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Integrability of Helmholtz conditions

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Abstract

A geometric algorithm of integrability for partial differential and algebraic equations (PDAEs), together with the *connection* approach and the covariant derivative machinery, is applied to the Helmholtz conditions for the inverse problem of the calculus of variations. A huge bundle of families of algebraic conditions for integrability is generated.

Key words: *Integrability of partial differential equations, Variational calculus, Inverse problem, Helmholtz conditions.*

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1 The Algorithm of integrability

For a system of linear PDAEs, defined by an affine subbundle $R_1 \subset J^1\pi$ associated to a fibre bundle $\pi : E \to B$, $\dim B = n$, with $\pi_1^o(R_1) = E_0 \subset E$ *algebraic* constraints, necessary conditions of integrability are determined by a recursive algorithm of consistency $J^1E_0 \cap R_1 = R_1^{(1)}$, $\pi_1^o(R_1^{(1)}) = E_1, \ldots$, such that, at each step we select form the family of n-dimensional distributions $R_1^{(l-1)}$ in E_{l-1}, a subfamily $R_1^{(l)} = J^1E_{l-1} \cap R_1^{(l-1)}$ of tangent distributions to E_{l-1}, and those points E_l where this subfamily exist. Once a consistent system $(R_1^{(f)} \subset J^1\pi_f, E_f, \pi_f = \pi|_{E_f})$ is obtained (if constraints appear at B the system is inconsistent, but this does not happen for linear homogeneous PDAEs), we can look for involution of the distributions.

By rubbing out the counter, $R_1 \equiv R_1^{(f)}$, $E \equiv E_f$, $\pi \equiv \pi_f$, we lift the PDEs $J^1R_1 \subset J^1\pi_1$, and add the holonomy conditions $R_2 = J^1R_1 \cap J^2\pi$, with $i : J^2\pi \to J^1\pi_1$ the natural injection [22, 6]. If new algebraic constraints appear $\pi_2^o(R_2) \subset E$, the former algorithm of consistency at the first level must be applied again. Otherwise, $R_1^1 = \pi_2^1(R_2)$

determines distributions *algebraically* involutive, where *all* possible crossed partial derivatives (p.d.) have been checked (all of them for normal form PDEs). By restricting the family of distributions, new crossed p.d. can be checked, $R_2^1 = J^1R_1^1 \cap J^2\pi$, ... Once again, by recursive liftings at the first and second level, a consistent system ($R_2 \equiv R_2^f \subset J^2\pi_f, R_1 \equiv R_1^f \subset J^1\pi_f, E \equiv E_f, \pi \equiv \pi_f$) is obtained. However, unless the final system is in normal form, *algebraic* involution does not guarantee the existence of solutions, and the third level must be checked, $J^2(R_1) \cap J^3\pi$, etc. A PDE system is formally integrable if $\pi_{l+1}^l(R_{l+1}) = R_l$, $\forall l > 0$. It means that in a formal power series expansion, each step of the recursive equations determining the coefficients of a higher level is a compatible (generically undetermined) system of linear equations. The infinite step algorithm of formal integrability becomes finite for PDEs with involutive symbol $N_1 = TR_1 \cap V(\pi_1^o)$ [2, 3, 7, 8, 10, 12, 16] (Cartan-Kuranishi algorithm).

There are two natural projections from $J^1(\pi_1) \equiv J^1(J^1\pi)$ to $J^1\pi$, $(\pi_1)_1^o$ by selecting just the base point for the jet bundle system associated to the fiber bundle $\pi_1 : J^1\pi \to B$, and $j^1\pi$, the prolongation of the projection π, $j^1\pi(j^1s_1(x)) = j^1(\pi_1^o \circ s_1)(x)$. In local coordinates

