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On the geometric stability criterion for low order resonances

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Abstract

It is known that a geometric counterpart for classical stability criteria for two degrees of freedom Hamiltonian systems exists for resonances of order bigger than two. In this paper we show that this geometric approach can be extended for resonances of order one and two, based on the same idea, by considering the flow on the integral manifold where the origin lies after a normalization procedure.

1 Introduction

The determination of nonlinear stability of equilibrium positions in Hamiltonian systems is a classical problem. The question is trivial for one degree of freedom Hamiltonian systems but it turns to be intricate for more degrees of freedom. In this paper we will focus in two degrees of freedom Hamiltonian systems. We will suppose that the origin is an equilibrium position and that the Hamiltonian function $\mathcal{H}$ can be expanded around it in the form

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3 + \cdots,$$

where each $\mathcal{H}_i$ is a homogeneous polynomial of degree i in coordinates and momenta.

It is well known that a necessary condition for the origin to be stable is that all the characteristic multipliers of the corresponding linear system, associated to the quadratic term $\mathcal{H}_2$, have zero real part. Besides, if $\mathcal{H}_2$ is sign defined Dirichlet's theorem ensures the stability of the origin in the Lyapunov sense [10]. For the remaining cases specialized results, based on KAM theory, are required. Thus, we are left with the case of characteristic multipliers of the form $\pm\omega_1$, $\pm\omega_2$ and $\mathcal{H}_2$ not sign defined.

The way this situation is treated strongly depends on the normal form of the quadratic part of $\mathcal{H}_2$. In fact the normal form of $\mathcal{H}_2$ determines the further reduction of the Hamiltonian function and the subsequent reduced phase space. In this way, several situations

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must be considered. The first one takes place when ω_1 and ω_2 are independent over the rationals. In this case Arnold's theorem ensures the stability of the origin if certain non-degeneracy condition is fulfilled [1, 7]. The second one takes place when ω_1 and ω_2 are not independent over the rationals and they satisfy a resonant condition of order s, that is there exist m and n, coprime integers, such that

$$m\omega_1 - n\omega_2 = 0$$

and $m + n = s$. For these cases, Markeev provided appropriate results for resonances of third and four order [6], and later on Sokolski gave conditions of stability for first and second order resonances [12, 11].

However, if $\omega_1, \omega_2 \neq 0$ and the corresponding linear system is semisimple, that is the canonic Jordan matrix is diagonal, the normal form of the quadratic part $\mathcal{H}_2$ can be written in polar coordinates as

$$\mathcal{H}_2 = \omega_1 \Psi_1 - \omega_2 \Psi_2, \tag{1}$$

it does not matter if a resonant condition is satisfied or not. This fact was exploited by Cabral and Meyer [3] to give a general stability criterion, including Arnold's theorem and Markeev's results. It was proven by Elipe et al. [4, 5] and Pascual [9] that this result has a geometric counterpart giving rise to a *geometric criterion* of stability; it is enough to characterize the phase flow of the Hamiltonian system, normalized up to an appropriate order, in a neighborhood of the origin on the manifold of the reduced phase space where the origin lies.

Nevertheless, if ω_1 or ω_2 are zero or the corresponding linear system is not semisimple, that is in the case of first order resonance and second order resonance in the not semisimple case, the normal form of $\mathcal{H}_2$ is no longer (1) and the previous results are not of applicability and Sokolski's theorems have to be taken into account. The question is if these theorems have the same geometric counterpart as that of Cabral and Meyer.

In this paper we will show that if we apply the simple ideas of the geometric criterion we find the same conditions of stability that in the Sokolski's theorems. To this end, we will consider the *Birkhoff normal form* [2] up to a certain order, and the corresponding set of invariants associated to the reduction that generates the reduced phase space. Finally we will study the phase flow on the integral manifold where the origin lies.

The paper is organized in four sections. In Section 2 we recall the geometric criterion for the semisimple case. Section 3 is devoted to the low order resonances. The conclusions are drawn in Section 4.

