Courant-Nijenhuis tensors and generalized geometries

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Dedicated to José F. Cariñena on his 60th birthday

Abstract

Nijenhuis tensors N on Courant algebroids compatible with the pairing are studied. This compatibility condition turns out to be of the form $N + N^* = \lambda I$ for irreducible Courant algebroids, in particular for the extended tangent bundles $\mathcal{T}M = \mathrm{T}M \oplus \mathrm{T}^*M$. It is proved that compatible Nijenhuis tensors on irreducible Courant algebroids must satisfy quadratic relations $N^2 - \lambda N + \gamma I = 0$, so that the corresponding hierarchy is very poor. The particular case $N^2 = -I$ is associated with Hitchin's generalized geometries and the cases $N^2 = I$ and $N^2 = 0$ – to other "generalized geometries". These concepts find a natural description in terms of supersymplectic Poisson brackets on graded supermanifolds.

Key words: *Nijenhuis tensor, Leibniz algebra, Courant bracket, Courant algebroid, generalized complex geometry.*

MSC (2000): *Primary 17B99; Secondary 17B62, 53C15, 53C56, 53D05, 53D17.*

1 Introduction

The theory of Nijenhuis tensors on Lie algebras goes back to a concept of contractions of Lie algebras introduced by E. J. Saletan [Sa]. The study of Nijenhuis tensors for Lie algebroids and Nijenhuis tensors on Poisson manifolds have been originated in [MM, KSM]. In [CGM1] it has been developed the theory of Nijenhuis tensors for associative products, and in [CGM2] contractions and Nijenhuis tensors have been studied for algebraic operations of arbitrary type on sections of vector bundles.

Recall that a *Nijenhuis tensor* N for a bilinear operation "$\circ$" on sections of a vector bundle A over M is a $(1, 1)$-tensor $N \in \mathrm{Sec}(A \otimes A^*)$, viewed also as vector bundle morphism

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$N : A \to A$ (or the corresponding $C^\infty(M)$-linear map $N : \mathrm{Sec}(A) \to \mathrm{Sec}(A)$ on sections), such that its *Nijenhuis torsion*

$$\mathrm{Tor}_N(X, Y) = N(X) \circ N(Y) - N(X \circ_N Y) \tag{1}$$

vanishes. Here "$\circ_N$" is the 'contracted' product:

$$X \circ_N Y = N(X) \circ Y + X \circ N(Y) - N(X \circ Y). \tag{2}$$

This general procedure has been applied in [CGM3] to Leibniz algebras and *Courant algebroids* [LWX] in their Leibniz algebra formulation. *Leibniz algebras* – non-skew-symmetric generalizations of Lie algebras – were studied first by J.-L. Loday [Lo] (they are called sometimes *Loday algebras*) and the (co)homology theory of Lie algebras was generalized to this framework.

Definition 1 A *Leibniz product (bracket)* on a vector space $\mathcal{A}$ is a bilinear operation "$\circ$" satisfying the Jacobi identity

$$(X \circ Y) \circ Z = X \circ (Y \circ Z) - Y \circ (X \circ Z) \tag{3}$$

for all $X, Y, Z \in \mathcal{A}$. The space $\mathcal{A}$ with a Leibniz product we call a *Leibniz algebra*.

Let now "$\circ$" be a local Leibniz product on the space $\mathrm{Sec}(A)$ of sections of a vector bundle A over M, i.e. a product which is locally defined by a bidifferential operator, and let $N : A \to A$ be a $(1,1)$-tensor over A. According to the general scheme in [CGM2], if the Nijenhuis torsion (1) vanishes, the contracted product (2) is a Leibniz product which is *compatible* with the original one, i.e. $X \circ_N Y + \lambda X \circ Y$ is a Leibniz product for any $\lambda \in \mathbb{R}$. Note that the compatibility is always satisfied.

Theorem 1 *[CGM3] The products "$\circ_N$" and "$\circ$" are always compatible. The contracted product (2) is still Leibniz if and only if the Nijenhuis torsion (1) is a 2-cocycle with respect to the Leibniz cohomology operator, i.e.*

$$\begin{aligned}(\delta\, Tor_N)(X, Y, Z) \;=\;\; & Tor_N(X, Y \circ Z) - Tor_N(X \circ Y, Z) - Tor_N(Y, X \circ Z) \\ & - Tor_N(X, Y) \circ Z + X \circ Tor_N(Y, Z) - Y \circ Tor_N(X, Z) = 0. \end{aligned} \tag{4}$$

In this case "$\circ_N$" and "$\circ$" are compatible Leibniz products.

