Monografías de la Real Academia de Ciencias de Zaragoza. **29:** 187–196, (2006).

The momentum equation

Eduardo Martínez

Departamento de Matemática Aplicada, Facultad de Ciencias,
Universidad de Zaragoza, 50009 Zaragoza, Spain
e-mail: emf@unizar.es

Dedicated to José F. Cariñena on the occasion of his 60th birthday.

Abstract

By using the theory of Lie algebroids, the momentum equation for a nonholonomically constrained mechanical system with symmetry is reinterpreted in terms of parallel transport with respect to a connection. Such connection is canonically asociated to the geometry of the problem.

Key words: *Lagrangian Mechanics, Nonholonomic systems, Momentum map, Lie algebroids*

MSC (2000): *70F25, 37J15, 22A22.*

1 Introduction

The concept of the momentum map is one of the most important concepts in the study of the differential geometric properties of Hamiltonian systems with symmetry(see [13, 17, 19]. In the case of a free Hamiltonian system the momentum map associated to the symmetry group is a constant of the motion, and hence the dynamics reduces to the level sets of the momentum map. In the case of a nonholonically constrained system this property does not hold. On the contrary, the momentum map satisfies a differential equation known as the momentum equation [2] and hence the dynamics is not easily reduced to a submanifold defined by the momentum.

The main problem in the constrained case is that the momentum map takes values on the dual of a bundle of Lie algebras, instead of taking value in the dual of a Lie algebra. This fact nearly forces us to use a more general structure, the Lie algebroid structure, in which tangent bundles, Lie algebras, bundles of Lie algebras, quotients of tangent bundles by symmetry groups and other more general situations fit in a natural way.

The purpose of this notes is to unveil the situation by using modern differential geometric tools. We will show that a momentum map can be associated to every ideal of a

Lie algebroid, and that there exists a natural linear connection (in the sense of Lie algebroid connection theory [10]) such that the momentum equation is just the expression of the vanishing of the covariant derivative of the momentum map with respect to such connection. In other words, the evolution of the momentum map is given by parallel transport along solutions of the dynamics. As a consequence of this fact, we will obtain that the dynamics restricts to the inverse image of each holonomy subbundle on the dual bundle of the ideal we have considered. Finally, the situation of the constrained case is also analysed.

Along this paper all manifolds and maps are assumed to be smooth. We will consider actions of Lie groups on manifolds and we will assume that the action of the Lie group is ***free and proper***. If G is a Lie group acting on a manifold Q, the fundamental vector field associated with an element $\xi \in \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ of G will be denoted by ξ_Q.

2 The momentum map

Lagrangian Mechanics. We consider a manifold Q, its tangent bundle $\tau_Q\colon TQ \to Q$ and a Lagrangian $L \in C^\infty(TQ)$ defined on it. Lagrange's equations are the equations for the critical points of the action functional $S = \int L dt$, and in local coordinates these equations read

$$\delta L = \left[\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i}\right] dq^i = 0. \tag{1}$$

To obtain them we just have to consider (infinitesimal) variations δq of the coordinates q with fixed endpoints, from where the variations $\delta \dot{q}$ of the velocities $\dot{q}$ are obtained as the derivative of the variations of the q coordinates, that is, $\delta\dot{q} = \frac{d}{dt}\delta q$.

In geometric terms, we can define the Cartan 1-form θ_L, which in this paper will be identified with a map $\theta_L\colon TQ \to T^*Q$ (the Legendre transformation), whose coordinate expression is $\theta_L = \frac{\partial L}{\partial v^i} dq^i$. The variation of coordinates defines a vector field $X = X^i \frac{\partial}{\partial q^i}$ on M, and the complete lift $X^{\mathsf{c}} = X^i \frac{\partial}{\partial q^i} + \dot{X}^i \frac{\partial}{\partial \dot{q}^i}$ of such vector field describes the joint variation of coordinates and velocities. Then the Euler-Lagrange equations can be conveniently written as the equations

$$\mathcal{L}_\Gamma \langle\, \theta_L \,, X \,\rangle = \mathcal{L}_{X^{\mathsf{c}}} L \qquad \text{for all } X \in \mathfrak{X}(Q), \tag{2}$$

where the unknown is the dynamical vector field $\Gamma \in \mathfrak{X}(TQ)$.