$$(\pi_1)_1^o(x^i, y^a, z_i^a; v_{,i}^a, w_{j,i}^a) = (x^i, y^a, z_i^a)$$

and

$$j^1\pi(x^i, y^a, z_i^a; v_{,i}^a, w_{j,i}^a) = (x^i, y^a, v_{,i}^a)$$

The fibres of both projections intersect in a manifold $K = [(\pi_1)_1^o]^{-1} \cap [j^1\pi]^{-1}$, with coordinates $(x^i, y^a, z_i^a; v_{,i}^a \equiv z_i^a, w_{j,i}^a)$. In the transformation of a second order PDE into a first order one, it represents the (*first*) holonomy condition $z_i^a = \frac{\partial y^a}{\partial x^i}$, and K **matches** $J^2\pi$ for $\dim B = 1$. In the general case $J^2\pi$ is a subset of K determined by the identity of crossed p.d. $w_{j,i}^a = w_{i,j}^a$, *second holonomy* or integrability conditions. The prolongation J^1R_1 can be restricted to $R_2 = J^1R_1 \cap J^2\pi$, or to $(R_1)_1 = J^1R_1 \cap K$. Both R_2 and $(R_1)_1$ are affine subbundles over (a subset of) R_1, R_2 representing either a second order PDAE over E or a first order PDAE over $J^1\pi$, and $(R_1)_1$ a first order PDAE over $J^1\pi$. In R_2 we have all the integrability conditions, and its solution sections (as a first order PDE over $J^1\pi$) will be the lifting of solution sections for R_1. The symbol $N_2 = TR_2 \cap V(\pi_2^1)$ is a subspace (at each fibre) of $N_{1,1} = T(R_1)_1 \cap V(K)$, and $\dim N_2 \leq \dim N_{1,1}$.

Geometric Cartan test of involution[11]

The symbol $N_1 = TR_1 \cap V(\pi_1^o)$ *of a PDE is involutive if* $N_2 = N_{1,1}$

We are next going to apply the algorithm of integrability for Helmoltz conditions up to the second level; a general classification scheme, as the one given by Douglas [5] for $n = 2$, would be possible if the resulting system is involutive.

2 The inverse Problem of the Calculus of Variations

Given a fibre bundle $\pi : E \to \mathbb{R}$, and a one form along the projection $\pi_1 : J^1\pi \to \mathbb{R}$, we have a Lagrangian dynamical system $L(t, x, v)dt$, with the well known associated Euler-Lagrange equations. The inverse problem is to determine if, for a system of second order ordinary differential equations in normal form $\frac{d^2x^i}{dt^2} = f^i(t, x, v = \frac{dx}{dt})$, there exist (regular) Lagrangian functions whose Euler-Lagrange equations are equivalent to the given system.

$$\frac{d}{dt}(\frac{\partial L}{\partial v^i}) = \frac{\partial L}{\partial x^i} \qquad \frac{d}{dt} \equiv= \Gamma = \partial_t + v^i\partial_{x^i} + f^i(t, x, v)\partial_{v^i}$$

with the substitutions $\frac{dx^i}{dt} = v^i$ and $\frac{dv^i}{dt} = f^i(t, x, v)$, becomes a system of second order linear homogeneous PDEs in one unknown $L(t, x, v)$, in the bundle $T^*\mathbb{R} \times_{\mathbb{R}} J^1\pi \to J^1\pi$, with $2n + 1$ independent variables, $\dim E = n + 1$. We can transform it into a system of first order PDEs, with new variables

$$P_i = \partial_{v^i}L \quad Q_i = \partial_{x^i}L \quad Q_o = \partial_t L$$

The equivalent system is

$$\Gamma P_i = Q_i \quad \partial_{v^j}P_i = \partial_{v^i}P_j \quad \partial_{x^j}P_i = \partial_{v^i}Q_j \quad \partial_{x^j}Q_i = \partial_{x^i}Q_j \quad \partial_t P_i = \partial_{v^i}Q_o \quad \partial_t Q_i = \partial_{x^i}Q_o$$

