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Geometrical Structures emerging from Quantum Mechanics

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Abstract

In this paper we show that a geometrical description of quantum mechanics is possible. A Jacobi bracket emerges quite naturally from the reduction of the Hilbert space to the complex projective space of pure states. The momentum map associated with the action of the unitary group allows the identification of the complex projective space with the minimal orbit of the co-adjoint action on the dual of the Lie algebra. From here, it is possible to build the convex body of density states which is found to carry many interesting new geometrical structures.

Key words: *Quantum mechanics, Kählerian structures, Inverse problem in quantum mechanics, Density states, Bi-Hamiltonian systems.*

1 Introduction

This paper is based on the view that our description of the external world is ultimately geometrical. We know that in some fields of physics such as general relativity or classical mechanics and classical field theories geometric ideas have been very useful. Nevertheless, the deepest physical theory we have today is quantum theory where geometric ideas are not readily available. More likely this state of affairs has to-do with the fact that our standard descriptions of quantum systems deals with Hilbert spaces and the algebra of linear operators acting on them. We shall argue, however, that there are interesting geometrical structures in quantum mechanics also, and that perhaps we should look at quantum theory as a geometric theory. To this aim we may be guided by the "correspondence principle" as stated by Dirac [1]:

"Classical Mechanics must be a limiting case of quantum mechanics. We should expect to find that important concepts in classical mechanics correspond to important concepts in quantum mechanics and, from the understanding of the general nature of the analogy between classical and quantum mechanics, we may hope to get laws and theorems in quantum mechanics appearing as simple generalizations of well-known results in classical mechanics."

According to this point of view, it is quite natural to investigate and unveil geometric structures in quantum mechanics which may be the analogue of analogous structures present in classical mechanics.

A very important aspect of quantum mechanics is the probabilistic interpretation, this requires that physical states should be interpreted as rays in the Hilbert space. This set may be given a manifold structure, the structure of a complex projective space. Being "non-linear", the notion of operator as a linear map will not make sense anymore, therefore we are obliged to introduce concepts and mathematical tools appropriate for a (differential) manifold. It is convenient to deal with the complex projective space considered as a real differential manifold. In this picture we may consider flows, infinitesimal generators, Hamiltonian vector fields and generating functions along with Poisson tensors, metric tensors and complex structures. To avoid all technicalities arising from infinite dimensions we shall restrict ourselves to complex projective spaces arising from finite dimensional Hilbert spaces. However we notice that by now manifold aspects of infinite dimensional Hilbert spaces are available in textbooks, as for instance Ref.s [2, 3, 4, 5, 6].

To clearly identify geometric structures on the complex projective space considered as a real differential manifold we shall consider related structures on the "realified" version of the complex Hilbert space and consider their projectability under the quotienting procedure.

2 Preliminaries

Let $\mathbb{H}$ be the Hilbert space of a quantum system, by $\mathbb{H}-\{0\}$ we denote the space of normalizable states. We define rays to be the equivalence classes of normalizable states differing only by multiplication by a nonzero complex number. We say that $|\psi_1\rangle$ and $|\psi_2\rangle$ are equivalent if $|\psi_1\rangle = \lambda\,|\psi_2\rangle$, where $\lambda \in \mathbb{C}_0$ (the group of nonzero complex numbers). The ray space $\mathbb{PH}$ is defined as the quotient by this equivalence relation

$$\mathbb{PH} = (\mathbb{H}-\{0\})/\mathbb{C}_0. \tag{1}$$

The natural projection $\pi : \mathbb{H}-\{0\} \rightarrow \mathbb{PH}$ maps each normalized state $|\psi\rangle$ to the ray $[\psi]$ on which it lies.