The tensor N we will call a *Nijenhuis tensor* (for the Leibniz algebra $\mathrm{Sec}(A)$) if the Nijenhuis torsion Tor_N vanishes and a *weak Nijenhuis tensor* if the Nijenhuis torsion Tor_N is a Leibniz 2-cocycle. In both cases the contracted product "$\circ_N$" is Leibniz and it is compatible with the original one.

Example 1 An interesting example of a Leibniz product is the following Leibniz algebra version of the *Courant bracket* (called sometimes also *Dorfman bracket*) on sections $X + \xi$ of the bundle $\mathcal{T}M = \mathrm{T}M \oplus \mathrm{T}^*M$:

$$(X + \xi) \circ (Y + \eta) = [X, Y] + (\pounds_X \eta - i_Y d\xi). \tag{5}$$

Here $[X, Y]$ is clearly the bracket of vector fields, $\pounds_X$ is the Lie derivative, etc. The extended tangent bundle $\mathcal{T}M$ with the canonical symmetric pairing, coming from the contraction, and with the Courant bracket is an example of a *Courant algebroid* (cf. [LWX, Ro1]).

Since a Courant algebroid (see the next section) is not only a Leibniz algebra on sections of a vector bundle but also a non-degenerate pairing with certain consistency conditions with the Leibniz product, it has been studied in [CGM3] what is the property of N that ensures the consistency conditions being satisfied also for "$\circ_N$". It turns out that it is sufficient to assume that $N + N^* = \lambda I$, $\lambda \in \mathbb{R}$, where N^* is dual to N with respect to the pairing. This implies in the particular case of $\mathcal{T}M$ that such N is associated with a triplet consisting of a $(1, 1)$-tensor, a 2-form, and a bivector field on M.

In this paper we prove that this condition is also necessary for so called *irreducible* Courant algebroids ($\mathcal{T}M$ is a canonical example). We prove also that such compatible Nijenhuis tensors on irreducible Courant algebroids must satisfy additionally a quadratic equation $N^2 - \lambda N + \gamma I = 0$, se the associated hierarchy is trivial. Particular cases: $N^2 = -I$, $N^2 = I$, and $N^2 = 0$ correspond to the so called *complex*, *product*, and *tangent Courant structures*, respectively. The complex Courant structures on $\mathcal{T}M$ were introduced recently by N. Hitchin [Hi] under the name of *complex generalized geometries* and they drew much attention among mathematicians and physicists. Our work shows that, in practice, due to the above quadratic equation, no more "generalized geometries" in this sense than complex, product, and tangent are possible. Since, according to [Ro2], any Courant algebroid is associated with a cubic homological Hamiltonian Θ on a symplectic $\mathcal{N}$-manifold of degree 2, we show that in this language complex Courant structures correspond to certain quadratic super-functions N such that $\{\{\Theta, N\}, N\} = -\Theta$, where the bracket is the corresponding Poisson superbracket.

2 Nijenhuis tensors for Courant algebroids

Let us recall briefly the structure of a Courant algebroid. We will use here the Leibniz product (bracket) version of the Courant bracket presented already in [Ro1] with some simplifications discussed already in [CGM3] (cf. also [GM, Definition1], [KS2, Definition 2.1], and [Uch]).

Definition 2 A *Courant algebroid* is a vector bundle $\tau : A \rightarrow M$ equipped with a

Leibniz product (bracket) "$\circ$" on $Sec(A)$, a vector bundle map (covering the identity) $\rho : A \rightarrow \mathrm{T}M$ and a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on A satisfying the identities

$$\rho(X)\langle Y, Y \rangle = 2\langle X, Y \circ Y \rangle, \tag{6}$$

$$\rho(X)\langle Y, Y \rangle = 2\langle X \circ Y, Y \rangle. \tag{7}$$

Note that (6) is equivalent to

$$\rho(X)\langle Y, Z \rangle = \langle X, Y \circ Z + Z \circ Y \rangle. \tag{8}$$