i.e., the original system plus the holonomy conditions. The bundle now is $\tau : T^*(J^1\pi) \to J^1\pi$, by identifying $T^*\mathbb{R}$ with $\mathbb{R}\times\mathbb{R}$, that is, the form Ldt with a function L in $J^1\pi$. The unknown functions are the components of a locally exact one form $\alpha \simeq dL$. The lifting of $\Gamma P_i = Q_i$ by ∂_{x^j} and ∂_{v^j}, commutation relations $[\Gamma, \partial_{v^j}]$ and $[\Gamma, \partial_{x^j}]$, and the holonomy conditions allow to obtain

$$\Gamma(\partial_{v^j}P_i) + (\partial_{x^j} + \partial_{v^j}f^k\partial_{v^k})P_i - \partial_{x^i}P_j = 0 \tag{1}$$

and

$$\Gamma(\partial_{x^j}P_i) + \partial_{x^j}f^k\partial_{v^k}P_i = \Gamma(\partial_{x^i}P_j) + \partial_{x^i}f^k\partial_{v^k}P_j \tag{2}$$

where only p.d. of the P_i components appear. Moreover, by splitting (1) into its symmetric and skew-symmetric components, and using (2), we find the Helmholtz conditions for the inverse problem [18] (symmetry of $g_{ij} = \partial_{v^i}P_j$ is understood)

$$\Gamma(g_{ij}) = g_{ik}\Gamma^k_j + g_{jk}\Gamma^k_i \qquad \frac{\partial g_{ij}}{\partial v^k} = \frac{\partial g_{ik}}{\partial v^j} \qquad g_{ik}\Phi^k_j = g_{kj}\Phi^k_i \quad , \tag{3}$$

with $\Phi^j_i = -\partial_{x^i}f^j - \Gamma^j_k\Gamma^k_i - \Gamma(\Gamma^j_i)$ the components of the Jacobi endomorphism $\Phi = \Phi^j_i d^i \otimes \partial_j$.

3 "Connection" approach to the Helmholtz conditions

The reader is invited to consult [1, 9, 15, 17, 21] for a panoramic review about the inverse problem of the calculus of variations. The main references followed in this work are [4, 20], where the *connection* approach to Helmholtz conditions is applied, and [13, 14, 19] for properties of the covariant derivations of tensors along the tangent bundle projection.

A system of SODEs in normal form $\frac{d^2x^i}{dt^2} = f^i(t, x, v \equiv \frac{dx}{dt})$ determines a non linear connection, with coefficients $\Gamma^i_k = -\frac{1}{2}\frac{\partial f^i}{\partial v^k}$. A geometric representation can be obtained through the bundle $\pi : E \to \mathbb{R}$ of space-time over time; in this framework, the SODE system $\Gamma = \partial_t + v^i\partial_{x^i} + f^i(t, x, v)\partial_{v^i}$ is a vector field on $J^1\pi$, the first jet bundle. There, the non linear connection can be read as a linear connection on $\pi_1^{o*}(\tau_E)$, with $\pi_1^o : J^1\pi \to E$ and $\tau_E : TE \to E$. The associated covariant derivative splits into a *dynamical* derivation $\bigtriangledown$, a vertical D^V and a horizontal D^H derivations. For the inverse problem, and many other questions of interest, the relevant operations are restricted to the space of π-vertical vector fields along π_1^o, and its dual space of one forms over the fibres. The covariant derivations are determined by its particular action on functions, a basis of vertical vector fields $\partial_i = \partial_{x^i}$, and a dual basis of one forms d^i (identified with the contact forms $\theta^i = dx^i - v^i dt$ for the restriction):

$$\begin{array}{lllllllll}
\bigtriangledown F & = & \Gamma(F) & D^V_i F & = & \partial_{v^i}F & D^H_i F & = & (\partial_{x^i} - \Gamma^j_i\partial_{v^j})F \\
\bigtriangledown \partial_i & = & \Gamma^j_i\partial_j & D^V_i\partial_j & = & 0 & D^H_i\partial_j & = & \frac{\partial\Gamma^k_j}{\partial v^i}\partial_k \\
\bigtriangledown d^i & = & -\Gamma^i_j d^j & D^V_i d^j & = & -\delta^j_i dt (\equiv 0) & D^H_i d^j & = & -\frac{\partial\Gamma^j_k}{\partial v^i}d^k
\end{array} \tag{4}$$