Momentum map. Noether's theorem can be easily deduced from the above equation. If X is a vector field such that the Lagrangian is invariant in the sense that $\mathcal{L}_{X^{\mathsf{c}}} L = 0$, then we deduce that the function $\langle\, \theta_L \,, X \,\rangle$ is a constant of the motion, that is $\mathcal{L}_\Gamma \langle\, \theta_L \,, X \,\rangle = 0$.

In many cases symmetries of the Lagrangian are associated to Lie groups and Lie algebras. When we have a Lie group G acting on the manifold Q by symmetries of the

Lagrangian, the Noether constants of motion can be collectively considered as components of the momentum map. Slightly more generally, if we have an action of a finite-dimensional Lie algebra $\mathfrak{g}$ on Q, given by a morphism of real Lie algebras $\xi \in \mathfrak{g} \mapsto X_\xi \in \mathfrak{X}(Q)$, and such that $\mathcal{L}_{X_\xi^{\mathsf{C}}} L = 0$ for every $\xi \in \mathfrak{g}$, then Noether's theorem implies that $\mathcal{L}_\Gamma \langle\, \theta_L \,, X_\xi \,\rangle = 0$. Taking into account the linearity of the action map, we can define the momentum map $J\colon TM \to \mathfrak{g}^*$ by $\langle\, \theta_L \,, X_\xi \,\rangle = \langle\, J \,, \xi \,\rangle$, or more exactly, $\langle\, J(v) \,, \xi \,\rangle = \langle\, \theta_L(v) \,, X_\xi(q) \,\rangle$ for every $v \in TQ$ and where $q = \tau(v)$. Then every component of J is a constant of the motion and hence conservation of the momentum can be expressed in the form

$$\mathcal{L}_\Gamma J = 0. \tag{3}$$

Nonholonomic mechanics. The situation is more complicated in nonholonomic Lagrangian Mechanics [1, 4]. In addition to the Lagrangian $L \in C^\infty(TQ)$ we have some constraints that are to be satisfied by the solution curves and the equations of motion are obtained by means of D'Alembert principle. We will consider only the case of linear constraints, which geometrically correspond to a subbundle $D \subset TM$, and the curves are constrained to have velocity in D. Virtual displacements are just elements of D and d'Alembert principle states that the work of the constraint forces vanishes along virtual displacements, from where we get Lagrange-D'Alembert equations in the form $\delta L \in D^\circ$, where D° is the annihilator of the distribution D. Equivalently, if we take only variations which are virtual displacements, we arrive to

$$\mathcal{L}_\Gamma \langle\, \theta_L \,, X \,\rangle = \mathcal{L}_{X^{\mathsf{C}}} L \qquad \text{for all } X \in \mathrm{Sec}(D). \tag{4}$$

This equation is similar to equation (2), but now it has to be satisfied only for vector fields X taking values in D, and in addition one has to take into account that the curves must remain in D, that is Γ must be tangent to the submanifold $D \subset Q$.

Nonholonomic momentum. The physical properties of a nonholonomic system are not determined by the value of the Lagrangian on the constraint manifold. They are determined by the constraint manifold D and by the value of Lagrangian on the full manifold TQ (or on an open neighborhood containing D). Consequently, by a symmetry of a nonholonomic system we mean a vector field X which is a symmetry of the Lagrangian $\mathcal{L}_{X^{\mathsf{C}}} L = 0$ and a symmetry of the constraint distribution, i.e X takes values in D and X^{C} is tangent to the submandifold $D \subset TQ$. For such a vector field, equation (4) implies that the function $\langle\, \theta_L \,, X \,\rangle$ is a constant of the motion, which is the statement of the nonholonomic Noether's theorem.

The difference with respect to the situation in the unconstrained case comes from the definition of the momentum map and its conservation properties. Assume that we have a Lie algebra $\mathfrak{g}$ acting on Q by symmetries of the Lagrangian L, i.e. $\mathcal{L}_{X_\xi^{\mathsf{C}}} L = 0$. In general, we cannot ensure that the momentum is conserved, $\mathcal{L}_\Gamma\langle\,\theta_L\,,X_\xi\,\rangle\neq 0$ because the vector field X_ξ does not take values in D.