2 The geometric criterion

Let us suppose that ω_1 and ω_2 satisfy a resonant condition of order greater or equal than two, and that $\mathcal{H}_2$ can be written as (1). Then, $\mathcal{H}$ can be brought into its Birkhoff's normal form, where $\mathcal{H}_2$ is a formal integral. Moreover, the normal form is generated by the so called invariants we denote M_1, M_2, C and S (see [5] for details) and the normal form up to order N is written as

$$\mathcal{H} = \mathcal{H}_2 + \sum_{j=3}^{N} \mathcal{H}_j$$

where $\mathcal{H}_2 = 2\omega M_2$ $(m = \omega n)$, and

$$\mathcal{H}_j = \sum_{2(\gamma_1+\gamma_2)+(n+m)(\gamma_3+\gamma_4)=j} a_{\gamma_1\gamma_2\gamma_3\gamma_4} M_1^{\gamma_1} M_2^{\gamma_2} C^{\gamma_3} S^{\gamma_4}, \qquad 3 \leq j \leq N.$$

The invariants are not independent and they satisfy the equation

$$C^2 + S^2 = (M_1 + M_2)^n (M_1 - M_2)^m, \tag{2}$$

together with the relation

$$M_1 \geq |M_2|. \tag{3}$$

Note that the reduced phase space is given by the equation (2) and the restriction (3). It is a set of surfaces of revolution, one for each constant value of M_2. Fixed a value for M_2, (2) is a surface of revolution with a vertex in the point $M_1 = |M_2|$, $C = S = 0$. In the figure 1 we see different surfaces of revolution for a 1:3 resonance depending on the value of M_2.

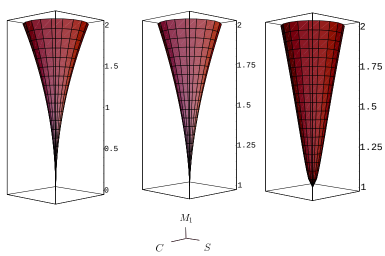

Figure 1.— The reduced phase space in 1:3 resonance for different values of M_2.

Once the reduced phase space is determined, it is possible to know the flow of the normalized system, when it is truncated to a prescribed order. Indeed, the flow results as the intersection of the normalized Hamiltonian function with the surface defined by (2). Based on this idea, the following stability result can be established (for more details, see [5, 9]).

Theorem 1 *Let us assume that the Hamiltonian is normalized up to a certain order $N \geq s$, being $\mathcal{H}_N$ the first term that does not vanish for $M_2 = 0$. Let us consider the two surfaces*

$$\mathcal{G}_1 = \{(C, S, M_1) \in \mathbb{R}^3; \quad \mathcal{H}_N(C, S, M_1, 0) = 0\},$$

and

$$\mathcal{G}_2 = \{(C, S, M_1) \in \mathbb{R}^3; \quad C^2 + S^2 = M_1^s\}.$$

If the origin is an isolated point of intersection, then it is stable. In other case, and the two surfaces are not tangent, the origin is unstable.

3 Low order resonances

This section is devoted to extend theorem 1 to the cases studied by Sokolski for low order resonances. In this way, we will look for the set of invariants associated to each resonance and the corresponding reduced phase space. Being I the new formal integral we define the two surfaces

- $\mathcal{G}_1$, defined by the first term in normal form does not vanish for $I = 0$.

- $\mathcal{G}_2$, the manifold of the reduced phase space for $I = 0$.

If the two surfaces have the origin as an isolated intersection point, the origin is stable. In other case, if they are not tangent, the origin is unstable.

Now we are in position to recover stability criteria of Sokolski for first and second order resonances from a geometric point of view. We will do this in the two following subsections.

3.1 Resonance of order 2

In this case we focus on the non semisimple case because the semisimple one is solved by the results given in section 2.

