Some topics concerning the theory of singular dynamical systems

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Abstract

Some subjects related to the geometric theory of singular dynamical systems are reviewed in this paper. In particular, the following two matters are considered: the theory of canonical transformations for presymplectic Hamiltonian systems, and the Lagrangian and Hamiltonian constraint algorithms and the time-evolution operator.

Key words: Presymplectic manifolds, singular systems, canonical transformations, Lagrangian formalism, Hamiltonian formalism.

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1 Introduction

The aim of this paper is to carry out a brief review of several topics concerning the theory of autonomous singular dynamical systems, from a geometrical perspective. In particular, our interest will be focused on two subjects, namely: the theory of canonical transformations for singular systems, and the problem of the compatibility of the dynamical equations of Lagrangian and Hamiltonian singular systems; more precisely, the analysis of the Lagrangian and Hamiltonian constraint algorithms and their relation.

The article is based on the developments made on the references [18] and [7], for the theory of canonical transformations, and [1], [12], [13], [32], [10] and [23] for aspects related with the constraint algorithms. Thus, we will refer to these articles for more details on all these results.

All manifolds are real and $C^\infty$. All maps are $C^\infty$. Sum over crossed repeated indices is understood. Throughout this paper $i(X)\omega$ denotes the contraction between the vector field $X$ and the differential form $\omega$, and $L(X)\omega$ the Lie derivative of $\omega$ with respect to the vector field $X$. 
2 Canonical transformations

2.1 Presymplectic Locally Hamiltonian Systems

Definition 1 A presymplectic locally Hamiltonian system (p.l.h.s.) is a triad \((M, \omega, \alpha)\), where \((M, \omega)\) is a presymplectic manifold, and \(\alpha \in Z^1(M)\) (i.e., it is a closed 1-form), which is called a Hamiltonian form.

A p.l.h.s has associated the following equation

\[ i(X)\omega = \alpha, \quad X \in \mathfrak{X}(M) \]

If \(X\) exists, it is called a presymplectic locally Hamiltonian vector field associated to \(\alpha\). Nevertheless, in the best cases, this equation has consistent solutions only in a submanifold \(\mathcal{C} \hookrightarrow M\), where there exist \(X \in \mathfrak{X}(M)\), tangent to \(\mathcal{C}\), such that

\[ [i(X)\omega - \alpha]|_{\mathcal{C}} = 0 \quad (1) \]

Furthermore, the solution \(X\) is not unique, in general, and this non-uniqueness is known as gauge freedom. In general \((\mathcal{C}, \omega_{\mathcal{C}} = \mathcal{J}_{\mathcal{C}}\omega)\) is a presymplectic manifold which is called the final constraint submanifold (f.c.s.).

The following theorem gives the local structure of p.l.h.s. (see [7]):

Theorem 1 Let \((M, \omega, \alpha)\) be a p.l.h.s., and \(\mathcal{J}_{\mathcal{C}}: \mathcal{C} \hookrightarrow M\) the f.c.s. Then:

1. There are a symplectic manifold \((P, \Omega)\) and a coisotropic embedding \(\mathcal{I}_C: \mathcal{C} \hookrightarrow P\) such that \(\omega_{\mathcal{C}} = \mathcal{I}_C^*\Omega\).

2. For every \(X \in \mathfrak{X}(M)\), tangent to \(\mathcal{C}\), solution to (1), there exists a family of vector fields \(\mathfrak{X}(P, \mathcal{C}) \subset \mathfrak{X}(P)\) such that, for every \(X_\xi \in \mathfrak{X}(P, \mathcal{C})\), (a) \(X_\xi\) are tangent to \(\mathcal{C}\), (b) \(X_\xi|_{\mathcal{C}} = X|_{\mathcal{C}}\), and (c) \(X_\xi\) are solutions to the equations \(i(X_\xi)\Omega = \alpha_P + \xi\), where \(\alpha_P \in Z^1(P)\) satisfies \(\mathcal{I}_C^*\alpha_P = \mathcal{J}_{\mathcal{C}}^*\alpha\), and \(\xi \in Z^1(P)\) is any first-class constraint form (i.e., \(\mathcal{I}_C^*\xi = 0\), and the Hamiltonian vector field associated with \(\xi\), \(X_\xi \in \mathfrak{X}(P)\), is tangent to \(\mathcal{C}\)).

3. The coisotropic embedding \(\mathcal{I}_C\) and the family \(\mathfrak{X}(P, \mathcal{C})\) are unique, up to an equivalence relation of local symplectomorphisms reducing to the identity on \(\mathcal{C}\).

