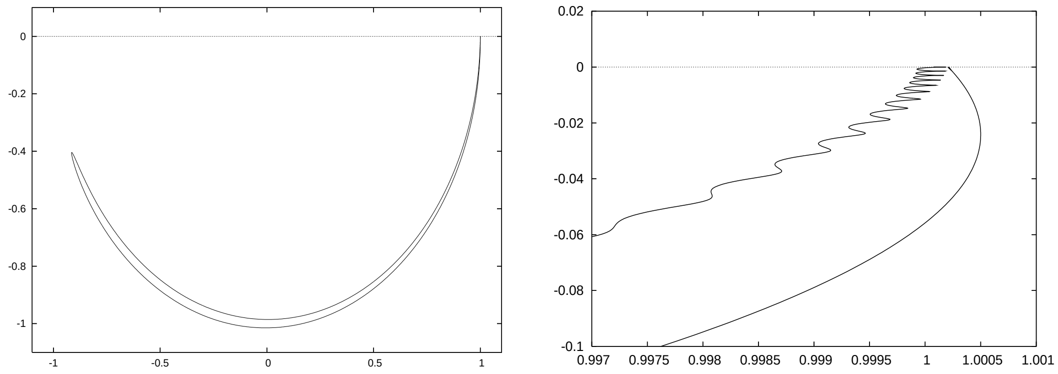

Figure 4.— Horsehsoe-shaped invariant manifold of L_3.

the case of the Lyapunov families of PO, when μ is fixed and C_J decreases, will be studied. In this case, it will be necessary to take into account the bifurcations that appear in both, planar and vertical Lyapunov families. Finally, we will consider the families of invariant tori.

3 Difficulties around L_3

On one hand, let us say that the invariant 1-dimensional manifolds of L_3 may be regarded as the skeleton of the 2-dimensional manifolds of the Lyapunov PO close to L_3 (when decreasing the Jacobi constant) and therefore of the 3-dimensional manifolds of the tori associated with the Lyapunov PO. On the other hand, the specific particularities of L_3 give rise to some numerical difficulties when analysing the homoclinic connections. Let us mention some of these particularities:

- For small μ, both the unstable and stable manifolds of L_3, $W^u(L_3)$ and $W^s(L_3)$, return to the neighborhood of L_3, see Figure 4.
- Let us take small values of μ and let us concentrate on planar homoclinic orbits to L_3. We will distinguish between symmetric and non-symmetric homoclinic orbits. Due to symmetry (1), if an invariant manifold of L_3 has an orthogonal crossing with the $\{y = 0\}$ axis (i.e. for a suitable time t, we have $y(t) = x'(t) = 0$), then it becomes a symmetric homoclinic connection of L_3. This homoclinic orbit will have a horseshoe shape for μ small enough. Actually, if one considers the homoclinic connections of L_3 with only one crossing with the x axis (the orthogonal one), there exists a sequence of values of $\mu_k \to 0$ for which this is the case (see [9] and [2]).
- The loops of the invariant manifolds that appear close to L_3 are inherited by the invariant manifolds of planar Lyapunov orbits. That is, if we decrease the Jacobi constant, the Lyapunov PO has 2-dimensional invariant manifolds with also a horseshoe shape and loops close to the Lyapunov orbit (see Figure 5). Therefore, when