In particular, Helmholtz conditions for the inverse problem represent a system of PDAEs on the unknown multipliers $g_{ij}(t, x, v)$, the components of a symmetric two co-variant tensor field $g = g_{ij}d^i.d^j$ along π_1^o, corresponding to the Hessian of the Lagrangian; we will mainly restrict the operations (contractions and derivatives) to this space of tensors and their derivatives. With this tools, Helmholtz conditions have a covariant formulation

$$\bigtriangledown g = 0 \quad D^V g(X, Y, Z) = D^V g(Y, X, Z) \quad (\Phi_1 - \Phi_2).g(X, Y) = 0 \tag{5}$$

where $D^V T(X, Y, \ldots) = (D^V_X T)(Y, \ldots)$ and $\Phi_k.T(X_1, X_2, \ldots) = T(X_1, \ldots, \Phi(X_k), \ldots)$.

The second order PDEs associated to the (second) holonomy conditions are obtained by taking into account the commutation relations of the dynamical, vertical and horizontal derivations:

$$\left[\bigtriangledown, D^V\right] = -D^H \quad \left[\bigtriangledown, D^H\right] = D^V_\Phi - \mu(\Psi) \tag{6}$$

$$\left[D^V, D^V\right] = 0 \quad \left[D^H, D^H\right] = D^V_R + \mu(Rie) \tag{7}$$

$$\left[D^V, D^H\right] = \mu(\theta) \quad , \tag{8}$$

with

$$R(X,Y) = \frac{1}{3}(D^V\Phi(X,Y) - D^V\Phi(Y,X)) \tag{9}$$

$$Rie(X,Y,Z) = -(D^V_Z R)(X,Y) \quad \Psi(X,Y) = R(X,Y) + D^V_Y\Phi(X) \tag{10}$$

and θ a $(1,3)$ tensor with components $\theta^l_{ijk} = -\frac{1}{2}\partial^3_{v^iv^jv^k} f^l$, and therefore totally symmetric (t.s.). Notice that all these *structural* tensors, except for θ, are derived from the Jacobi endomorphism Φ. The former expressions are understood acting on $(1,p)$ or $(0,p)$ type tensor fields T according to the following rules

$$D^V_A T(X_1,\cdots,X_q,Y,\cdots) = D^V_{A(X_1,\cdots,X_q)}T(Y,\cdots) \tag{11}$$

for a $(1,q)$ tensor field A; $\mu(A)T = a(A)T - i(A)T$ with

$$a(A)T(X_1,\cdots,X_{q-1},Y_1,\cdots,Y_p) = A(X_1,\cdots,X_{q-1},T(Y_1,\cdots,Y_p)) \tag{12}$$

(and vanishing for $(0,p)$ type T), and

$$i(A)T(X_1,\cdots,X_{q-1},Y_1,\cdots,Y_p) = \sum_{i=1}^{p} T(Y_1,\cdots,A(X_1,\cdots,X_{q-1},Y_i),\cdots,Y_p) \tag{13}$$

Bianchi identities for the connection represent some relations among the *structural* tensors, e.g. $\sum_{\mathrm{cycl}} D^V R(X,Y,Z) = 0$, and will be introduced when necessary.