The Hilbert space structure carries an Hermitian product denoted as usual in Dirac's notation of bra and ket as $h(\psi,\varphi) = \langle\psi|\varphi\rangle$. This Hermitian structure may be decomposed into real and imaginary part, giving rise to an Euclidean product and a symplectic product respectively. This Hermitian product allows to define a realization of the unitary group in terms of isometries $h(\Phi(\psi),\Phi(\varphi)) = h(\psi,\varphi)$. Being symplectic, this action carries along a momentum map $\mu : \mathbb{H}-\{0\} \rightarrow u^*(\mathbb{H})$, where by $u(\mathbb{H})$ we denote the Lie algebra of the unitary group and by $u^*(\mathbb{H})$ we denote its dual. Here the finite dimensionality of

$\mathbb{H}$ is crucial to identify $u^*(\mathbb{H})$ uniquely from the linearity requirement. We may collect these various spaces in the following commutative diagram:

$$
\begin{array}{ccccc}
U(\mathbb{H}) & & & & \\
 & \searrow & & & \\
\mathbb{C}_0 & \longrightarrow & \mathbb{H}-\{0\} & & \\
 & & \pi\downarrow & \searrow^{\mu} & \\
 & & \mathbb{PH} & \longrightarrow & u^*(\mathbb{H})
\end{array}
\tag{2}
$$

To make sense of the geometrical structures arising from the Hermitian structure on $\mathbb{H}$ at the manifold level of $\mathbb{PH}$, we have to promote it to a tensor field. After we have introduced tensor fields on $\mathbb{H}$, we may consider the projectability properties with respect to $\mathbb{C}_0$ or to the real version of it, $S^1 \times \mathbb{R}_+$. The association of operators and vectors with tensorial quantities, already at the level of $\mathbb{H}$, will allow us to perform non-linear transformations and will pave the way to the transition to $\mathbb{PH}$ where concepts linked to the linear structure do not make sense. Let us recall two examples of non-linear transformations which are already widely considered in the quantum framework.

3 Non linear transformations

3.1 The eikonal transformation

This transformation [7] is usually considered when one is interested in the WKB approximation or, more generally, in the quantum-classical transition. It amounts to write the wave function in terms of its amplitude and phase:

$$\Psi = A(\vec{r},t)\, \mathrm{e}^{iS(\vec{r},t)/\hbar}. \tag{3}$$

It is well known that by setting

$$\pi = \frac{S}{2\hbar J}, \quad \chi = A^2, \tag{4}$$

where χ is the probability density and J is the current density

$$J = \hbar\chi\frac{\nabla S}{m}, \tag{5}$$

the equation of Schrödinger may be recast in the form

$$
\begin{aligned}
\frac{d\pi}{dt} &= \left\{\frac{\hbar}{2m}\frac{\triangle\sqrt{\chi}}{\sqrt{\chi}}\right\} - \frac{\hbar}{m}(\nabla\pi)^2 - \frac{U}{\hbar} \\
\frac{d\chi}{dt} &= -\frac{2\hbar}{m}\mathrm{div}\,(\chi\nabla\pi)
\end{aligned}
\tag{6}
$$

where the term in curl brackets is usually called the quantum potential and it is the starting point to obtain an approximate solution in terms of the solutions of the associated Hamilton-Jacobi equation.

3.2 Wigner function

Another instance of a widely used non-linear transformation is the description of the wave function in terms of the Wigner function [8]. We recall that this function (quasi-distribution on phase space) is defined in the following way:

$$W(q,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \mathrm{e}^{-ipy} \left\langle q + \frac{\hbar}{2} y | \Psi \right\rangle \left\langle \Psi | q - \frac{\hbar}{2} y \right\rangle , \tag{7}$$

it is the Weyl transform of the projector $|\Psi\rangle\langle\Psi|$.

This correspondence allows, more generally, to associate functions on phase space with operators. We have that

$$W: \quad \hat{\mathcal{A}}(\mathbb{H}) \longrightarrow \mathfrak{F}(T^*M) \tag{8}$$

allows to represent operators as functions on phase space.

Indeed

$$W(q,p) = W(|\Psi\rangle\langle\Psi|) \tag{9}$$

provides charts for the complex projective space, and

$$A(q,p) = \int_{-\infty}^{\infty} dz \mathrm{e}^{ipz} \left\langle q - \frac{z}{2} \middle| \hat{A} \middle| q + \frac{z}{2} \right\rangle \tag{10}$$

represents the operator $\hat{A}$ as a function on phase space.