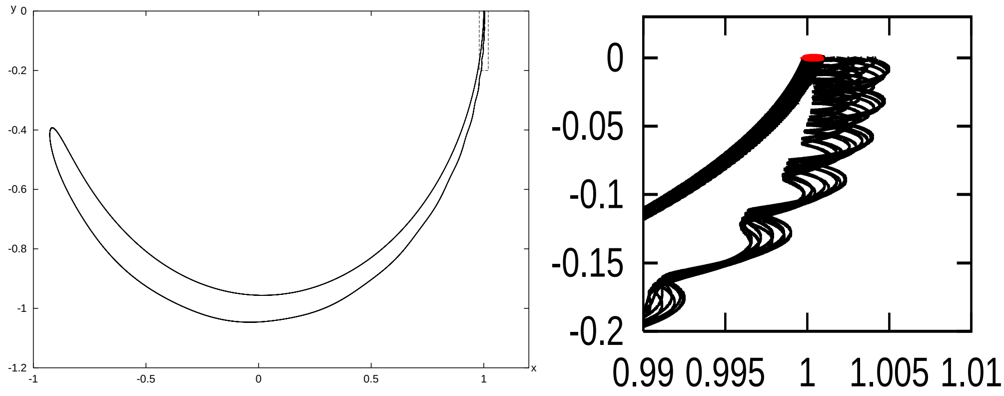

Figure 5.— Left: horsehsoe-shaped invariant manifold of a Lyapunov PO. Right: zoom.

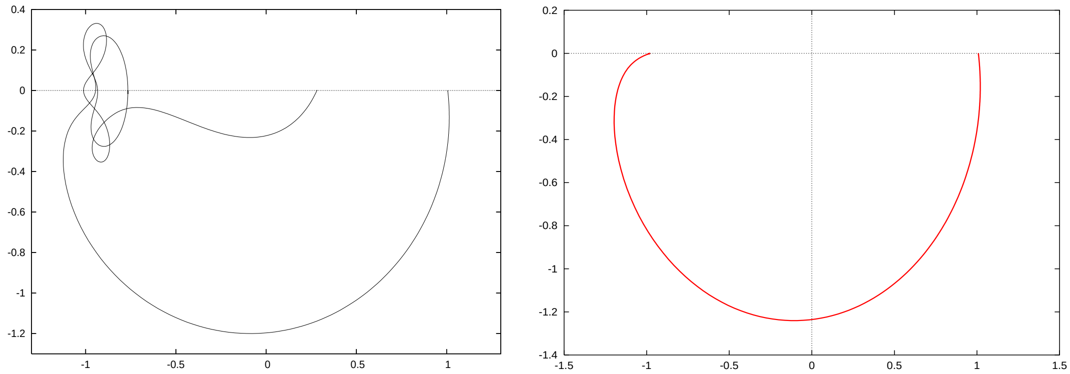

Figure 6.— Left: invariant manifold of L_3 that also surrounds L_2 and the small primary. Right: invariant manifold of L_3 that collides with the small primary.

analysing the *symmetric* homoclinic orbits of the Lyapunov PO by means of the intersection between the invariant manifold and the $y = x' = 0$ plane at a certain crossing, a careful analysis that takes into account the number of crossings must be done. Alternatively, a method that gives the homoclinic connections, regardless the number of crossings, should be designed.

- For bigger values of μ, the neighborhood of the small primary and $L_{1,2}$ play a role (see Figure 6 left) and we can even have manifolds that go to collision with the small primary, so a regularization must be done (we have used Levi-Civita coordinates). We show in Figure 6 right a collision manifold. We refer the interested reader to [2] where a systematic study of the invariant manifolds of L_3, when varying μ, is carried out.

4 Further work

Once the behaviour of the invariant manifolds of L_3 is studied, the next step consists of consider the families of Lyapunov orbits and tori. For each one of these objects we will take into account the following two situations:

- Symmetric homoclinic connections. As before, due to the symmetry (1), it will be enough to take into account the branches of one of the invariant manifolds, W^u or W^s, associated to the object and their intersections with the section $\Sigma = \{y = 0\}$. As we have mention before, to develop a numerical method to find $W^{u,s} \cap \Sigma$ without considering the number of crossings with the section will be an important tool.

- Non-symmetric homoclinic connections. In this case, we will look for intersections between both W^u and W^s invariant manifolds. One way to do this could be to find, firstly, the intersections of each manifold (separately) with a suitable section Σ (for example, $x = 0.5 - \mu$), and then the intersections of the manifolds on Σ. Nevertheless, again in this case will be interesting to develop a numerical tool in order to find homoclinic connections as zeros of a suitable function.

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