At every point $q\in Q$, we have to select those symmetry directions $\xi\in\mathfrak{g}$ such that $X_\xi(q)$ is in D_q. Thus one can proceed as in [2] by defining for every $q\in Q$ the vector subspace,

$$\mathfrak{g}^q=\{\,\xi\in\mathfrak{g}\mid X_\xi(q)\in D_q\,\}\tag{5}$$

and $\mathfrak{g}^D=\cup_{q\in Q}\mathfrak{g}^q$. If the dimension of $\mathfrak{g}^q$ is constant (does not depend of q), then $\mathfrak{g}^D\to Q$ is a vector bundle.

The ***nonholonomic momentum map*** is the map $J^{nh}\colon TQ\to(\mathfrak{g}^D)^*$ defined by

$$\langle\,J^{nh}(v)\,,\xi\,\rangle=\langle\,\theta_L(v)\,,X_\xi(q)\,\rangle\qquad\text{for every } q\in Q,\ v\in T_qQ \text{ and } \xi\in\mathfrak{g}^q.\tag{6}$$

Notice that J^{nh} takes values on a vector bundle instead of on the dual of a Lie algebra as it is the case in the unconstrained counterpart.

The momentum equation. As explained above, in general we cannot find an element ξ of the Lie algebra $\mathfrak{g}$ such that $X_\xi(q)\in D_q$ for every $q\in Q$. In other words, if we find a vector field X symmetry of the nonholonomic systems, the symmetry direction ξ changes from point to point, so that $X_{\xi(q)}(q)\in D_q$. We are therefore forced to treat with a section of $\mathfrak{g}^D\to Q$ instead of with a fixed element of $\mathfrak{g}$. This implies that the momentum map is not conserved, and the evolution of the momentum is governed by an equation which is known as the momentum equation [2].

Following [7], we take a local basis $\{e_a\}$ of sections of $\mathfrak{g}^D\to Q$ and hence we can write

$$J^{nh}=p_be^b.\tag{7}$$

One can show (see [7]) that the components p_a of the nonholonomic momentum map satisfy the equation

$$\frac{dp_b}{dt}=\langle\,p\,,[v,e_b]+\dot e_b\,\rangle.\tag{8}$$

which is known as the ***nonholonomic momentum equation***.

While this equation has a very clear origin (the symmetry direction ξ changes from point to point), it has an unclear geometrical meaning. In the next section we will write a similar momentum equation in a more clear geometrical set up, by making use of the geometry of Lie algebroids. We will concentrate in the unconstrained case, and at the end of the paper we consider the nonholonomically constrained case.

3 Mechanics on Lie algebroids

A natural setting. As we are assuming that the action of the group is free and proper, the bundle $\pi\colon Q\to M=Q/G$ is a principal bundle. The lifted action of G on TQ is also free and proper and the quotient $\tau\colon E = TQ/G \to M$ is a vector bundle. The Lagrangian L being invariant defines a function $l \in C^\infty(E)$ by projection onto the quotient. The tangent map to the principal bundle projection π induces a map $\rho\colon E \to TM$ given by

$$\rho([v]) = T\pi(v).$$

Moreover, a section of E can be identified with an equivariant vector field, and it is well known that the bracket of equivariant vector fields is also equivariant, from where we get a bracket $[\![\ ,\ ]\!]$ canonically defined on the set of sections of E which endows the $C^\infty(M)$-module $\mathrm{Sec}(E)$ with a Lie algebra structure. In addition such a Lie bracket satisfy the following property

$$[\![\sigma, f\eta]\!] = f[\![\sigma,\eta]\!] + \big(\rho(\sigma)f\big)\eta,$$

for every $\sigma,\eta \in \mathrm{Sec}(E)$ and $f \in C^\infty(M)$, and where $\rho(\sigma) \in \mathfrak{X}(M)$ is the vector field $\rho(\sigma)(m) = \rho(\sigma(m))$. It follows that the vector bundle $\tau\colon E \to M$ has a structure of Lie algebroid [3, 12] with anchor ρ and bracket $[\![\,,\,]\!]$, known as the ***Atiyah or gauge Lie algebroid***.