4 Generating integrability conditions

Helmholtz conditions represent a system of PDAEs for the multipliers, the components of a symmetric $(0,2)$ vertical tensor g along the π^o_1 projection. The covariant framework is therefore the bundle $\nu : V \equiv S^2(V^*E)\times_E J^1\pi \to J^1\pi \equiv M$, with $S^2(V^*E)$ the manifold of symmetric two covariant tensors on the fibres of $\pi : E \to \mathrm{I\!R}$ (E is the configuration space-time), $g = g_{ij}d^i.d^j$, and sections of ν are symmetric $(0,2)$ tensor fields (acting on π-vertical vectors) along the tangent bundle projection $\pi^o_1 : J^1\pi \to E$. Elements of V will be denoted by g (the base point is understood); elements in the fibres of $\nu^o_1 : J^1\nu \to V$ can be described by three tensors $z^o = \bigtriangledown g$, $z^v = D^V g$ and $z^h = D^H g$, of $(0,2)$ and $(0,3)$ type respectively; finally, elements in the fibres of $(\nu_1)^o_1 : J^1\nu_1 \to J^1\nu$ $(\nu_1 : J^1\nu \to M)$ are described by three tensors $w^o = \bigtriangledown g$, $w^v = D^V g$ and $w^h = D^h g$ (identified with the corresponding z in $K \subset J^1\nu_1$), and nine tensors $u^{oo} = \bigtriangledown z^o$, $u^{oh} = D^H z^o$, $u^{ov} = D^V z^o$,

$u^{ho} = \bigtriangledown z^h$, $u^{vo} = \bigtriangledown z^v$, $u^{hh} = D^H z^h$, $u^{hv} = D^V z^h$, $u^{vh} = D^H z^v$ and $u^{vv} = D^V z^v$ of types $(0,2)$, $(0,3)$ and $(0,4)$. All tensors are symmetric in their last two indices.

The initial algebraic Helmholtz condition in V is expressed as

$$(\Phi_1 - \Phi_2).g = 0 \tag{14}$$

and the initial PD Hemlholtz equations in $J^1\nu$ are determined by

$$z^o = 0 \qquad z^v \text{ t.s.} \tag{15}$$

We also have to consider the holonomy conditions, the equations defining the subbundle $J^2\nu \subset K \subset J^1\nu_1$. The first family of equations, for $K \subset J^1\nu_1$, identify the tensors w with the corresponding z, and is trivially applied. The second family, the one corresponding to the identification of the mixed partial derivatives, is determined by the commutation relations (6), (7) and (8), when applied to g:

$$\begin{aligned}
u^{vo}(X,Y,Z) - u^{ov}(X,Y,Z) &= -z^h(X,Y,Z) && (16)\\
u^{ho}(X,Y,Z) - u^{oh}(X,Y,Z) &= z^v(\Phi(X),Y,Z) + \\
+g(\Psi(X,Y),Z) &+ g(Y,\Psi(X,Z)) && (17)\\
u^{vv}(X,Y,Z,W) - u^{vv}(Y,X,Z,W) &= 0 && (18)\\
u^{hh}(X,Y,Z,W) - u^{hh}(Y,X,Z,W) &= z^v(R(X,Y),Z,W) - \\
g(Rie(X,Y,Z),W) &- g(Z,Rie(X,Y,W)) && (19)\\
u^{hv}(X,Y,Z,W) - u^{vh}(Y,X,Z,W) &= \\
-g(\theta(X,Y,Z),W) &- g(Z,\theta(X,Y,W)) && (20)
\end{aligned}$$

Prolongations of the equations to the next upper level is obtained by applying the dynamical, vertical and horizontal derivations, and a systematic use of Leibnitz rule. We will, by aesthetic reasons, make first the lifting $D^V(z^o) = u^{ov} = 0$, and $\bigtriangledown(z^v) = u^{ov}$ t.s. to obtain from (16) a new first order PDE: z^h t.s. The following properties are easily proven by systematic computation:

Property 1 *For any algebraic condition, i.e., some contraction $A \odot g = 0$ of g with another tensor field A, new algebraic conditions $(\bigtriangledown^p A) \odot g = 0$, $p \geq 0$ are generated by $\bigtriangledown$ lifting and the $z^o = 0$ equation.*

Property 2 *For any $(1,q)$ tensor field A, the algebraic condition*

$$(A_1 - A_2)\odot g(X_1,\cdots,X_{q-1},Y_1,Y_2) \equiv A\odot g(X_1,\cdots,X_{q-1},Y_1,Y_2) - A\odot g(X_1,\cdots,X_{q-1},Y_2,Y_1)$$

generates, by D^V lifting and the z^v t.s. equation, the new condition

$$\sum_{cycl(1,2,3)} \Big([D^V_{Y_1}A] \odot g(X_1, \cdots, X_{q-1}, Y_2, Y_3) - [D^V_{Y_2}A] \odot g(X_1, \cdots, X_{q-1}, Y_1, Y_3)\Big) = 0$$

where $\sum_{cycl(1,2,3)}$ *means a cyclic sum for the three vectors* (Y_1, Y_2, Y_3)*.*