4 Tensorial quantities in the abstract setting

Our aim in this section is to associate tensorial quantities with standard objects on the Hilbert space (states, observables and evolutionary equations). This association will be useful to consider the possibility of defining corresponding objects on $\mathbb{PH}$. In the real manifold setting we have the following diagram

$$\begin{array}{ccc} (S^1 \times \mathbb{R}_+) & \longrightarrow & \mathbb{H}_{\mathbb{R}} - \{0\} \\ & & \downarrow \\ & & (\mathbb{PH})_{\mathbb{R}} \end{array} \tag{11}$$

The Hermitian structure on $\mathbb{H}$

$$h(x,y) = g(x,y) + i\omega(x,y) \tag{12}$$

gives rise to contravariant corresponding structures, i.e. defined on $\mathbb{H}^*_{\mathbb{R}} = \mathrm{Lin}_{\mathbb{R}}(\mathbb{H}_{\mathbb{R}}, \mathbb{R})$. We shall denote by G the one corresponding to g, and by Λ the one corresponding to ω. The composition of G with ω gives the complex structure. The two structures G and Λ allow to define an Hermitian structure on $\mathbb{H}^*$. We have

$$J = G \circ \omega \quad \text{with} \quad J^2 = -\mathbb{I}. \tag{13}$$

On $\mathbb{H}^*$ we have the contravariant Hermitian structure:

$$G + i\Lambda \tag{14}$$

To make more smooth the transition to real differential manifolds, we consider $\mathbb{H}_{\mathbb{R}}$ as a realification of $\mathbb{H}$ and as a Kähler manifold

$$(\mathbb{H}_{\mathbb{R}}, J, g, \omega). \tag{15}$$

The transition to tensorial objects requires preliminarily the introduction of the tangent bundle of $\mathbb{H}_{\mathbb{R}}$:

$$T\mathbb{H}_{\mathbb{R}} \rightleftharpoons \mathbb{H}_{\mathbb{R}} \times \mathbb{H}_{\mathbb{R}}. \tag{16}$$

As it is well known, maps

$$X : \mathbb{H}_{\mathbb{R}} \longrightarrow T\mathbb{H}_{\mathbb{R}} \tag{17}$$

are vector fields. With any vector x, y we associate constant vector fields by setting

$$\begin{aligned} X_x &: \mathbb{H}_{\mathbb{R}} \longrightarrow T\mathbb{H}_{\mathbb{R}}, \quad \psi \mapsto (\psi, x) \quad \forall \psi \in \mathbb{H}_{\mathbb{R}}, \\ X_y &: \mathbb{H}_{\mathbb{R}} \longrightarrow T\mathbb{H}_{\mathbb{R}}, \quad \psi \mapsto (\psi, y) \quad \forall \psi \in \mathbb{H}_{\mathbb{R}}. \end{aligned} \tag{18}$$

Therefore with the Hermitian structure

$$h(x, y) = \langle x|y \rangle = g(x, y) + i\omega(x, y) \tag{19}$$

we associate the tensor

$$h(\psi)(X_x(\psi), X_y(\psi)) = h(x, y) = \langle x|y \rangle. \tag{20}$$

By using

$$J(x) = ix; \quad J^2 = -\mathbb{I}, \tag{21}$$

we define a $(1, 1)$ −tensor field

$$J : T\mathbb{H}_{\mathbb{R}} \longrightarrow T\mathbb{H}_{\mathbb{R}}, \tag{22}$$

by setting

$$J(\psi)(\psi, x) = (\psi, ix). \tag{23}$$

Thus, we may consider all expressions defined on $\mathbb{H}_{\mathbb{R}}$, say

$$\omega(x, Jy) = g(x, y), \quad g(Jx, Jy) = g(x, y), \quad \omega(Jx, Jy) = \omega(x, y), \tag{24}$$

as the expressions of the corresponding tensor fields evaluated at a generic point $\psi \in \mathbb{H}_{\mathbb{R}}$. Similarly we may write the contravariant form:

$$\langle \alpha|\beta \rangle_{\mathbb{H}^*_{\mathbb{R}}} = G(\alpha, \beta) + i\Lambda(\alpha, \beta) \tag{25}$$

on the dual real Hilbert space $\mathbb{H}_{\mathbb{R}}^{*}$.