To begin with, we state the Sokolski's theorem [11] for the non semisimple case.

Theorem 2 *Let us consider a Hamiltonian system under a 1:1 resonance whose normal form up to order 4 is written in terms of the cartesian variables as*

$$\mathcal{H} = \frac{d}{2}(x^2+y^2) + \omega(xY - yX) + (X^2+Y^2)[A(X^2+Y^2) + B(xY - yX) + C(x^2+y^2)] + \overline{\mathcal{H}},$$

where $d = \pm 1$ and $\overline{\mathcal{H}} = O(x, X, y, Y, 6)$. If $dA > 0$, then the origin is stable. If $dA < 0$, then the equilibrium is unstable.

Now, we will show that the same result is obtained from a geometric point of view. To do this, we follow the work of Palacián and Yanguas [8] about the reduction of polynomial planar Hamiltonians with quadratic unperturbed part. In this way, we will introduce the semisimple part of $\mathcal{H}_2$, namely $xY - yX$, as a formal integral in order to reduce the Hamiltonian system to another one with one degree of freedom. In this case, there are four linearly independent invariants I_1, I_2, I_3, I_4 that in terms of cartesian variables can be written as

$$I_1 = x^2 + y^2, \qquad I_3 = xX + yY,$$

$$I_2 = X^2 + Y^2, \qquad I_4 = xY - yX.$$

Using invariants, the Hamiltonian normal form (up to order 4) is written as

$$\mathcal{H} = \frac{d}{2} I_1 + \omega I_4 + A I_2^2 + B I_2 I_4 + C I_1 I_2 + \overline{\mathcal{H}}.$$

It is worth to note that invariants are not independent but they verify the relation

$$I_1 I_2 = I_3^2 + I_4^2. \tag{4}$$

Moreover,

$$I_1, I_2 \geq 0. \tag{5}$$

Because of the formal integral, the reduced phase space is regarded as a family of elliptic hyperboloids defined by (4) and (5) and labeled by I_4. In figure 2 we show the two different types of surfaces defined by (4); for $I_4 \neq 0$ we have a two sheet elliptic hyperboloid and for $I_4 = 0$ the two sheets meet at the origin. It is worth to note that, because of (5), only one of the sheets corresponds to the reduced phase space.

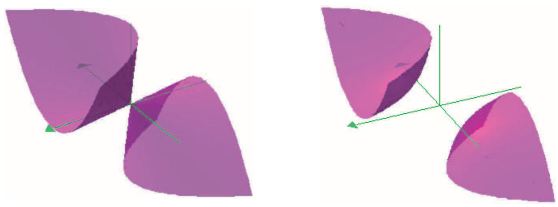

Figure 2.— $I_1 I_2 = I_3^2 + I_4^2$ for $I_4 = 0$ (left) and $I_4 \neq 0$ (right).

In order to derive a geometric criterion we focus on the phase flow on the manifold where the origin lies, the corresponding to $I_4 = 0$. In this way, we define the following two surfaces

$$\mathcal{G}_1 = \{(I_1, I_2, I_3) \in \mathbb{R}^3; \quad \mathcal{H}(I_1, I_2, I_3, 0) = 0\},$$

and

$$\mathcal{G}_2 = \{(I_1, I_2, I_3) \in \mathbb{R}^3; \quad I_1 I_2 = I_3^2, \quad I_1, I_2 \geq 0\}.$$

