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# Some topics concerning the theory of singular dynamical systems

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## Abstract

Some subjects related to the geometric theory of singular dynamical systems are reviewed in this paper. In particular, the following two matters are considered: the theory of canonical transformations for presymplectic Hamiltonian systems, and the Lagrangian and Hamiltonian constraint algorithms and the time-evolution operator.

**Key words:** Presymplectic manifolds, singular systems, canonical transformations, Lagrangian formalism, Hamiltonian formalism.

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# 1 Introduction

The aim of this paper is to carry out a brief review of several topics concerning the theory of autonomous singular dynamical systems, from a geometrical perspective. In particular, our interest will be focused on two subjects, namely: the theory of canonical transformations for singular systems, and the problem of the compatibility of the dynamical equations of Lagrangian and Hamiltonian singular systems; more precisely, the analysis of the Lagrangian and Hamiltonian constraint algorithms and their relation.

The article is based on the developments made on the references [18] and [7], for the theory of canonical transformations, and [1], [12], [13], [32], [10] and [23] for aspects related with the constraint algorithms. Thus, we will refer to these articles for more details on all these results.

All manifolds are real and  $C^{\infty}$ . All maps are  $C^{\infty}$ . Sum over crossed repeated indices is understood. Throughout this paper  $i(X)\omega$  denotes the contraction between the vector field X and the differential form  $\omega$ , and  $L(X)\omega$  the Lie derivative of  $\omega$  with respect to the vector field X.

#### 2 Canonical transformations

#### 2.1 Presymplectic Locally Hamiltonian Systems

**Definition 1** A presymplectic locally Hamiltonian system (p.l.h.s.) is a triad  $(M, \omega, \alpha)$ , where  $(M, \omega)$  is a presymplectic manifold, and  $\alpha \in Z^1(M)$  (i.e., it is a closed 1-form), which is called a Hamiltonian form.

A p.l.h.s has associated the following equation

$$i(X)\omega = \alpha$$
 ,  $X \in \mathfrak{X}(M)$ 

If X exists, it is called a presymplectic locally Hamiltonian vector field associated to  $\alpha$ . Nevertheless, in the best cases, this equation has consistent solutions only in a submanifold  $j_C \colon C \hookrightarrow M$ , where there exist  $X \in \mathfrak{X}(M)$ , tangent to C, such that

$$[i(X)\omega - \alpha]|_C = 0 \tag{1}$$

Furthermore, the solution X is not unique, in general, and this non-uniqueness is known as gauge freedom. In general  $(C, \omega_C = j_C^* \omega)$  is a presymplectic manifold which is called the final constraint submanifold (f.c.s.).

The following theorem gives the local structure of p.l.h.s. (see [7]):

**Theorem 1** Let  $(M, \omega, \alpha)$  be a p.l.h.s., and  $j_C \colon C \hookrightarrow M$  the f.c.s. Then:

1. There are a symplectic manifold  $(P, \Omega)$  and a coisotropic embedding  $\iota_C : C \hookrightarrow P$ such that  $\omega_C = \iota_C^* \Omega$ .

2. For every  $X \in \mathfrak{X}(M)$ , tangent to C, solution to (1), there exists a family of vector fields  $\mathfrak{X}(P,C) \subset \mathfrak{X}(P)$  such that, for every  $X_{\xi} \in \mathfrak{X}(P,C)$ , (a)  $X_{\xi}$  are tangent to C, (b)  $X_{\xi}|_{C} = X|_{C}$ , and (c)  $X_{\xi}$  are solutions to the equations  $i(X_{\xi})\Omega = \alpha_{P} + \xi$ , where  $\alpha_{P} \in Z^{1}(P)$  satisfies  $i_{C}^{*}\alpha_{P} = j_{C}^{*}\alpha$ , and  $\xi \in Z^{1}(P)$  is any first-class constraint form (i.e.,  $i_{C}^{*}\xi = 0$ , and the Hamiltonian vector field associated with  $\xi$ ,  $X_{\xi} \in \mathfrak{X}(P)$ , is tangent to C).

3. The coisotropic embedding  $\iota_C$  and the family  $\mathfrak{X}(P,C)$  are unique, up to an equivalence relation of local symplectomorphisms reducing to the identity on C.

 $(P, C, \Omega)$  is the coisotropic canonical system associated to  $(M, \omega, \alpha)$ .

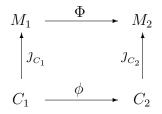
#### 2.2 Canonical Transformations for p.l.h.s.