The metric tensor G will define the duality map $T^{*}\mathbb{H}_{\mathbb{R}} \longrightarrow T^{*}\mathbb{H}_{\mathbb{R}}$. The contravariant tensor associated with the imaginary part Λ will allow the definition of Poisson brackets

$$\{f, h\}_{\omega} = \Lambda\left(df, dh\right), \tag{26}$$

while the symmetric tensor G allows for the definition of a symmetric (commutative product) bracket

$$(f, h)_{g} = G\left(df, dh\right). \tag{27}$$

This bracket is called the Riemann-Jordan bracket. Putting together the two products gives us a new product on functions

$$((f, h))_{\mathbb{H}_{\mathbb{R}}} = (f, h)_{g} + i\left\{f, h\right\}_{\omega}. \tag{28}$$

Now we are ready to consider the projectability of these tensorial structure to the quotient manifold $(\mathbb{PH})_{\mathbb{R}}$. We consider the quotient map

$$\pi: \quad (\mathbb{H} - \{0\})_{\mathbb{R}} \longrightarrow (\mathbb{PH})_{\mathbb{R}} \tag{29}$$

along with the associated tangent map

$$T\pi: \quad T\left(\mathbb{H} - \{0\}\right)_{\mathbb{R}} \longrightarrow T\left(\mathbb{PH}\right)_{\mathbb{R}}. \tag{30}$$

The kernel of this map defines the quotienting distribution generated by the vector field

$$\triangle: \quad \mathbb{H}_{\mathbb{R}} \longrightarrow T\mathbb{H}_{\mathbb{R}}, \quad |\Psi\rangle \longrightarrow \left(|\Psi\rangle, |\Psi\rangle\right) \tag{31}$$

infinitesimal generator of the action of the $\mathbb{R}_{+}$ group, and the vector field

$$\Gamma = J(\Delta) \tag{32}$$

infinitesimal generator of the action of S^{1}. Having the involutive distribution responsible for the quotienting procedure, we may look for projectable functions

$$L_{\Delta} f = 0, \quad L_{\Gamma} f = 0. \tag{33}$$

These functions allow to investigate the projectability of the contravariant tensors associated with the Hermitian structure. It is not difficult to see that

$$L_{\Delta}\left\{\pi^{*} f, \pi^{*} h\right\}_{\omega} = -2\left\{\pi^{*} f, \pi^{*} h\right\}_{\omega} \quad . \tag{34}$$

Thus, the Poisson bracket on $(\mathbb{H} - \{0\})_{\mathbb{R}}$ is not projectable onto $(\mathbb{PH})_{\mathbb{R}}$.

5 Recovering quantum mechanics from functions and vector fields

A large class of projectable functions has the form

$$f_A(\psi) = \frac{\langle\psi|A\psi\rangle}{\langle\psi|\psi\rangle} \tag{35}$$

for A any complex linear operator on $\mathbb{H}$. When A is Hermitian, f_A is a real valued function.

The Jacobi bracket on these functions reduces to the Poisson bracket and therefore defines the Poisson bracket on the complex projective space. A similar statement holds true for the commutative bracket associated with the contravariant tensor G.

Proposition. *On $\mathbb{PH}$, critical points of f_A are the "eigenvectors" and their values at these points are the "eigenvalues" corresponding to the Hermitian operator A.*

The Poisson bracket on these functions corresponds to the commutator $i[A, B]$, while the Riemann-Jordan bracket corresponds to the Jordan product $AB + BA$. The Hamiltonian vector field Γ_A associated by the Poisson bracket with f_A will define the Schrödinger equation on the space of pure space (more precisely, will be the evolution equation in the form given by Von Neumann).

Proposition.[9] *A generic function on $\mathbb{PH}$ defines a quantum evolution, via the associated Hamiltonian vector field, if and only if the vector field is a derivation for the Riemann-Jordan product.*

In this case, thanks to Wigner's theorem [10], we may show that any such function has necessarily the form f_A for some Hermitian operator A. Therefore, any such vector field is complete and gives rise to a one-parameter groups of "unitary transformations". These one-parameter group of unitary transformations provide us with a realization of the unitary group in terms of "non-linear" transformations.