The phase flow on the manifold $I_4 = 0$ is given by the intersection of these two surfaces. In order to have stability it is necessary the orbits around the origin are closed, which implies the origin is an isolated intersection point of $\mathcal{G}_1$ and $\mathcal{G}_2$. The intersection of the two surfaces is given by the set of points

$$\mathcal{G}_1 \cap \mathcal{G}_2 = \left\{ (I_1, I_2, I_3) \in \mathbb{R}^3; \quad \frac{d}{2} I_1 + A I_2^2 + C I_1 I_2 = 0, \quad I_1 I_2 = I_3^2, \quad I_1, I_2 \geq 0 \right\},$$

so it is clear that a point belonging to $\mathcal{G}_1 \cap \mathcal{G}_2$ must satisfy the second degree polynomial equation in I_2

$$\frac{d}{2} I_1 + A I_2^2 + C I_1 I_2 = 0 \tag{6}$$

for $I_1, I_2 \geq 0$. As it is expected, the origin is one of the solutions, but we are interested in determine whether this solution is isolated or not. To solve this question we consider the discriminant D of equation (6),

$$D = I_1 (C^2 I_1 - 2dA)$$

If $A = 0$, the set of points $I_1 = I_3 = 0$ belongs to $\mathcal{G}_1 \cap \mathcal{G}_2$ and the two surfaces are tangent; no information about stability is obtained. If $A \neq 0$, we have intersection points different to the origin if $D \geq 0$. Here two cases must be distinguished.

On the one hand, if $dA < 0$ it follows that $D \geq 0$ it does not matter the value of $I_1 \geq 0$. As a consequence, for each value of I_1 we obtain an intersection point and the origin is not isolated. Because the two surfaces $\mathcal{G}_1$ and $\mathcal{G}_2$ are not tangent there are asymptotic orbits to the origin and it is unstable.

On the other hand, if $dA > 0$ the discriminant is greater or equal than zero if $I_1 \geq \frac{2dA}{C^2}$ (when $C \neq 0$) or if $I_1 = 0$ (when $C = 0$). Thus, it is possible to find a neighborhood of the origin U such that $U \cap (\mathcal{G}_1 \cap \mathcal{G}_2) = \{(0,0,0)\}$ and the origin is isolated. In this case a family of closed orbits exists around the origin and it is stable.

In the figures 3 and 4 we see the four different situations for the surfaces $\mathcal{G}_1$ and $\mathcal{G}_1 \cap \mathcal{G}_2$ when they are projected onto the plane $I_3 = 0$. We note that if $dA > 0$, the origin is an isolated point of the intersection, and therefore stable. Otherwise, if $dA < 0$, the origin is not an isolated intersection point, and therefore unstable. These conditions completely agree with those given by Sokolski.

3.2 Resonance of order 1

For a resonance of order one, two situations must be considered depending if the corresponding linear system is semisimple or not. Both situations were studied by Sokolski [12] where he provided two stability criteria, one for each case. Following the same line as in the previous section we will see that a geometric counterpart can be given for these results.

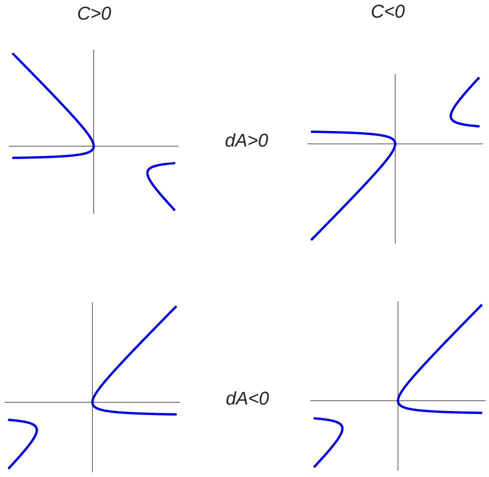

Figure 3.— $\mathcal{G}_1$ projected onto the plane $I_3 = 0$.