Property 3 *For any* $(1, q)$ *tensor field* A*, the algebraic condition*

$$(A_1 - A_2) \odot g(X_1, \cdots, X_{q-1}, Y_1, Y_2) \equiv A \odot g(X_1, \cdots, X_{q-1}, Y_1, Y_2) - A \odot g(X_1, \cdots, X_{q-1}, Y_2, Y_1)$$

generates, by D^H *lifting and the* z^h *t.s. equation, the new condition*

$$\sum_{cycl(1,2,3)} \Big([D^H_{Y_1}A] \odot g(X_1, \cdots, X_{q-1}, Y_2, Y_3) - [D^H_{Y_2}A] \odot g(X_1, \cdots, X_{q-1}, Y_1, Y_3)\Big) = 0$$

where $\sum_{cycl(1,2,3)}$ *means a cyclic sum for the three vectors* (Y_1, Y_2, Y_3)*.*

Property 4 *For any algebraic condition* $A \odot g = 0$*, with* $A \odot g$ *a* $(0, q)$ *type tensor, a new algebraic condition* $i(\theta)[A \odot g] = 0$ *is generated by applying the second holonomy condition* $\left[D^V, D^H\right] = \mu(\theta)$*.*

Property 1 applied to the initial condition $(\Phi_1 - \Phi_2).g = 0$ generates the family

$$F_1 = \{(\bigtriangledown^p \Phi_1 - \bigtriangledown^p \Phi_2).g = 0\}_{p=0} \tag{21}$$

A second family

$$F_2 = \{\sum_{cycl(1,2,3)} (\bigtriangledown^p R)_{12} \odot g = 0\}_{p=0} \tag{22}$$

is generated by applying Property 2 to $(\Phi_1 - \Phi_2).g = 0$, taking into account the expression (9) for R ($[R_{12} \odot g](X, Y, Z) = g(R(X, Y), Z)$), and Property 1. Property 3 over $(\Phi_1 - \Phi_2).g = 0$ does not give new independent conditions because of the Bianchi identity $(D^H \Phi)(X, Y) - (D^H \Phi)(Y, X) = (\bigtriangledown R)(X, Y)$.

In [20] it is proven that, up to degeneration of the second order holonomy condition concerning θ, no new algebraic conditions appear when taking into account the appropriate Bianchi identities. For example, D^V and D^H lifting of $F_2|_{p=0}$ and appropriate use of the total symmetry of z^v and z^h do not generate new conditions because of the Bianchi identities $\sum_{cycl} D^V R(X, Y, Z) = 0$ and $\sum_{cycl} D^H R(X, Y, Z) = 0$. Similarly, Property 3 over elements $p > 0$ of F_1, commutation of $\bigtriangledown$ and D^H, and the expression for Ψ reproduces again elements of F_2. Notice that when making use of the commutation relations we are in fact lifting to the second level and making use of the holonomy conditions.

Application of Proposition 4 to the first family F_1 generates

$$F_3^o \;=\; \{0 = i(\theta)[(\bigtriangledown^p \Phi_1 - \bigtriangledown^p \Phi_2).g](X_1, X_2, Y_1, Y_2) =$$

$$
\begin{aligned}
&= [(\nabla^p\Phi_1 - \nabla^p\Phi_2).g](\theta(X_1,X_2,Y_1),Y_2) + [(\nabla^p\Phi_1 - \nabla^p\Phi_2).g](Y_1,\theta(X_1,X_2,Y_2)) = \\
&= g(\nabla^p\Phi[\theta(X_1,X_2,Y_1)],Y_2) - g(\theta(X_1,X_2,Y_1),\nabla^p\Phi(Y_2)) + \\
&+ g(\nabla^p\Phi(Y_1),\theta(X_1,X_2,Y_2)) - g(Y_1,\nabla^p\Phi[\theta(X_1,X_2,Y_2)]) \equiv \\
&\quad \Omega(\theta,\nabla^p\Phi)\odot g(X_1,X_2,Y_1,Y_2) - \Omega(\theta,\nabla^p\Phi)\odot g(X_1,X_2,Y_2,Y_1)\}
\end{aligned} \tag{23}
$$