\((P, \mathcal{C}, \Omega)\) is the coisotropic canonical system associated to \((M, \omega, \alpha)\).

2.2 Canonical Transformations for p.l.h.s.

Let \(\mathfrak{X}_{lh}(\mathcal{C})\) be the set of locally Hamiltonian vector fields in \((\mathcal{C}, \omega_{\mathcal{C}})\); that is, \(\mathfrak{X}_{lh}(\mathcal{C}) = \{X_\mathcal{C} \in \mathfrak{X}(\mathcal{C}) \mid L(X_\mathcal{C})\omega_{\mathcal{C}} = 0\}\).

Definition 2 Let \((M_i, \omega_i, \alpha_i)\) \((i = 1, 2)\) be a p.l.h.s., with f.c.s. \(\mathcal{J}_{\mathcal{C}_i}: \mathcal{C}_i \hookrightarrow M_i\), and \(\omega_{\mathcal{C}_i} = \mathcal{J}_{\mathcal{C}_i}^*\omega_i\), such that \(\dim M_1 = \dim M_2\), \(\dim C_1 = \dim C_2\), and \(\operatorname{rank} \omega_{\mathcal{C}_1} = \operatorname{rank} \omega_{\mathcal{C}_2}\).
A canonical transformation between these systems is a pair \((\Phi, \phi)\), with \(\Phi \in \text{Diff}(M_1, M_2)\) and \(\phi \in \text{Diff}(C_1, C_2)\), such that:

1. \(\Phi \circ j_{C_1} = j_{C_2} \circ \phi\); that is, we have the commutative diagram

\[
\begin{array}{ccc}
M_1 & \xrightarrow{\Phi} & M_2 \\
\downarrow{j_{C_1}} & & \downarrow{j_{C_2}} \\
C_1 & \xrightarrow{\phi} & C_2
\end{array}
\]

2. \(\phi_*(x_{ih}(C_1)) \subset x_{ih}(C_2)\).

The generalization of Lee Hwa Chung’s theorem to presymplectic manifolds allows us to prove that (see [18]):

**Proposition 1** Condition 2 is equivalent to saying that there exists \(c \in \mathbb{R}\) such that

\[
\int_{C_1} (\Phi^*\omega_2 - c\omega_1) = \phi^*\omega_2 - c\omega_1 = 0.
\]

\(c\) is called the valence of the canonical transformation. So, **univalent canonical transformations** are the presymplectomorfisms between \(C_1\) and \(C_2\).

Let \((P_i, C_i, \Omega_i)\) be the coisotropic canonical systems associated with the p.l.h.s \((M_i, \omega_i, \alpha_i)\), \(i = 1, 2\). The class of \(\phi\) is defined by \(\{\phi\} = \{\Psi \in \text{Diff}(P_1, P_2) \mid \Psi \circ \iota_{C_1} = \iota_{C_2} \circ \phi\}\). So we have the diagram

\[
\begin{array}{ccc}
P_1 & \xrightarrow{\Psi} & P_2 \\
\downarrow{\iota_{C_1}} & & \downarrow{\iota_{C_2}} \\
M_1 & \xrightarrow{\Phi} & M_2 \\
\downarrow{j_{C_1}} & & \downarrow{j_{C_2}} \\
C_1 & \xrightarrow{\phi} & C_2
\end{array}
\]

And therefore we have (see [7]):

**Theorem 2** There exists \(\psi \in \{\phi\}\) which is a symplectomorphism between the symplectic manifolds \((P_1, \Omega_1)\) and \((P_2, \Omega_2)\).

As a particular situation, we can analyze the canonical transformations in a p.l.h.s. Thus, let \((M, \omega, \alpha)\) be a p.l.h.s., with f.c.s. \(j_C : C \hookrightarrow M\). Consider the involutive distribution \(\ker \omega_C\) in \(C\), and assume that the quotient space \(\hat{C} = C / \ker \omega_C\) is a manifold with natural projection \(\hat{\pi} : C \rightarrow \hat{C}\) (it is called the reduced phase space associated to the p.l.h.s.). Then the form \(\omega_C\) is \(\hat{\pi}\)-projectable; hence there exists \(\hat{\Omega} \in \Omega^2(\hat{C})\) such that \(\omega_C = \hat{\pi}^*\hat{\Omega}\). Furthermore, \((\hat{C}, \hat{\Omega})\) is a symplectic manifold, and we have the following result (see [7]):
Proposition 2  Every canonical transformation \((\Phi, \phi)\) in \((M, \omega, \alpha)\) leaves the distribution \(\ker \omega_C\) invariant. As a consequence, there exists a unique \(\hat{\phi} \in \text{Diff}(\hat{C})\) such that:

1. \(\hat{\phi} \circ \hat{\pi} = \hat{\pi} \circ \phi\).
2. \(\hat{\phi}\) is a symplectomorphism.