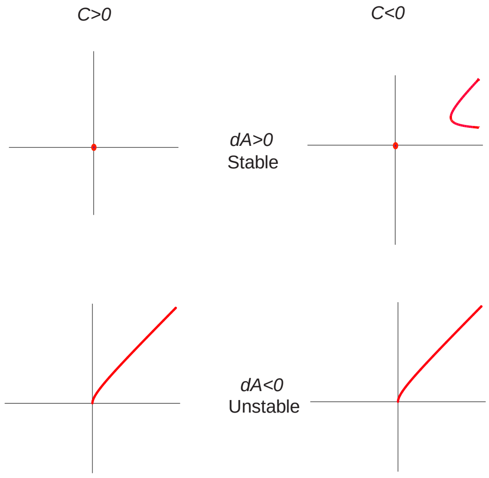

Figure 4.— $\mathcal{G}_1 \cap \mathcal{G}_2$ projected onto the plane $I_3 = 0$.

3.2.1 Semisimple case

For the semisimple case Sokolski established the following theorem

Theorem 3 *Let us consider a Hamiltonian system under a 0:1 resonance whose normal form up to order N is written in terms of the cartesian variables as*

$$\mathcal{H}(x, y, X, Y) = \mathcal{H}_2(x, y, X, Y) + \mathcal{H}_3(x, y, X, Y) + \cdots + \mathcal{H}_N(x, y, X, Y) + \overline{\mathcal{H}},$$

where

$$\mathcal{H}_2 = \frac{d}{2}(\omega_2^2 y^2 + Y^2), \qquad \mathcal{H}_j = \sum_{k=0}^{[j/2]} h_{j-2k}^{(k)} (y^2 + Y^2)^k, \quad 3 \leq j \leq N,$$

being $d = \pm 1$ and $h_{j-2k}^{(k)}$ a homogeneous polynomial of degree $j - 2k$ in x, X and $\overline{\mathcal{H}} = O(x, X, y, Y, N+1)$. If at least one coefficient of the polynomial $h_N^{(0)}$ is nonzero and $h_N^{(0)}$ is a sign–defined function, then the origin is stable. If at least one coefficient of the polynomial $h_N^{(0)}$ is a sign–variable function, then the origin is unstable. In particular, if N is an odd number, then the origin is unstable.

As N is not explicitly specify in the theorem it is supposed to be the first term in the normal form that is not the null function. Now, we are in conditions to recover the conclusions of the theorem from the geometric approach. First of all we carry out a normalization procedure by reducing the number of degrees of freedom by means of a formal integral. Following [8], we take $\omega_2^2 y^2 + Y^2$ as the formal integral. Besides, a set of three independent invariants I_1, I_2, I_3 is obtained, that in terms of cartesian variables can be written as

$$\begin{aligned} I_1 &= x, \\ I_2 &= X, \\ I_3 &= \omega_2^2 y^2 + Y^2. \end{aligned} \tag{7}$$

Note that $\mathcal{H}_2$ becomes

$$\mathcal{H}_2 = \frac{d}{2} I_3,$$

and it is no more than a multiple of the formal integral. Also note that the reduced phase space is defined by $I_3 = c$, with c a constant, and it is regarded to a family of parallel planes, one for each constant value of I_3.

In order to derive a geometric criterion, we take into account that the origin lies on the plane $I_3 = 0$ and we define the two surfaces

$$\mathcal{G}_1 = \{(I_1, I_2, I_3) \in \mathbb{R}^3; \quad \mathcal{H}(I_1, I_2, 0) = 0\},$$

and

$$\mathcal{G}_2 = \{(I_1, I_2, I_3) \in \mathbb{R}^3; \quad I_3 = 0\}.$$

As $I_3 = 0$ it follows that $y = Y = 0$ and therefore, $\mathcal{H}(I_1, I_2, 0) = h_N^{(0)}(I_1, I_2)$. In this way, the intersection of $\mathcal{G}_1$ and $\mathcal{G}_2$ can be described by the set

$$\mathcal{G}_1 \cap \mathcal{G}_2 = \left\{(I_1, I_2, I_3) \in \mathbb{R}^3; \quad h_N^{(0)}(I_1, I_2) = 0, \quad I_3 = 0\right\}.$$