Notice that the kernel of the map ρ is precisely the set of infinitesimal symmetry directions (equivariant vector fields tangent to the orbits of the group) and can be identified with the adjoint bundle $(Q\times\mathfrak{g})/G$. Moreover, we have an exact sequence of Lie algebroids (the Atiyah sequence)

$$0 \to \ker\rho \xrightarrow{i} TQ/G \xrightarrow{j} TM \to 0, \tag{9}$$

where i is the canonical inclusion and $j = \rho$. In other words, the relevant object is $K = \ker\rho$ and the relevant property of K is that it is an ***ideal*** of the Lie algebroid $E = TQ/G$. Finally, in the constrained case, the constraint subbundle $D \subset TQ$ projects to a subbundle $D/G \subset E$. The following sections develop the idea indicated in this motivating example.

Lagrangian Mechanics on Lie algebroids. On a Lie algebroid E one can also define Lagrangian systems. This theory was introduced by A. Weinstein [18] and later developed by this author [14, 11, 9]. For such a system, Lagrange's equations are the equations for the critical points of the action functional defined on an appropriate Banach manifold of admissible curves with fixed endpoints [18, 5, 15, 16, 6].

If we fix a local coordinate system (x^i) on M and we choose a basis of local sections $\{e_\alpha\}$, then we have a local coordinate system (x^i, y^α) on E and the structure of Lie algebroid is locally determined by the so-called structure functions ρ^i_α and $C^\alpha_{\beta\gamma}$, given by

$$\rho(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i} \qquad \text{and} \qquad [\![e_\alpha, e_\beta]\!] = C^\gamma_{\alpha\beta} e_\gamma. \tag{10}$$

With this conventions the ***Euler-Lagrange equations*** read $\delta L = 0$, where

$$\delta L = \left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) - \frac{\partial L}{\partial y^\gamma} C^\gamma_{\beta\alpha} y^\beta - \rho^i_\alpha \frac{\partial L}{\partial x^i} \right\} e^\alpha, \tag{11}$$

together with the admissibility constraint $\dot{x}^i = \rho^i_\alpha y^\alpha$. In geometric terms, we can define the Cartan 1-form θ_L (again interpreted as a map form E to E^*) whose local components are $\partial L/\partial y^\alpha$, and hence θ_L is related to the momenta. The variations are defined by the complete lift σ^{c} of a section σ of E and the Euler-Lagrange equations can be written in the form

$$d_\Gamma \langle\, \theta_L \,, \sigma \,\rangle = d_{\sigma^{\mathsf{c}}} L, \tag{12}$$

which resembles that of a standard Lagrangian system on the tangent bundle. In this expression, Γ is a SODE section [14] whose integral curves are the solution of the dynamics.

Momentum map on Lie algebroids. Following the ideas in our motivating example, we consider an ideal $K \subset E$ of the Lie algebroid E, that is $[\![\sigma, \eta]\!] \in \mathrm{Sec}(K)$ for every $\sigma \in \mathrm{Sec}(E)$ and every $\eta \in \mathrm{Sec}(K)$. If E is a regular Lie algebroid (the rank of ρ is constant) we can take $K = \ker \rho$ but in the general case we have to choose a subbundle $K \subset \ker \rho$ which is invariant under the formation of brackets.

Associated to K we can define a map $J\colon E \to K^*$ by the restriction of θ_L to K. In other words

$$\langle\, J(a) \,, k \,\rangle = \langle\, \theta_L(a) \,, k \,\rangle = \frac{d}{ds} L(a + sk)\big|_{s=0}, \tag{13}$$

for every $a \in E$ and $k \in K$ with $\tau(a) = \tau(k)$.

The map J is said to be the ***momentum map*** with respect to the ideal K.

The canonical connection. Given a Lie algebroid $\tau\colon E \to M$, a linear E-connection on a vector bundle $\nu\colon F \to M$ is a $C^\infty(M)$-linear map $\sigma \in \mathrm{Sec}(E) \mapsto \nabla_\sigma \in \mathrm{Der}(\mathrm{Sec}(E))$ such that the associated vector field is $\rho(\sigma)$, that is, it satisfies the condition $\nabla_\sigma(f\zeta) = \big(\rho(\sigma)f\big)\zeta + f\nabla_\sigma\zeta$. See [10] for details about E-connections.