where

$$\Omega(\theta,\nabla^p\Phi)\odot g(X_1,X_2,Y_1,Y_2) = g(\nabla^p\Phi[\theta(X_1,X_2,Y_1)],Y_2) + g(\nabla^p\Phi(Y_1),\theta(X_1,X_2,Y_2))$$

When applying ∇ to $(\Omega_1-\Omega_2)(\theta,\nabla^p\Phi)\odot g = 0$ we find a term $(\Omega_1-\Omega_2)(\theta,\nabla^{p+1}\Phi)\odot g$ plus a new condition $(\Omega_1-\Omega_2)(\nabla\theta,\nabla^p\Phi)\odot g = 0$, which can be rewritten through the Bianchi identity $\nabla\theta = -\frac{1}{3}\sum D^V D^V\Phi$. Recursive application of ∇ by Proposition 1 gives the family

$$F_3 = \{\Omega(\nabla^q\theta,\nabla^p\Phi)\odot g(X_1,X_2,Y_1,Y_2) - \Omega(\nabla^q\theta,\nabla^p\Phi)\odot g(X_1,X_2,Y_2,Y_1) = 0\} \tag{24}$$

which perhaps contains not independent integrability conditions because of some Bianchi identities.

Application of Proposition 4 to the second family F_2 generates

$$
\begin{aligned}
F_4^o &= \{\sum_{cycl(1,2,3)} i(\theta)[(\nabla^pR)_{12}\odot g](X_1,X_2,Y_1,Y_2,Y_3) = \\
&= \sum_{cycl(1,2,3)} [(\nabla^pR)_{12}\odot g(\theta(X_1,X_2,Y_1),Y_2,Y_3) + \\
&+ (\nabla^pR)_{12}\odot g(Y_1,\theta(X_1,X_2,Y_2),Y_3) + \\
&+ (\nabla^pR)_{12}\odot g(Y_1,Y_2,\theta(X_1,X_2,Y_3))] = 0 \equiv \\
&\equiv \Delta(\nabla^pR,\theta)\odot g(X_1,X_2,Y_1,Y_2,Y_3)\}
\end{aligned} \tag{25}
$$

Again, Proposition 1 generates

$$F_4 = \{\Delta(\nabla^pR,\nabla^q\theta)\odot g(X_1,X_2,Y_1,Y_2,Y_3) = 0\} \tag{26}$$

We can also apply Propositions 2 and 3 to F_3 by skew-symmetry, determining the corresponding families F_3^V and F_3^H, and recursively Proposition 4 to the new conditions $F_{3,T} = i(\theta)^r F_3$, $F_{4,T} = i(\theta)^r F_4$, $F_{3,T}^V = i(\theta)^r F_3^V$, $F_{3,T}^H = i(\theta)^r F_3^H$. Moreover, $F_{3,T}$ being also in skew-symmetric form, Propositions 2 and 3 should give additional conditions, $[F_{3,T}]^V$ and $[F_{3,T}]^H$, and again $i(\theta)^r[F_{3,T}]^V$, etc.

It is a non trivial task to clean the former conditions of possible dependencies, by the use of Bianchi identities and commutation relations. The desirable objective of determining the consistency of the complete system up to second order, and eventually testing for the involution condition, seems to be far form the results already presented. Together with the algebraic conditions, new first and second order PDE conditions are determined along the path, either by prolongation of lower order (zero or first) conditions to the next level, or by projection of second order ones through combinations that eliminate the second order terms.

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References

[1] I. Anderson and G. Thompson; "The inverse problem of the calculus of variations for ordinary differential equations", *Memoirs Amer. Math. Soc.***98** No. 473 (1992).