Some applications of the geometric theory of p.l.h.s. and their canonical transformations are the following: the study of canonical transformations for regular autonomous systems (it includes a new geometrical description of these kinds of systems based on the coisotropic embedding theorem) [8], the analysis of the geometric structure and construction of canonical transformations for the free relativistic massive particle [8], the discussion about time scaling transformations in Hamiltonian dynamics and their application to celestial mechanics [9], and the construction of realizations of symmetry groups for singular systems and, as an example, the Poincaré realizations for the free relativistic particle [35].

3  Constraints and the Evolution Operator

3.1  Lagrangian Dynamical Systems

(See [3] for details).

Let \(Q\) be a \(n\)-dimensional differential manifold which constitutes the configuration space of a dynamical system. Its tangent and cotangent bundles, \(\tau_Q: TQ \to Q\) and \(\pi_Q: T^*Q \to Q\), are the (phase spaces of velocities and momenta) of the system.

A Lagrangian dynamical system is a couple \((TQ, L)\), where \(L \in C^\infty(TQ)\) is the Lagrangian function of the system. Using the canonical elements of \(TQ\), the vertical endomorphism \(S \in \mathfrak{T}_1(TQ)\) and the Liouville vector field \(\Delta \in \mathfrak{X}(TQ)\), we can define the Lagrangian 2-form \(\omega_L = -d(dL \circ S)\), and the Lagrangian energy function \(E_L = \Delta(L) - L\). Moreover, we define the Legendre transformation associated with \(L\), \(\mathcal{F}L: TQ \to T^*Q\), as the fiber derivative of the Lagrangian.

\(L\) is a singular Lagrangian if \(\omega_L\) is a presymplectic form (which is assumed to have constant rank) or, what is equivalent, \(\mathcal{F}L\) is no longer a local diffeomorphism. In particular, \(L\) is an almost-regular Lagrangian if: (i) \(M_0 = \mathcal{F}L(TQ)\) is a closed submanifold of \(T^*Q\), (ii) \(\mathcal{F}L\) is a submersion onto \(M_0\), and (iii) for every \(p \in TQ\), the fibres \(\mathcal{F}L^{-1}(\mathcal{F}L(p))\) are connected submanifolds of \(TQ\). Then \((TQ, L)\) is an almost-regular Lagrangian system.

For almost-regular Lagrangian systems, \((TQ, \Omega_L, dE_L)\) is a p.l.h.s., and we have the so-called Lagrangian dynamical equation

\[
i(\Gamma_L)\omega_L = dE_L
\]  \(\text{(2)}\)

Variational considerations lead us to impose that, solutions \(\Gamma_L \in \mathfrak{X}(TQ)\) to (2) must be second-order differential equations (SODE); that is, holonomic vector fields in \(TQ\).
Geometrically this means that
\[ S(\Gamma_L) = \Delta \]  \hspace{1cm} (3)

For singular Lagrangians this does not hold in general, and (3) must be imposed as an additional condition (sode-condition). Integral curves of vector fields satisfying (2) and (3) are solutions to the Euler-Lagrange equations.

The Lagrangian problem consists in finding a submanifold \( j_{S_f} : S_f \hookrightarrow TQ \), and \( \Gamma_L \in \mathfrak{X}(TQ) \), tangent to \( S_f \), such that

\[ [i(\Gamma_L)\omega_L - dE_L]|_{S_f} = 0 \quad , \quad [S(\Gamma_L) - \Delta]|_{S_f} = 0 \]