Proposition *Functions corresponding to integer powers of A, say $f_k(\psi) = f_{A^k}(\psi)$, provide commuting functions with respect to the Poisson bracket. They are a maximal set of commuting functions if and only if eigenvalues are simple, i.e. critical points on $\mathbb{PH}$ are isolated.*

It is not difficult to see that to make $\{\cdot,\cdot\}_\omega$ projectable we have to introduce a quadratic factor for the natural Poisson bracket available on $\mathbb{H}$

$$\{f, h\}_D(\Psi) = \langle\Psi|\Psi\rangle \{f, h\}_\omega(\Psi). \tag{36}$$

Indeed in this way we obtain a bracket which is a binary product on projectable functions. Having introduced a conformal factor we find that the bracket $\{\cdot,\cdot\}_D$ is projectable but does not satisfy the Jacobi identity. It defines a Jacobi bracket

$$[f, h] = \{f, h\}_D + fL_X h - hL_X f \tag{37}$$

where

$$X = \Lambda(d\langle\Psi|\Psi\rangle) \tag{38}$$

is the Hamiltonian vector field of $\langle\Psi|\Psi\rangle$.

Thus the Poisson bracket on the complex projective space $\mathbb{PH}$ is not related to the constant Poisson bracket on $\mathbb{H}$ but it is a reduction of a Jacobi bracket. For more details on the reduction of Jacobi manifolds, see Ref.[11]. With the same trick we will be able to consider the projection of the commutative bracket associated with G, we have to multiply it by the conformal factor $\langle\Psi|\Psi\rangle$.

6 The inverse problem for quantum systems

In this section we would like to consider the inverse problem for quantum systems, i.e. under which conditions a given vector field will preserve some Hermitian structure. We consider first the inverse problem directly on $\mathbb{H}$. Let X be a vector field on $\mathbb{H}$

$$X: \quad \mathbb{H} \longrightarrow T\mathbb{H} = \mathbb{H} \times \mathbb{H}, \quad \Psi \longrightarrow \left(\Psi, -\frac{i}{\hbar}\hat{H}\Psi\right) \tag{39}$$

with corresponding equation of motion

$$\frac{d}{dt}\Psi = -\frac{i}{\hbar}\hat{H}\Psi. \tag{40}$$

We have [12, 13, 14] the following

Proposition. *A complex linear field X generates a flow $\phi_t : \mathbb{H} \to \mathbb{H}$ preserving some Hermitian scalar product h,*

$$\phi_t^* h = h,$$

if and only if any one of the following equivalent conditions is satisfied:

i) $\hat{H} = \hat{H}^\dagger$ with respect to some h, i.e.

$$L_X h = 0;$$

ii) $\hat{H}$ is diagonalizable and has a real spectrum;

iii) all the orbits

$$\mathrm{e}^{-it\hat{H}}\Psi_0$$

are bounded sets for any initial condition Ψ_0.

Vector fields preserving alternative Hermitian structures are called biHamiltonian quantum systems. According to previous considerations on functions in involution we deduce that any quantum system admits alternative Hermitian structures.

7 Compatible Hermitian structures

In analogy with compatible Poisson structures arising from completely integrable systems in classical mechanics, we may define compatible Hermitian structures [13].

Definition Two Hermitian tensors h_1 and h_2 are compatible when

$$L_{\Gamma_1} h_2 = L_{\Gamma_2} h_1 = 0, \tag{41}$$

where the Hamiltonian of Γ_1 is $\frac{1}{2}h_1(\Delta, \Delta)$ and the Hamiltonian of Γ_2 is $\frac{1}{2}h_2(\Delta, \Delta)$.

Proposition. *The* $(1,1)-$*tensor field*

$$F = (G_1 + i\Lambda_1) \circ (g_2 + i\omega_2)$$

is bounded, positive and self-adjoint with respect to both the Hermitian structures.

Remark. For h_1 and h_2 in generic position, the commutant and the bi-commutant of F coincide. In this generic situation, F generates a maximal set of commuting observables.

A further consequence is the following :

Given two Hermitian tensors h_1 and h_2, we obtain two alternative realizations of the unitary group, the common subgroup

$$U(\mathbb{H}, h_1) \cap U(\mathbb{H}, h_2) \tag{42}$$

is maximal Abelian if h_1 and h_2 are compatible and in generic position.