Let  $\mathfrak{X}_{lh}(C)$  be the set of locally Hamiltonian vector fields in  $(C, \omega_C)$ ; that is,  $\mathfrak{X}_{lh}(C) = \{X_C \in \mathfrak{X}(C) \mid L(X_C)\omega_C = 0\}.$ 

**Definition 2** Let  $(M_i, \omega_i, \alpha_i)$  (i = 1, 2) be a p.l.h.s., with f.c.s.  $j_{C_i}: C_i \to M_i$ , and  $\omega_{C_i} = j_{C_i}^* \omega_i$ , such that dim  $M_1 = \dim M_2$ , dim  $C_1 = \dim C_2$ , and rank  $\omega_{C_1} = \operatorname{rank} \omega_{C_2}$ .

A canonical transformation between these systems is a pair  $(\Phi, \phi)$ , with  $\Phi \in \text{Diff}(M_1, M_2)$ and  $\phi \in \text{Diff}(C_1, C_2)$ , such that:

1.  $\Phi \circ j_{c_1} = j_{c_2} \circ \phi$ ; that is, we have the commutative diagram



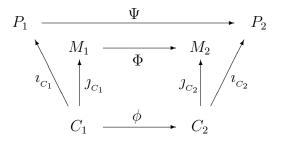
2.  $\phi_*(\mathfrak{X}_{lh}(C_1)) \subset \mathfrak{X}_{lh}(C_2).$ 

The generalization of Lee Hwa Chung's theorem to presymplectic manifolds allows us to prove that (see [18]):

**Proposition 1** Condition 2 is equivalent to saying that there exists  $c \in \mathbb{R}$  such that  $j_{C_1}^*(\Phi^*\omega_2 - c\omega_1) = \phi^*\omega_{C_2} - c\omega_{C_1} = 0.$ 

c is called the valence of the canonical transformation. So, univalent canonical transformations are the presymplectomorfisms between  $C_1$  and  $C_2$ .

Let  $(P_i, C_i, \Omega_i)$  be the coisotropic canonical systems associated with the p.l.h.s  $(M_i, \omega_i, \alpha_i), i = 1, 2$ . The class of  $\phi$  is defined by  $\{\phi\} = \{\Psi \in \text{Diff}(P_1, P_2) \mid \Psi \circ \imath_{C_1} = \imath_{C_2} \circ \phi\}$ . So we have the diagram



And therefore we have (see [7]):

**Theorem 2** There exists  $\psi \in \{\phi\}$  which is a symplectomorphism between the symplectic manifolds  $(P_1, \Omega_1)$  and  $(P_2, \Omega_2)$ .

As a particular situation, we can analyze the canonical transformations in a p.l.h.s. Thus, let  $(M, \omega, \alpha)$  be a p.l.h.s., with f.c.s.  $j_C \colon C \hookrightarrow M$ . Consider the involutive distribution ker  $\omega_C$  in C, and assume that the quotient space  $\hat{C} = C/\ker \omega_C$  is a manifold with natural projection  $\hat{\pi} \colon C \to \hat{C}$  (it is called the *reduced phase space* associated to the p.l.h.s.). Then the form  $\omega_C$  is  $\hat{\pi}$ -projectable; hence there exists  $\hat{\Omega} \in \Omega^2(\hat{C})$  such that  $\omega_C = \hat{\pi}^* \hat{\Omega}$ . Furthermore,  $(\hat{C}, \hat{\Omega})$  is a symplectic manifold, and we have the following result (see [7]): **Proposition 2** Every canonical transformation  $(\Phi, \phi)$  in  $(M, \omega, \alpha)$  leaves the distribution ker  $\omega_C$  invariant. As a consequence, there exists a unique  $\hat{\phi} \in \text{Diff}(\hat{C})$  such that:

- 1.  $\hat{\phi} \circ \hat{\pi} = \hat{\pi} \circ \phi$ .
- 2.  $\hat{\phi}$  is a symplectomorphism.

Some applications of the geometric theory of p.l.h.s. and their canonical transformations are the following: the study of canonical transformations for regular autonomous systems (it includes a new geometrical description of these kinds of systems based on the *coisotropic embedding theorem*) [8], the analysis of the geometric structure and construction of canonical transformations for the free relativistic massive particle [8], the discussion about time scaling transformations in Hamiltonian dynamics and their application to celestial mechanics [9], and the construction of realizations of symmetry groups for singular systems and, as an example, the Poincaré realizations for the free relativistic particle [35].

## 3 Constraints and the Evolution Operator

## 3.1 Lagrangian Dynamical Systems

(See [3] for details).