If K is an ideal of E, on the bundle $K \to M$, we can define a ***canonical linear E-connection*** be means of

$$\nabla_\sigma \eta = [\![\sigma, \eta]\!], \tag{14}$$

for σ a section of E and η a section of K. This connection can be locally described as follows. We can take a local basis $\{e_\alpha\} = \{e_I, e_a\}$ of sections of E such that the first elements $\{e_I\}$ are a local basis of K. We have that the connection coefficients are just a subset of the structure functions,

$$\nabla_{e_\alpha} e_I = [\![e_\alpha, e_I]\!] = C^J_{\alpha I} e_J. \tag{15}$$

As any linear E-connection, it can be extended to the dual bundle by the rule

$$\langle \nabla_\sigma \theta \,, \eta \rangle = d_\sigma \langle \theta \,, \eta \rangle - \langle \theta \,, \nabla_\sigma \rangle \eta, \tag{16}$$

where θ is a section of E^*. If $\{e^I\}$ is the dual basis of $\{e_I\}$ then

$$\nabla_{e_\alpha} e^I = -C^I_{\alpha J} e^J. \tag{17}$$

The canonical connection ∇ is ***flat***, that is,

$$[\nabla_\sigma, \nabla_\xi] = \nabla_{[\![\sigma,\xi]\!]}.$$

Indeed, the difference between both terms applied to a section η of K gives

$$[\nabla_\sigma, \nabla_\xi]\eta - \nabla_{[\![\sigma,\xi]\!]}\eta = [\![\sigma, [\![\xi, \eta]\!]]\!] - [\![\xi, [\![\sigma, \eta]\!]]\!] - [\![[\![\sigma, \xi]\!], \eta]\!],$$

which vanishes by the Jacobi identity for σ, ξ and η.

The momentum equation. With the help of the above tools we can easily find a generalization of the momentum equation which unveils its geometrical origin.

We first notice that the map J can be understood as a section of the vector bundle $\tau^*(K^*) \to E$, i.e. as a section of K^* along τ. In this way we can take the covariant derivative of J with respect to the dynamical section Γ.

Theorem 1 *The momentum map satisfies the* ***momentum equation***

$$\nabla_\Gamma J = 0. \tag{18}$$

Proof. If we take local coordinates associated to a local basis of sections $\{e_I, e_a\}$ as explained above, then we have

$$\langle \nabla_\Gamma J \,, e_I \rangle = d_\Gamma \langle J \,, e_I \rangle - \langle J \,, \nabla_\Gamma e_I \rangle.$$

Taking into account that $\langle J \,, e_I \rangle = \frac{\partial L}{\partial y^I}$ and $\nabla_\Gamma e_I = y^\alpha C^J_{\alpha I} e_J$ we have that the covariant derivative of J with respect to Γ has the local expression

$$\nabla_\Gamma J = \left[d_\Gamma \left(\frac{\partial L}{\partial y^J} \right) - \frac{\partial L}{\partial y^I} C^I_{\alpha J} y^\alpha \right] e^J.$$

On the other hand, taking into account that K is an ideal we have that the structure functions ρ^i_I and $C^a_{\alpha J}$ vanish, in view of which the first set of Euler-Lagrange equations (11) reads

$$\begin{aligned} \langle \delta L \,, e_I \rangle &= d_\Gamma \left(\frac{\partial L}{\partial y^J} \right) - \frac{\partial L}{\partial y^\gamma} C^\gamma_{\beta J} y^\beta - \rho^i_J \frac{\partial L}{\partial x^i} \\ &= d_\Gamma \left(\frac{\partial L}{\partial y^J} \right) - \frac{\partial L}{\partial y^I} C^I_{\beta J} y^\beta. \end{aligned}$$

Therefore $\langle \nabla_\Gamma J \,, \eta \rangle = \langle \delta L \,, \eta \rangle = 0$ for every section η of K, and the result follows. $\square$

In general, the momentum equation does not provide with constants of the motion. For a parallel section η of K, i.e., $\nabla\eta = 0$, we have that $\langle J, \eta \rangle$ is a constant of the motion. More generally, if η is a section of K along the projection τ such that $\nabla_\Gamma\eta = 0$, then the function $\langle J, \eta \rangle \in C^\infty(E)$ is a constant of the motion. In adapted coordinates, the components $\eta^J \in C^\infty(E)$ of such a section η must satisfy

$$d_\Gamma\eta^I + C^I_{\alpha J} y^\alpha \eta^J = 0. \tag{19}$$

In case we have one or more of such sections η, the solutions of the Euler-Lagrange equations remains on level sets of the function $\langle J, \eta \rangle$. To reduce the dynamics on the general case we must proceed as follows.