Similarly, (7) easily implies the invariance of the pairing $\langle \cdot, \cdot \rangle$ with respect to the left multiplication

$$\rho(X)\langle Y, Z \rangle = \langle X \circ Y, Z \rangle + \langle Y, X \circ Z \rangle \tag{9}$$

and that ρ is the anchor map for the left multiplication:

$$X \circ (fY) = fX \circ Y + \rho(X)(f)Y. \tag{10}$$

A rather unpleasant constatation is that, even when the Nijenhuis torsion Tor_N of a $(1,1)$-tensor $N \in \mathrm{Sec}(A^* \otimes A)$ vanishes (so the contracted bracket is a Leibniz bracket), the conditions (6) and (7) need not to be satisfied automatically for the 'contracted' product (2). Assume therefore that N is just a $(1,1)$-tensor on A (do not assume that N is Nijenhuis at the moment) and repeat in short from [CGM3] the checking under what conditions the identities (6) and (7) are still satisfied for "$\circ_N$". Exactly as in the classical case of a Lie algebroid contraction [CGM2, Lemma 2], we have the anchor $\rho_N = \rho \circ N$ for the contracted multiplication

$$X \circ_N (fY) = f(X \circ_N Y) + \rho(NX)(f)Y. \tag{11}$$

Let N^* be the adjoint of N with respect to the pairing: $\langle NX, Y \rangle = \langle X, N^*Y \rangle$ and let $\Delta = N + N^*$. Using the invariance (7) we get easily

$$\begin{aligned}
\langle X \circ_N Y, Z \rangle &= \langle NX \circ Y + X \circ NY - N(X \circ Y), Z \rangle \\
&= \rho(NX)\langle Y, Z \rangle - \langle Y, NX \circ Z \rangle + \langle Y, N^*(X \circ Z) \rangle + \langle Y, X \circ N^*Z \rangle,
\end{aligned}$$

which equals $\rho(NX)\langle Y, Z \rangle - \langle Y, X \circ_N Z \rangle$ if and only if $\langle Y, X \circ \Delta Z - \Delta(X \circ Z) \rangle = 0$ for all X, Y, Z, i.e. if and only if Δ commutes with the left multiplication

$$X \circ \Delta Z - \Delta(X \circ Z) = 0. \tag{12}$$

Thus (12) is equivalent to the invariance of the pairing with respect to "$\circ_N$":

$$\rho_N(X)\langle Y, Z \rangle = \langle X \circ_N Y, Z \rangle + \langle Y, X \circ_N Z \rangle.$$

Similarly, checking (6) for "$\circ_N$", we get

$$\langle X, Y \circ_N Y \rangle = \frac{1}{2}\rho(X)\langle Y, \Delta Y \rangle - \frac{1}{2}\rho(N^*X)\langle Y, Y \rangle$$

which equals $\frac{1}{2}\rho(NX)\langle Y, Y \rangle$ if and only if $\rho(X)\langle Y, \Delta Y \rangle = \rho(\Delta X)\langle Y, Y \rangle$. The latter can be rewritten in the form

$$\langle X, Y \circ \Delta Y + \Delta Y \circ Y \rangle = 2\langle \Delta X, Y \circ Y \rangle$$

or

$$Y \circ \Delta Y + \Delta Y \circ Y = 2\Delta(Y \circ Y).$$

Using (12) we get finally the condition

$$\Delta(Y \circ Y) = \Delta Y \circ Y. \tag{13}$$

Theorem 2 ([CGM3]) *If $N : A \rightarrow A$ is a $(1,1)$-tensor on a Courant algebroid, then the contracted product (2) is compatible with the symmetric pairing $\langle \cdot, \cdot \rangle$ of the Courant algebroid, in the sense that (6) and (7) are satisfied for "$\circ_N$" and ρ_N, if and only if*

$$X \circ (N + N^*)Y = (N + N^*)(X \circ Y) \quad and \quad (N + N^*)(Y \circ Y) = (N + N^*)Y \circ Y$$

for all sections X, Y of A.

It is clear that, how restrictive the above conditions are, depends on 'irreducibility' of the Courant product. However, there is one (and only one) case which works for any Courant algebroid, namely the case $N + N^* = \lambda I$, $\lambda \in \mathbb{R}$. A Courant algebroid we call *irreducible* if λI are the only $(1,1)$-tensors $\Delta : A \rightarrow A$ satisfying (12) and (13).