Now, if \( \mathcal{F}_0L : TQ \rightarrow M_0 \) is the restriction of \( \mathcal{F}L \) to \( M_0 \), we have that \( \omega_L \) and \( E_L \) are \( \mathcal{F}_0L \)-projectable: there exist \( \omega_0 \in \Omega^2(M_0) \), and \( h_0 \in C^\infty(M_0) \) such that \( \omega_L = \mathcal{F}_0L^*\omega_0 \), \( E_L = \mathcal{F}_0L^*h_0 \). Then, \( (M_0, \omega_0, dh_0) \) is a p.l.h.s which is called the canonical Hamiltonian system associated with the Lagrangian system \( (TQ, \omega_L, dE_L) \) So we have the Hamiltonian dynamical equation

\[ i(X_0)\omega_0 = dh_0 \quad ; \quad X_0 \in \mathfrak{X}(M_0) \]

and the Hamiltonian problem consists in finding a submanifold \( j_{M_f} : M_f \hookrightarrow M_0 \) and \( X_0 \in \mathfrak{X}(M_0) \) tangent to \( M_f \) such that

\[ [i(X_0)\omega_0 - dh_0]|_{M_f} = 0 \]

### 3.2 Constraint Algorithms

In order to solve the Hamiltonian problem stated for an almost-regular system different kinds of Hamiltonian constraint algorithms were developed. The first was the local-coordinate Dirac constraint algorithm [16], but there were also geometric algorithms: the Presymplectic Constraint Algorithm (PCA) of Gotay, Nester, Hinds [21], and others by Marmo, Tulczyjew et al. [30], [31], etc. All of them give a sequence of submanifolds which, in the best cases, stabilizes giving the f.c.s.: \( T^*Q \leftarrow M_0 \leftarrow M_1 \leftarrow \ldots \leftarrow M_f \).

For the Lagrangian problem, the first attempt was not to consider the SODE-problem (3), and then develop a Lagrangian constraint algorithm by simply applying the P.C.A. to the Lagrangian dynamical equation (2), obtaining a sequence of submanifolds \( TQ \leftarrow P_1 \leftarrow \ldots \leftarrow P_f \) [19]. The SODE problem is studied later in [20], obtaining a submanifold \( S_f \) which solves the lagrangian problem, but which is not defined by constraints and is not maximal. Later, Kamimura [28] and Batlle, Gomis, Pons, Román-Roy [1] developed local-coordinate Lagrangian constraint algorithms in which the Lagrangian dynamical equation (2) and the SODE-condition (3) were both considered at the same time. The f.c.s. \( S_f \) obtained at the end of the corresponding sequence, \( T^*Q \leftarrow S_1 \leftarrow \ldots \leftarrow S_f \), is
maximal and is defined by constraints. The relation between the aforementioned sequences of submanifolds is explained in the following diagram

\[
\begin{array}{c}
T^*Q \leftarrow P_1 \leftarrow \ldots \leftarrow P_f \\
\downarrow \downarrow \downarrow \downarrow \\
S_1 \leftarrow \ldots \leftarrow S_f
\end{array}
\]

(4)

In particular, the submanifolds \( P_i, i = 1 \ldots f \), are defined by constraints that can be expressed as \( \mathcal{FL} \)-projectable functions which give all the Hamiltonian constraints, and are related with the first-class Hamiltonian constraints. Furthermore, the submanifolds \( S_i, i = 1 \ldots f \), are defined by adding constraints that are not \( \mathcal{FL} \)-projectable, and they are related with the second-class Hamiltonian constraints.

The remaining question was how to describe geometrically the submanifolds \( S_i \) and their properties. First, this problem was solved in [12], [13] for \( S_1 \), i.e.; the submanifold of compatibility conditions for (2) and (3), obtaining as the main result that:

**Theorem 3** For every SODE \( \Gamma \in \mathfrak{X}(TQ) \), we have:

\[
S_1 = \{ p \in TQ \mid [i(Z)(i(\Gamma)\omega_L - dE_L)](p) = 0, \forall Z \in \mathcal{M} \}
\]

where \( \mathcal{M} = \{ Z \in \mathfrak{X}(TQ) \mid S(Z) \in \ker \mathcal{FL}_* = \ker \omega_L \cap \mathfrak{X}^{(\tau Q)}(TQ) \} \).

In particular, for every \( Z \in \ker \omega_L \subset \mathcal{M}, i(Z)(i(\Gamma)\omega_L - dE_L) = i(Z)(dE_L) \) define the submanifold \( P_1 \) where (2) is compatible. They are called dynamical constraints, and are related to the existence of primary first-class Hamiltonian constraints.

For every \( Z \not\in \ker \omega_L, Z \in \mathcal{M} \), these functions are called SODE-constraints, and are related to the existence of primary second-class Hamiltonian constraints.