Holonomy and orbit reduction. A linear E-connection on a vector bundle $\pi\colon F \to M$ is equivalent to a horizontal lifting map $\lambda^{\mathsf{H}}\colon \pi^*E \to TF$ such that $T\pi(\lambda^{\mathsf{H}}(v, a)) = \rho(a)$. Given a section σ of E, the horizontal lift of σ is the vector field $\sigma^h \in \mathfrak{X}(F)$ determined by the condition $\mathcal{L}_{\sigma^h}\hat{\theta} = \widehat{\nabla_\sigma\theta}$, and it is related to λ^{H} by $\sigma^h(v) = \lambda^{\mathsf{H}}(v, \sigma(\pi(v)))$ for every $v \in F$. A curve $v(t)$ is horizontal if there exists a curve $a(t)$ in E such that $\dot{v} = \lambda^{\mathsf{H}}(v(t), a(t))$.

The following theorem is an obvious consequence of the momentum equation, and it is the base of the reduction result that follows.

Theorem 2 *Let $a(t)$ be an integral curve of the* SODE *Γ and let $\mu(t)$ be the curve in K^* defined by $\mu = J \circ a$. Then the curve $(\mu(t), a(t))$ is a horizontal curve in $\tau^*K^* \to E$.*

Proof. First notice that the curve $(\mu(t), a(t))$ is a horizontal curve if and only if $\dot{\mu}(t) = \lambda^{\mathsf{H}}(\mu(t), a(t))$, where λ^{H} is the horizontal lifting maps defined by the connection, or equivalently $\nabla_a\mu = 0$. Thus, in coordinates

$$\begin{aligned}
\nabla_{a(t)}\mu(t) &= \left[\frac{d}{dt}J_A(a(t)) - J_B(a(t))C^B_{\alpha A}a^\alpha(t)\right] e^A \\
&= \left[d_\Gamma J_A - J_B C^B_{\alpha A} y^\alpha(t)\right]\Big|_{a(t)} e^A \\
&= (\nabla_\Gamma J) \circ a,
\end{aligned}$$

that is $\nabla_a(J \circ a) = (\nabla_\Gamma J) \circ a$, from where the result follows. □

Therefore, the solutions of the Euler-Lagrange equations are contained in subsets of the form $J^{-1}(\mathcal{O})$, where $\mathcal{O}$ is a 'holonomy bundle', that is, it is the set of points that can be reached by a horizontal curve, or in other words, the orbits of a point under parallel transport.

Thus, if $\mathcal{O}$ is regular in the sense that $N = J^{-1}(\mathcal{O})$ is a submanifold of E, then we can restrict the dynamical vector field $\rho^1(\Gamma)$ to N. On the Lie algebroid setting, one must impose that N is prolongable, that is $\mathcal{T}^E N \equiv \rho^{-1}(TN)$ has constant dimension, so that the section Γ restricts to a section of $\mathcal{T}^E N$.

Nonholonomic momentum equation. When we have a nonholonomically constrained system, we must restrict everything to a subbundle $\mathcal{D} \subset E$. The momentum equation is now $\nabla_\Gamma J \in D^\circ$. Therefore, we have to consider the intersection $\mathcal{S} = K \cap D$ and we define the nonholonomic momentum map $J^{nh}\colon E \to \mathcal{S}^*$ by restriction to $\mathcal{S}$, i.e. $J^{nh}(a) = J(a)\big|_{\mathcal{S}}$. The connection ∇ restricts to $\mathcal{S}$ if and only if $\mathcal{S}$ is also an ideal of E, so that the nonholonomic momentum map satisfies an equation of the same type as in the unconstrained case, that is $\nabla_\Gamma J^{nh} = 0$.