Theorem 3 *The classical Courant algebroid structure on $\mathcal{T}M = TM \otimes T^*M$ is irreducible.*

Proof.- Suppose that the $(1,1)$-tensor Δ commutes with the left multiplication. In local coordinates (x^i) we can write $\Delta(\partial_j) = \sum_i \left(\Delta^i_j(x)\partial_i + \Delta^{*i}_j(x)dx^i \right)$ and $\Delta(dx^j) = \sum_i \left(\Delta^i_{*j}(x)\partial_i + \Delta^{*i}_{*j}(x)dx^i \right)$. In view of

$$0 = \Delta(\partial_k \circ \partial_j) = \partial_k \circ \Delta(\partial_j) = \sum_i \left(\frac{\partial \Delta^i_j}{\partial x^k}(x)\partial_i + \frac{\partial \Delta^{*i}_j}{\partial x^k}(x)dx^i \right)$$

and

$$0 = \Delta(\partial_k \circ dx^j) = \partial_k \circ \Delta(dx^j) = \sum_i \left(\frac{\partial \Delta^i_{*j}}{\partial x^k}(x)\partial_i + \frac{\partial \Delta^{*i}_{*j}}{\partial x^k}(x)dx^i \right)$$

we get that $\Delta^i_j(x) = \Delta^i_j$, $\Delta^i_{j*}(x) = \Delta^i_{*j}$, $\Delta^{*i}_j(x) = \Delta^{*i}_j$, and $\Delta^{*i}_{*j}(x) = \Delta^{*i}_{*j}$ are constant. Now, since

$$(x^k\partial_j) \circ \partial_k = [x^k\partial_j, \partial_k] = -\partial_j \quad and \quad (x^j\partial_k) \circ dx^k = \pounds_{x^j\partial_k} dx^k = dx^j,$$

we have

$$-\Delta(\partial_j) = -\sum_i \left(\Delta^i_j \partial_i + \Delta^{*i}_j dx^i\right) = x^k \partial_j \circ \sum_i \left(\Delta^i_k \partial_i + \Delta^{*i}_k dx^i\right) = -\Delta^k_j \partial_j + \Delta^{*j}_k dx^k \quad (14)$$

and

$$\Delta(dx^j) = \sum_i \left(\Delta^i_{*j} \partial_i + \Delta^{*i}_{*j} dx^i\right) = x^k \partial_j \circ \sum_i \left(\Delta^i_{*k} \partial_i + \Delta^{*i}_{*k} dx^i\right) = -\Delta^k_{*k} \partial_j + \Delta^{*j}_{*k} dx^k. \quad (15)$$

The identity (14) implies that $\Delta^i_j = \delta^i_j \Delta^k_k$ and $-\Delta^{*i}_j = \delta^i_k \Delta^{*j}_k$. Since the indices i, j, k are arbitrary, we conclude that $\Delta^i_j = \lambda \delta^i_j$ for some $\lambda \in \mathbb{R}$ and $\Delta^{*i}_j = 0$, i.e., $\Delta(\partial_j) = \lambda \partial_j$. Similarly, from the identity (15) we conclude that $\Delta(dx^j) = \lambda' dx^j$. But now $\lambda = \lambda'$ follows from (13). Indeed, $(X + \xi) \circ (X + \xi) = di_X \xi$, so that $(\lambda X + \lambda' \xi) \circ (X + \xi) = \lambda \mathcal{L}_X \xi - \lambda' i_X d\xi = (\lambda - \lambda') i_X d\xi + \lambda di_X \xi$ equals $\lambda' di_X \xi$ for all vector fields X and all 1-forms ξ, thus $\lambda = \lambda'$. $\qquad\square$

Definition 3 A $(1,1)$-tensor on a Courant algebroid we call *orthogonal* if $N + N^* = 0$. A (weak) Nijenhuis tensor N which is compatible with the symmetric pairing $\langle \cdot, \cdot \rangle$ of the Courant algebroid, in the sense that (6) and (7) are satisfied for "$\circ_N$" and ρ_N, we call a *(weak) Courant-Nijenhuis tensor*.

Thus weak Courant-Nijenhuis tensors give rise to contractions of Courant algebroids. Note however, that the structure of a Courant algebroid is extremely rigid and that there are very few true Courant-Nijenhuis tensors. First, observe that N is a Courant-Nijenhuis tensor if and only if $N - \frac{\lambda}{2}I$ is Courant-Nijenhuis (cf. [CGM2, Theorem 8]), so we can always reduce paired tensors to the case when $N + N^* = 0$, i.e. to the case of orthogonal N. Second, we have the following.