[2] R.L. Bryant, S.S. Chern, R.B. Gardner, H.L. Goldschmidt and P.A. Griffiths;Exterior Differentuial Systems. Mathematical Sciences Research Institute Publications 18. Springer-Verlag, NY, 1991.

[3] E. Cartan; Les Systèmes Différentielles Extérieurs et leurs Applications Géométriques. Hermann, 1945.

[4] M. Crampin, W. Sarlet, E. Martínez, G.B. Byrnes and G.E. Prince; "Toward a geometrical understanding of Douglas's solution of the inverse problem in the calculus of variations", *Inverse Problems***10** (1994), 245–260.

[5] J. Douglas; "Solution of the inverse problem of the calculus of variations", *Trans. Am. Math. Soc.***50** (1941), 71–128.

[6] C. Ehresmann, "Les prolongements d'une variété différentiable", *Comp. Rend. Acad. Sci.***233** 598–600, 777–779, 1081–1083, **234** 1028–1030, 1424–1425, (1951). 306–445.

[7] H. Goldschmidt; "Existemce Theorems for Analitic Linear Partial Differential Equations", *The Ann. of Math.* 2nd Ser., **86**, No 2, (1967), 246–270.

[8] H. Goldschmidt; "Integrability criteria for systems of non-linear partial differential equations", *J. Diff. Geom.* **1**, (1967), 269–307.

[9] J. Grifone and Z. Muzsnay; Variational principles for second-order differential equations. (World Scientific, 2000)

[10] M. Hausdorf and W.M. Seiler; "An Efficient Algorithm for the Geometric Completion to Involution", *Appl. Alg. Eng. Comm. Comp.* **13** (2002), 163–207.

[11] C. Lopez; "Geometry of involutive PDEs", unpublished.

[12] E. Mansfield; "A simple criterion for Involutivity", *J. London Math. Soc.* **2** (54), pp. 323–345.

[13] E. Martínez, J.F. Cariñena and W. Sarlet; "Derivations of differential forms along the tangent bundle projection", *Diff. Geometry and its Applications***2** (1992), 17–43.

[14] E. Martínez, J.F. Cariñena and W. Sarlet; "Derivations of differential forms along the tangent bundle projection. Part II", *Diff. Geometry and its Applications***3** (1993), 1–29.

[15] G. Morandi, C. Ferrario, G. Lo Vecchio, G. Marmo and C. Rubano; "The inverse problem in the calculus of variations and the geometry of the tangent bundle", *Phys. Rep.***188** (1990), 147–284.

[16] J.F. Pommaret; Systems od Partial Differential Equations and Lie Pseudo-groups. Gordon and Breach, 1978.

[17] G.E. Prince; "The inverse problem in the calculus of variations and its ramifications", In *Geometric Approaches to Differential Equations* ed. P. Vassiliou and I. Lisle. Lecture Notes of the Australian Mathematical Society, CPU (2000).

[18] W. Sarlet; "The Helmholtz conditions revisited. A new approach to the inverse problem of Lagrangian dynamics", *J. Phys. A: Math. Gen.***15** (1982), 1503–1517.

[19] W. Sarlet, A. Vandecasteele, F. Cantrijn and E. Martínez; "Derivations of forms along a map: the framework for time-dependent second-order equations", *Diff. Geometry and its Applications***5** (1995), 171–203.

[20] W. Sarlet, M. Crampin and E. Martínez; "The integrability conditions in the inverse problem of the calculus of variations for second order ordinary differential equations", *Acta Appl. Math.***54** (1998), 233–273.

[21] W. Sarlet; "Integrability aspects of the inverse problem of the calculus of variations", In: Differential Geometry and its Applications, Proc. 7th Int. Conf. (Brno 1998), I. Kolár et al. (Eds.) (1999) 563–574.

[22] D.J. Saunders;The geometry of Jet Bundles. London Mathematical Society Lecture Notes Series 142. Cambr. Univ. Press, 1989.