Nevertheless, in general $\mathcal{S}$ is not an ideal, and then we can write the momentum equation in terms of a constrained connection. Given a projector P onto $\mathcal{S}$, and the complementary projector $Q = I - P$, we define the connection $\check{\nabla}$ on K by means of

$$\check{\nabla}_\sigma \eta = P(\nabla_\sigma \eta) + \nabla_\sigma (Q\eta). \tag{20}$$

This connection restricts to $\mathcal{S}$ and its restriction is but $\check{\nabla}_\sigma \eta = P \nabla_\sigma \eta$ for $\eta \in \mathrm{Sec}(\mathcal{S})$. In terms of this projected connection, the nonholonomic momentum map satisfies the nonholonomic momentum equation

$$\check{\nabla}_\Gamma J^{nh} = J \circ H \tag{21}$$

where H is the restriction of $\nabla_\Gamma P$ to $\mathcal{S}$.

Acknowledgments: Partial financial support from MEC (Spain) grant BFM 2003-02532 is acknowledged.

References

[1] Arnold VI, *Dynamical Systems III*, Springer-Verlag, 1988.

[2] Bloch AM, Krishnaprasad PS, Marsden JE and Murray RM, *Nonholonomic mechanical systems with symmetry*, Arch. Rational Mech. Anal. **136** (1996) 21–99.

[3] Cannas da Silva A and Weinstein A, *Geometric models for noncommutative algebras*, Amer. Math. Soc., Providence, RI, 1999; xiv+184 pp.

[4] Cariñena JF and Rañada MF, *Lagrangian systems with constraints: a geometric approach to the method of Lagrange multipliers*, J. Phys. A: Math. Gen. **26** (1993) 1335–1351.

[5] Cariñena JF and Martínez E, *Lie algebroid generalization of geometric mechanics*, In Lie Algebroids and related topics in differential geometry (Warsaw 2000). Banach Center Publications 54 2001, p. 201.

[6] Cendra H, Ibort A and Marsden JE, *Variational principal fiber bundles: a geometric theory of Clebsch potentials and Lin constraints*, J. Geom. Phys. **4** (1987) 183–206.

[7] Cendra H, Marsden JE and Ratiu TS, *Geometric Mechanics, Lagrangian Reduction and Nonholonomic Systems*, in *Mathematics Unlimited, Springer-Verlag, New York, 2000.*

[8] Cortés J, de León M, Marrero JC and Martínez E, *Nonholonomic Lagrangian systems on Lie algebroids*, Preprint 2005, arXiv:math-ph/0512003.

[9] Cortés J, de León M, Marrero JC, Martín de Diego D and Martínez E, *A survey of Lagrangian mechanics and control on Lie algebroids and groupoids*, Int. Jour. on Geom. Meth. in Math. Phys. **3** (2006) 509-558.

[10] Fernandes, RL, *Lie algebroids, holonomy and characteristic classes*, Adv. Math. **170** (2002) 119–179.

[11] de León M, Marrero JC and Martínez E, *Lagrangian submanifolds and dynamics on Lie algebroids*, J. Phys. A: Math. Gen. **38** (2005), R241–R308.

[12] Mackenzie KCH, *General Theory of Lie Groupoids and Lie Algebroids*, Cambridge University Press, 2005.

[13] Marsden JE and Ratiu TS, *Introduction to Mechanics and symmetry*, Springer-Verlag, 1999.

[14] Martínez E, *Lagrangian Mechanics on Lie algebroids*, Acta Appl. Math., **67** (2001), 295-320.

[15] Martínez E, *Geometric formulation of Mechanics on Lie algebroids*, In Proceedings of the VIII Fall Workshop on Geometry and Physics, Medina del Campo, 1999, *Publicaciones de la RSME*, **2** (2001), 209–222.

[16] Martínez E, *Variational Calculus on Lie Algebroids*, preprint, arXiv:math-ph/0603028.

[17] Ortega JP and Ratiu TS, *Momentum maps and Hamiltonian Reduction*, Birkhäuser, 2004.

[18] Weinstein A, *Lagrangian Mechanics and groupoids*, Fields Inst. Comm. **7** (1996), 207-231.

[19] Zenkov D, Bloch A and Marsden JE, *The energy momentum method for the stability of nonholonomic systems*, Dynamic Stability of Systems **13** (1998), 123–166.