Let Q be a *n*-dimensional differential manifold which constitutes the configuration space of a dynamical system. Its tangent and cotangent bundles,  $\tau_Q \colon TQ \to Q$  and  $\pi_Q \colon T^*Q \to Q$ , are the (phase spaces of velocities and momenta of the system.

A Lagrangian dynamical system is a couple  $(TQ, \mathcal{L})$ , where  $\mathcal{L} \in C^{\infty}(TQ)$  is the Lagrangian function of the system. Using the canonical elements of TQ, the vertical endomorphism  $\mathcal{S} \in \mathfrak{T}_1^1(TQ)$  and the Liouville vector field  $\Delta \in \mathfrak{X}(TQ)$ , we can define the Lagrangian 2-form  $\omega_{\mathcal{L}} = -d(d\mathcal{L} \circ \mathcal{S})$ , and the Lagrangian energy function  $E_{\mathcal{L}} = \Delta(\mathcal{L}) - \mathcal{L}$ . Moreover, we define the Legendre transformation associated with  $\mathcal{L}$ ,  $\mathcal{FL}: TQ \to T^*Q$ , as the fiber derivative of the Lagrangian.

 $\mathcal{L}$  is a singular Lagrangian if  $\omega_{\mathcal{L}}$  is a presymplectic form (which is assumed to have constant rank) or, what is equivalent,  $\mathcal{FL}$  is no longer a local diffeomorphism. In particular,  $\mathcal{L}$  is an almost-regular Lagrangian if: (i)  $M_0 = \mathcal{FL}(\mathrm{T}Q)$  is a closed submanifold of  $\mathrm{T}^*Q$ , (ii)  $\mathcal{FL}$  is a submersion onto  $M_0$ , and (iii) for every  $\mathbf{p} \in \mathrm{T}Q$ , the fibres  $\mathcal{FL}^{-1}(\mathcal{FL}(\mathbf{p}))$  are connected submanifolds of  $\mathrm{T}Q$ . Then ( $\mathrm{T}Q, \mathcal{L}$ ) is an almost-regular Lagrangian system.

For almost-regular Lagrangian systems,  $(TQ, \Omega_{\mathcal{L}}, dE_{\mathcal{L}})$  is a p.l.h.s., and we have the so-called Lagrangian dynamical equation

$$i(\Gamma_{\mathcal{L}})\omega_{\mathcal{L}} = \mathrm{d}E_{\mathcal{L}} \tag{2}$$

Variational considerations lead us to impose that, solutions  $\Gamma_{\mathcal{L}} \in \mathfrak{X}(TQ)$  to (2) must be second-order differential equations (SODE); that is, holonomic vector fields in TQ. Geometrically this means that

$$S(\Gamma_{\mathcal{L}}) = \Delta \tag{3}$$

For singular Lagrangians this does not hold in general, and (3) must be imposed as an additional condition (SODE-condition). Integral curves of vector fields satisfying (2) and (3) are solutions to the Euler-Lagrange equations.

The Lagrangian problem consists in finding a submanifold  $j_{S_f} \colon S_f \hookrightarrow TQ$ , and  $\Gamma_{\mathcal{L}} \in \mathfrak{X}(TQ)$ , tangent to  $S_f$ , such that

$$[i(\Gamma_{\mathcal{L}})\omega_{\mathcal{L}} - dE_{\mathcal{L}}]|_{S_f} = 0 \quad , \quad [\mathcal{S}(\Gamma_{\mathcal{L}}) - \Delta]|_{S_f} = 0$$

Now, if  $\mathcal{FL}_0$ :  $\mathrm{T}Q \to M_0$  is the restriction of  $\mathcal{FL}$  to  $M_0$ , we have that  $\omega_{\mathcal{L}}$  and  $E_{\mathcal{L}}$  are  $\mathcal{FL}_0$ -projectable: there exist  $\omega_0 \in \Omega^2(M_0)$ , and  $h_0 \in \mathrm{C}^{\infty}(M_0)$  such that  $\omega_{\mathcal{L}} = \mathcal{FL}_0^*\omega_0$ ,  $E_{\mathcal{L}} = \mathcal{FL}_0^*h_0$ . Then,  $(M_0, \omega_0, \mathrm{d}h_0)$  is a p.l.h.s. which is called the *canonical Hamiltonian* system associated with the Lagrangian system ( $\mathrm{T}Q, \omega_{\mathcal{L}}, \mathrm{d}E_{\mathcal{L}}$ ) So we have the *Hamiltonian* dynamical equation