Since $h_N^{(0)}$ is a homogeneous polynomial of degree N in I_1, I_2 it can be written as

$$h_N^{(0)}(I_1, I_2) = a_{N,0} I_1^N + a_{N-1,1} I_1^{N-1} I_2 + \cdots + a_{0,N} I_2^N.$$

It is clear that the origin, $I_1 = I_2 = I_3 = 0$, belongs to $\mathcal{G}_1 \cap \mathcal{G}_2$. Even more, if we fix the value $I_1 = 0$ it must be $I_2 = I_3 = 0$ unless $a_{0,N} = 0$. If $a_{0,N} = 0$ and at least one coefficient in $h_N^{(0)}$ is not zero, $\mathcal{G}_1$ and $\mathcal{G}_2$ intersect transversely along the line $I_1 = I_3 = 0$. Therefore, there is an asymptotic orbit to the origin and it is unstable.

Now, we are interested in intersection points such that $I_1 \neq 0$. In this way, we introduce a new variable z such that $I_2 = z I_1$ $(z \neq 0)$. Dividing by I_1^N the function $h_N^{(0)}(I_1, I_2)$ we obtain the polynomial

$$p_N(z) = a_{N,0} + a_{N-1,1} z + \cdots + a_{0,N} z^N.$$

We note that if $p_N(z)$ has a real root, z_0, then the straight line defined by $I_3 = 0$, $I_2 = z_0 I_1$ belongs to $\mathcal{G}_1 \cap \mathcal{G}_2$. As a consequence, there are asymptotic lines to the origin and it is unstable. On the contrary, if $p_N(z)$ has no real roots, the origin is the unique intersection point and a family of closed orbits exists around it. Then it is stable. However, the existence or not of real roots for $p_N(z)$ depends if $h_N^{(0)}(I_1, I_2)$ is sign defined or not. In particular if $h_N^{(0)}(I_1, I_2)$ is sign defined $p_N(z)$ has no real roots and the origin is stable. On the other hand, if $h_N^{(0)}(I_1, I_2)$ changes the sign, $p_N(z)$ has at least one real root and the origin is unstable. We note that if N is an odd number, the polynomial $p_N(z)$ has at least one real root, and therefore the origin is unstable.

3.2.2 Non semisimple case

For this case the result of Sokolski reads as

Theorem 4 *Let us consider a Hamiltonian system under a 0:1 resonance whose normal form up to order N is written in terms of the cartesian variables as*

$$\mathcal{H}(x, y, X, Y) = \mathcal{H}_2(x, y, X, Y) + \mathcal{H}_3(x, y, X, Y) + \cdots + \mathcal{H}_N(x, y, X, Y) + \overline{\mathcal{H}},$$

where

$$\mathcal{H}_2 = \frac{d_1}{2}x^2 + \frac{d_2}{2}(\omega_2^2 y^2 + Y^2), \qquad \mathcal{H}_j = \sum_{k=0}^{[j/2]} a_{j-2k,k} X^{j-2k}(y^2 + Y^2)^k, \quad 3 \leq j \leq N,$$

being $d_1, d_2 = \pm 1$ and $\overline{\mathcal{H}} = O(x, X, y, Y, N + 1)$. If $a_{N,0} \neq 0$, and N is an odd number, then the origin is unstable. If $a_{N,0} \neq 0$, N is an even number and $d_1 a_{N,0} < 0$, then the origin is unstable. If $a_{N,0} \neq 0$, N is an even number and $d_1 a_{N,0} > 0$, then the origin is stable.

Note that, as in theorem 3, it is supposed that $\mathcal{H}_N$ is the first term of the normal form that is not the null function. Under this implicit hypothesis we proceed in the same way as in the previous cases. As it is shown in [8], both the formal integral and the invariants are the same that in the semisimple case, and are given by (7).

Now, the quadratic part of the Hamiltonian function $\mathcal{H}_2$ is written as

$$\mathcal{H}_2 = \frac{d_1}{2} I_1^2 + \frac{d_2}{2} I_3,$$

and the reduced phase space is again a collection of parallel planes, $I_3 = c$ with c a constant.