Compatible Hermitian tensors generate Jacobi brackets which are compatible: i.e., the Jacobi bracket associated with

$$\{f, h\}_{D_1} = g_1(\Delta, \Delta)\,\{f, h\}_{\omega_1} \tag{43}$$

is compatible with the Jacobi bracket associated with

$$\{f, h\}_{D_2} = g_2(\Delta, \Delta)\,\{f, h\}_{\omega_2} \tag{44}$$

8 Momentum map and density states

Because the action of the unitary group is a symplectic action, it is possible to define an associated momentum map. With the action

$$\phi : U(\mathbb{H}) \times \mathbb{H} \longrightarrow \mathbb{H} \tag{45}$$

we associate a momentum map

$$\mu : \quad \mathbb{H} \longrightarrow u^*(\mathbb{H}). \tag{46}$$

Let us first say something on $u^*(\mathbb{H})$ [15]. The space of Hermitian operators is identified with $u^*(\mathbb{H})$, the dual of the real Lie algebra $u(\mathbb{H})$, with duality map furnished by the trace

$$\langle A|T\rangle = \frac{i}{2}\mathrm{Tr}(\hat{A}\hat{T}), \tag{47}$$

and the duality map

$$u(\mathbb{H}) \ni T \longrightarrow iT \in u^*(\mathbb{H}) \tag{48}$$

identifies adjoint and co-adjoint action of $U(\mathbb{H})$. This implies that $u^*(\mathbb{H})$ becomes a Lie algebra with the following bracket

$$\left[\hat{A}, \hat{B}\right] = -i(\hat{A}\hat{B} - \hat{B}\hat{A}). \tag{49}$$

The scalar product, making $u^*(\mathbb{H})$ into a real Hilbert space, is the following

$$\langle A|B\rangle_{u^*(\mathbb{H})} = \frac{1}{2}\mathrm{Tr}(\hat{A}\hat{B}). \tag{50}$$

Eventually, we are able to write the momentum map in the form

$$\mu : \quad |\Psi\rangle \in \mathbb{H} \longrightarrow |\Psi\rangle\langle\Psi| \in u^*(\mathbb{H}). \tag{51}$$

On $u^*(\mathbb{H})$ the Riemann-Jordan tensor

$$R(\xi)\left(\hat{A}, \hat{B}\right) = \left\langle \xi|\left[\hat{A}, \hat{B}\right]_+\right\rangle = \frac{1}{2}\mathrm{Tr}\left(\xi(\hat{A}\hat{B} + \hat{B}\hat{A})\right) \tag{52}$$

and the Poisson tensor

$$\Lambda(\xi)\left(\hat{A}, \hat{B}\right) = \left\langle \xi|\left[\hat{A}, \hat{B}\right]\right\rangle_{u^*(\mathbb{H})} = \frac{1}{2i}\mathrm{Tr}\left(\xi(\hat{A}\hat{B} - \hat{B}\hat{A})\right) \tag{53}$$

together form a complex tensor

$$(R + i\Lambda)(\xi)\left(\hat{A}, \hat{B}\right) = \left\langle \xi|\hat{A}\hat{B}\right\rangle_{u^*(\mathbb{H})} = \mathrm{Tr}\left(\xi\hat{A}\hat{B}\right). \tag{54}$$

The momentum map relates this complex tensor with the dual Hermitian product

$$\mu_*(g^{-1} + i\omega^{-1}) = R + i\Lambda. \tag{55}$$

The $(1,1)-$tensor field

$$J(\xi)(A) = \frac{1}{\|\xi\|}[A, \xi] \tag{56}$$

satisfies

$$J^3 = -J \tag{57}$$

and induces a complex structure on every symplectic orbit. If the norm of a generic vector is r, $\|\xi\| = r$, the symplectic orbit passing through r is denoted by $D_r^1(\mathbb{H})$. When we consider $D_1^1(\mathbb{H})$ we get the symplectic orbit of pure states which is diffeomorphic with the complex projective space $\mathbb{PH}$. As a subset of $u^*(\mathbb{H})$ it is possible to consider all convex combinations of elements in $D_1^1(\mathbb{H})$ to obtain the set of density states. further details on this set may be found in Ref.[15].

9 Conclusions

Our geometric version of quantum mechanics has allowed to unveil many interesting geometrical structures. In particular we have obtained:

1) Alternative Jacobi structures

2) Bi-Kählerian manifolds

3) Generalized complex structures $J^3 = -J$, on the space of density states.

We believe that the extension of these considerations to quantum field theories may pave the way to better tackle a sound formulation of quantum gravity.

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