$$i(X_0)\omega_0 = dh_0$$
;  $X_0 \in \mathfrak{X}(M_0)$ 

and the Hamiltonian problem consists in finding a submanifold  $\mathcal{I}_{M_f}: M_f \hookrightarrow M_0$  and  $X_0 \in \mathfrak{X}(M_0)$  tangent to  $M_f$  such that

$$[i(X_0)\omega_0 - dh_0]|_{M_f} = 0$$

## 3.2 Constraint Algorithms

In order to solve the Hamiltonian problem stated for an almost-regular system different kinds of Hamiltonian constraint algorithms were developed. The first was the localcoordinate Dirac constraint algorithm [16], but there were also geometric algorithms: the Presymplectic Constraint Algorithm (PCA) of Gotay, Nester, Hinds [21], and others by Marmo, Tulczyjew et al. [30], [31], etc. All of them give a sequence of submanifolds which, in the best cases, stabilizes giving the f.c.s.:  $T^*Q \leftrightarrow M_0 \leftrightarrow M_1 \leftrightarrow \ldots \leftrightarrow M_f$ .

For the Lagrangian problem, the first attempt was not to consider the SODE-problem (3), and then develop a Lagrangian constraint algorithm by simply applying the P.C.A. to the Lagrangian dynamical equation (2), obtaining a sequence of submanifolds  $TQ \leftrightarrow P_1 \leftrightarrow \ldots \leftrightarrow P_f$  [19]. The SODE problem is studied later in [20], obtaining a submanifold  $S_f$  which solves the lagrangian problem, but which is not defined by constraints and is not maximal. Later, *Kamimura* [28] and *Batlle, Gomis, Pons, Román-Roy* [1] developed local-coordinate Lagrangian constraint algorithms in which the Lagrangian dynamical equation (2) and the SODE-condition (3) were both considered at the same time. The f.c.s.  $S_f$  obtained at the end of the corresponding sequence,  $T^*Q \leftrightarrow S_1 \leftrightarrow \ldots \leftrightarrow S_f$ , is

maximal and is defined by constraints. The relation between the aforementioned sequences of submanifolds is explained in the following diagram

In particular, the submanifolds  $P_i$ ,  $i = 1 \dots f$ , are defined by constraints that can be expressed as  $\mathcal{FL}$ -projectable functions which give all the Hamiltonian constraints, and are related with the first-class Hamiltonian constraints. Furthermore, the submanifolds  $S_i$ ,  $i = 1 \dots f$ , are defined by adding constraints that are not  $\mathcal{FL}$ -projectable, and they are related with the second-class Hamiltonian constraints.

The remaining question was how to describe geometrically the submanifolds  $S_i$  and their properties. First, this problem was solved in [12], [13] for  $S_1$ , i.e.; the submanifold of compatibility conditions for (2) and (3), obtaining as the main result that:

**Theorem 3** For every SODE  $\Gamma \in \mathfrak{X}(TQ)$ , we have:

$$S_1 = \{ \mathbf{p} \in \mathbf{T}Q \mid [i(Z)(i(\Gamma)\omega_{\mathcal{L}} - \mathrm{d}E_{\mathcal{L}})](\mathbf{p}) = 0, \ \forall Z \in \mathcal{M} \}$$

where  $\mathcal{M} = \{ Z \in \mathfrak{X}(\mathrm{T}Q) \mid \mathcal{S}(Z) \in \ker \mathcal{FL}_* = \ker \omega_{\mathcal{L}} \cap \mathfrak{X}^{\mathrm{V}(\tau_Q)}(\mathrm{T}Q) \}.$ 

In particular, for every  $Z \in \ker \omega_{\mathcal{L}} \subset \mathcal{M}$ ,  $i(Z)(i(\Gamma)\omega_{\mathcal{L}} - dE_{\mathcal{L}}) = i(Z)(dE_{\mathcal{L}})$  define the submanifold  $P_1$  where (2) is compatible. They are called dynamical constraints, and are related to the existence of primary first-class Hamiltonian constraints.

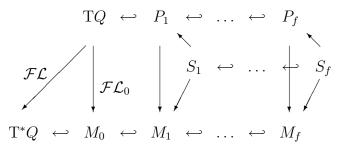
For every  $Z \notin \ker \omega_{\mathcal{L}}$ ,  $Z \in \mathcal{M}$ , these functions are called SODE-constraints, and are related to the existence of primary second-class Hamiltonian constraints.