Since the formal integral is I_3 and the origin lies on the plane $I_3 = 0$, we pay attention to the flow on this manifold. Thus, we consider the two surfaces

$$\mathcal{G}_1 = \{(I_1, I_2, I_3) \in \mathbb{R}^3; \quad \mathcal{H}(I_1, I_2, 0) = 0\},$$

and

$$\mathcal{G}_2 = \{(I_1, I_2, I_3) \in \mathbb{R}^3; \quad I_3 = 0\}.$$

To know their intersection it is worth to note that if $I_3 = 0$, then $y = Y = 0$, and therefore $\mathcal{H}(I_1, I_2, 0) = \frac{d_1}{2} I_1^2 + a_{N,0} I_2^N$. In this way the intersection is the set of points

$$\mathcal{G}_1 \cap \mathcal{G}_2 = \left\{ (I_1, I_2, I_3) \in \mathbb{R}^3; \quad \frac{d_1}{2} I_1^2 + a_{N,0} I_2^N = 0, \quad I_3 = 0 \right\}.$$

If N is an odd number, then the origin is not an isolated point of the intersection because

$$\mathcal{G}_1 \cap \mathcal{G}_2 = \left\{ (I_1, I_2, I_3) \in \mathbb{R}^3; \quad I_3 = 0, \quad I_2 = \sqrt[N]{\frac{-d_1}{2a_{N,0}} I_1^2} \right\}.$$

Therefore, as the surfaces intersect transversely, the origin is unstable. In the figure 5 we depict this set of points projected onto the plane $I_3 = 0$ depending on the sign of $d_1 a_{N,0}$.

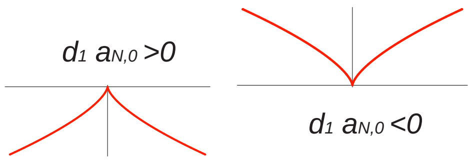

Figure 5.— $\mathcal{G}_1 \cap \mathcal{G}_2$ projected onto the plane $I_3 = 0$ for N an odd number.

If N is an even number, the intersection $\mathcal{G}_1 \cap \mathcal{G}_2$ changes depending on the sign of $d_1 a_{N,0}$. In this way, if $d_1 a_{N,0} < 0$, the intersection is given by

$$\mathcal{G}_1 \cap \mathcal{G}_2 = \left\{ (I_1, I_2, I_3) \in \mathbb{R}^3; \quad I_3 = 0, \quad I_2 = \sqrt[N]{\frac{-d_1}{2a_{N,0}} I_1^2} \right\}.$$

Therefore, the origin is unstable. In the figure 6 we depict this set of points projected onto the plane $I_3 = 0$.

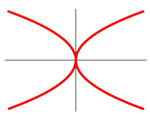

Figure 6.— $\mathcal{G}_1 \cap \mathcal{G}_2$ projected onto the plane $I_3 = 0$ for N an even number.

If $d_1 a_{N,0} > 0$, the origin is an isolated point of intersection and then it is a stable equilibrium point.

4 Conclusions

For a two degrees of freedom Hamiltonian system it was known that stability criteria for resonances of order bigger than two can be obtained from a geometric point of view [5, 9]. In this paper the cases of low order resonances, those of order one and two, have been analyzed from a geometric approach, and it has been shown that the criteria given by Sokolski [12, 11] can be recovered. The idea is based on the structure of the phase flow after a normalization procedure. In this way the normal form of the quadratic part of the Hamiltonian function plays an important role. In fact, this is the reason why the general criterion of Cabral and Meyer [3], and its geometric counterpart [5, 9], is not valid for low order resonances and ad hoc criteria must be given. Nevertheless, the geometric approach is the same does not matter the order of the resonance. In this way it is revealed as a powerful tool for studying stability properties of equilibrium positions.

Acknowledgments

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