For \( S_i, i = 2 \ldots f \), the problem was studied in [32], by imposing tangency conditions for solutions to (2) and (3). The main results are the following: there are two kinds of Lagrangian constraints defining every \( S_i, i = 1, \ldots f \):

- **Dynamical constraints**, which define the submanifolds \( P_i, i = 1, \ldots f \), in the sequence (4). They are related with the solutions to the eq. (2), and all of them can be expressed as \( \mathcal{FL} \)-projectable functions which give all the Hamiltonian constraints.
- **SODE (non-dynamical) constraints**, coming from the SODE-condition (3). They are not \( \mathcal{FL} \)-projectable.

Hence, the relation among the sequences of submanifolds in the Lagrangian and Hamiltonian formalisms is given in the following diagram

\[
\begin{array}{c}
T^*Q \leftarrow M_1 \leftarrow \ldots \leftarrow M_f \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
T^*Q \leftarrow M_0 \leftarrow M_1 \leftarrow \ldots \leftarrow M_f
\end{array}
\]
Furthermore, the tangency conditions for dynamical constraints give dynamical and SODE constraints, while the tangency conditions for SODE-constraints remove gauge degrees of freedom and do not give new constraints.

3.3 The Time Evolution Operator $K$

As a final remark, the complete relation between Hamiltonian and Lagrangian constraints is given by the so-called time evolution operator $K$, which relates Lagrangian and Hamiltonian constraints, and also solutions to the Lagrangian and Hamiltonian problems. (It is also known by the name of the relative Hamiltonian vector field [33]). It was introduced and studied for the first time by J. Gomis et al. [1], developing some previous ideas of K. Kamimura [28]. They give the local-coordinate definition of this operator, whose expression in coordinates is

$$K = v^A \left( \frac{\partial}{\partial q^A} \circ FL \right) + \frac{\partial L}{\partial q^A} \left( \frac{\partial}{\partial p^A} \circ FL \right)$$

The intrinsic definition and the geometric study of its properties was carried out independently in [10] and [23]. In the first work, $K$ was defined using the Skinner-Rusk unified formalism in $TQ \oplus T^*Q$. In the second article, the concept of section along a map plays the crucial role, and so we have:

**Definition 3** Let $(TQ, \omega_L, E_L)$ be a Lagrangian system, and $\Omega \in \Omega^2(T^*Q)$ the canonical form. The time-evolution operator $K$ associated with $(TQ, \omega_L, E_L)$ is a map $K : TQ \rightarrow TT^*Q$ verifying the following conditions:

1. **(Structural condition):** $K$ is vector field along $FL$, $\pi_{T^*Q} \circ K = FL$.
2. **(Dynamical condition):** $FL^*[i(K)(\Omega \circ FL)] = dE_L$.
3. **(SODE condition):** $T\tau_Q \circ K = Id_{TQ}$.

![Diagram](image)

Then, the relation between Lagrangian and Hamiltonian constraints is established as follows (see [1], [10], [23]):

**Proposition 3** If $\xi \in C^\infty(T^*Q)$ is a $i$th-generation Hamiltonian constraint, then $L(K)\xi$ is a $(i+1)$th-generation Lagrangian constraint.

In particular, if $\xi$ is a first-class constraint (resp. a second-class constraint) for $M_f$, then $L(K)\xi$ is a dynamical constraint (resp. a SODE constraint).
The time-evolution operator has also been used for studying different kinds of problems concerning singular systems. For instance, the operator $K$ has been extended for analyzing higher-order singular dynamical systems [2], [11], [26], [27]. It is also used for treating constrained systems in general (linearly singular systems) [24]. It has been defined and its properties studied for non-autonomous dynamical systems [6]. Finally, $K$ has been applied for analyzing gauge symmetries and other structures for singular systems [22], [25]. Furthermore, sections along maps in general are analyzed and used in different kinds of physical and geometrical problems in [4], [14], [15].

4 Discussion and outlook

Some of the previous problems have been studied in the sphere of first-order classical field theories, specially their multisymplectic formalism [5]. So, a geometric constraint algorithm has recently been completed for Lagrangian and Hamiltonian singular field theories [29], and the definition and properties of the operator $K$ have been carried out for field theories [17], [34].

Other potentially interesting topics could be the generalization of some of the above results; such as: to study the local structure of pre-multisymplectic Hamiltonian field theories (previous generalization of the coisotropic embedding theorem for premultisymplectic manifolds); the study of canonical transformations for Hamiltonian field theories (multisymplectomorphisms and pre-multisymplectomorphisms), and the application of the operator $K$ to analyze the relation between Lagrangian and Hamiltonian constraints of singular field theories (which could require prior development of the non-covariant formulation, i.e., space-time splitting, of field theories).

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