For  $S_i$ ,  $i = 2 \dots f$ , the problem was studied in [32], by imposing tangency conditions for solutions to (2) and (3). The main results are the following: there are two kinds of Lagrangian constraints defining every  $S_i$ ,  $i = 1, \dots f$ :

- Dynamical constraints, which define the submanifolds  $P_i$ , i = 1, ..., f, in the sequence (4). They are related with the solutions to the eq. (2), and all of them can be expressed as  $\mathcal{FL}$ -projectable functions which give all the Hamiltonian constraints.

- SODE (non-dynamical) constraints, coming from the SODE-condition (3). They are not  $\mathcal{FL}$ -projectable.

Hence, the relation among the sequences of submanifolds in the Lagrangian and Hamiltonian formalisms is given in the following diagram



Furthermore, the tangency conditions for dynamical constraints give dynamical and SODE constraints, while the tangency conditions for SODE-constraints remove gauge degrees of freedom and do not give new constraints.

# 3.3 The Time Evolution Operator K

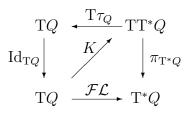
As a final remark, the complete relation between Hamiltonian and Lagrangian constraints is given by the so-called *time evolution operator* K, which relates Lagrangian and Hamiltonian constraints, and also solutions to the Lagrangian and Hamiltonian problems. (It is also known by the name of the *relative Hamiltonian vector field* [33]). It was introduced and studied for the first time by *J. Gomis* et al. [1], developing some previous ideas of *K. Kamimura* [28]. They give the local-coordinate definition of this operator, whose expression in coordinates is

$$K = v^A \left( \frac{\partial}{\partial q^A} \circ \mathcal{FL} \right) + \frac{\partial \mathcal{L}}{\partial q^A} \left( \frac{\partial}{\partial p^A} \circ \mathcal{FL} \right)$$

The intrinsic definition and the geometric study of its properties was carried out independently in [10] and [23]. In the first work, K was defined using the *Skinner-Rusk unified* formalism in  $TQ \oplus T^*Q$ . In the second article, the concept of section along a map plays the crucial role, and so we have:

**Definition 3** Let  $(TQ, \omega_{\mathcal{L}}, E_{\mathcal{L}})$  be a Lagrangian system, and  $\Omega \in \Omega^2(T^*Q)$  the canonical form. The time-evolution operator K associated with  $(TQ, \omega_{\mathcal{L}}, E_{\mathcal{L}})$  is a map  $K: TQ \longrightarrow TT^*Q$  verifying the following conditions:

- 1. (Structural condition): K is vector field along  $\mathcal{FL}$ ,  $\pi_{T^*Q} \circ K = \mathcal{FL}$ .
- 2. (Dynamical condition):  $\mathcal{FL}^*[i(K)(\Omega \circ \mathcal{FL})] = dE_{\mathcal{L}}$ .
- 3. (SODE condition):  $T\tau_Q \circ K = Id_{TQ}$ .



Then, the relation between Lagrangian and Hamiltonian constraints is established as follows (see [1], [10], [23]):

**Proposition 3** If  $\xi \in C^{\infty}(T^*Q)$  is a ith-generation Hamiltonian constraint, then  $L(K)\xi$  is a (i + 1)th-generation Lagrangian constraint.

In particular, if  $\xi$  is a first-class constraint (resp. a second-class constraint) for  $M_f$ , then  $L(K)\xi$  is a dynamical constraint (resp. a SODE constraint). The time-evolution operator has also been used for studying different kinds of problems concerning singular systems. For instance, the operator K has been extended for analyzing higher-order singular dynamical systems [2], [11], [26], [27]. It is also used for treating constrained systems in general (*linearly singular systems*) [24]. It has been defined and its properties studied for non-autonomous dynamical systems [6]. Finally, K has been applied for analyzing gauge symmetries and other structures for singular systems [22], [25]. Furthermore, sections along maps in general are analyzed and used in different kinds of physical and geometrical problems in [4], [14], [15].

# 4 Discussion and outlook

Some of the previous problems have been studied in the sphere of first-order classical field theories, specially their multisymplectic formalism [5]. So, a geometric constraint algorithm has recently been completed for Lagrangian and Hamiltonian singular field theories [29], and the definition and properties of the operator K have been carried out for field theories [17], [34].

Other potentially interesting topics could be the generalization of some of the above results; such as: to study the local structure of pre-multisymplectic Hamiltonian field theories (previous generalization of the *coisotropic embedding theorem* for premultisymplectic manifolds); the study of canonical transformations for Hamiltonian field theories (*multisymplectomorphisms* and *pre-multisymplectomorphisms*), and the application of the operator K to analyze the relation between Lagrangian and Hamiltonian constraints of singular field theories (which could require prior development of the non-covariant formulation, i.e., *space-time splitting*, of field theories).

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