Theorem 4 *([CGM3]) If N is an orthogonal Courant-Nijenhuis tensor, then*

$$X \circ N^2 Y = N^2(X \circ Y), \quad and \quad N^2(Y \circ Y) = (N^2 Y) \circ Y.$$

Proof.- Using $N^* = -N$ and the invariance of the pairing, we get

$$\langle N(X \circ_N Y), Z \rangle = -\langle X \circ_N Y, NZ \rangle = -\rho(NX)\langle Y, NZ \rangle + \langle Y, X \circ_N NZ \rangle \quad (16)$$

and

$$\langle NX \circ NY, Z \rangle = \rho(NX)\langle NY, Z \rangle + \langle Y, N(NX \circ Z) \rangle, \quad (17)$$

so N is Nijenhuis implies that the r.h. sides of (16) and (17) are equal, i.e.

$$X \circ_N NZ - N(NX \circ Z) = 0. \quad (18)$$

But the l.h.s of (18) is

$$NX \circ NZ - N(X \circ_N Z) - N^2(X \circ Z) + X \circ N^2 Z$$

and vanishing of the Nijenhuis torsion implies $N^2(X \circ Z) = X \circ N^2 Z$. The second identity one proves analogously, see the proof of (13). $\qquad\square$

Corollary 1 *Any Courant-Nijenhuis tensor N on an irreducible Courant algebroid satisfies:*

   (a) $N + N^* = \lambda I,$

   (b) $N^2 - \lambda N + \gamma I = 0,$

for certain $\lambda, \gamma \in \mathbb{R}$, so that the algebra with involution generated by N, thus the corresponding hierarchy, is trivial.

Proof.- According to Theorem 2, $N + N^* = \lambda I$. Then, applying Theorem 4 to $N := N - \frac{\lambda}{2}I$, we get $(N - \frac{\lambda}{2}I)^2 = \lambda' I$ which yields (b) with $\gamma = \frac{\lambda^2}{4} - \lambda'$.   □

Definition 4 An orthogonal Courant-Nijenhuis tensor N on a Courant algebroid we call (for the terminology see [BC])

  (i) a *complex Courant structure*, if $N^2 = -I$;

  (ii) a *product Courant structure*, if $N^2 = I$;

  (iii) a *tangent Courant structure*, if $N^2 = 0$.

Remark. Note that complex Courant structures on the canonical Courant algebroid $\mathcal{T}M$ from Example 1 have been introduced by N. Hitchin [Hi] under the name of *generalized complex geometries*. They have been then studied by M. Gualtieri [Gu] and have drawn an attention of other authors (see e.g. [Cr, LMTZ, Zu1, Zu2]). One can say, not very precisely, that a *generalized geometry* is a geometry of contractions in which we replace a Nijenhuis tensor on the tangent bundle (with the standard bracket of vector fields) with a similar Nijenhuis tensor on the 'extended tangent bundle' (with the Courant bracket). When generalizing this scheme to an arbitrary Courant algebroid, we can speak about a *Courant geometry.*

Corollary 2 *Any orthogonal Courant-Nijenhuis tensor on an irreducible Courant algebroid is proportional to either a complex Courant structure, or to a product Courant structure, or to a tangent Courant structure.*

3   Courant geometries as supergeometries

There is another approach to Courant algebroids, proposed by D. Roytenberg [Ro1, Ro2] (cf. also [Vo]), in which the Courant algebroid corresponds to a symplectic $\mathcal{N}$-manifold $(\widetilde{A}, \Omega)$ of degree 2 with the associated (graded) Poisson bracket $\{\cdot, \cdot\}$, equipped additionally with a cubic Hamiltonian Θ which is *homological*, i.e. $\{\Theta, \Theta\} = 0$. The symplectic $\mathcal{N}$-manifold $\widetilde{A}$ of degree 2 is here the pullback of $T^*[2]A[1]$ (fibered canonically

over $(A \oplus A^*)[1])$ with respect to the embedding $A \hookrightarrow A \oplus A^*$ given by $X \mapsto (X, \langle X/2, \cdot \rangle)$, i.e. it completes the commutative diagram

$$
\begin{array}{ccc}
\widetilde{A} & \longrightarrow & \mathrm{T}^*[2]A[1] \\
\downarrow & & \downarrow \\
A[1] & \longrightarrow & (A \oplus A^*)[1]
\end{array}
$$

Here we use the standard convention and, for a graded vector bundle E over a graded manifold $\mathcal{M}$, write $E[n]$ for the graded manifold obtained by shifting the fibre degrees by n. In this picture, the corresponding Leibniz bracket is a *derived bracket* (cf. [KS1, KS2]) for which Θ is a *generating Hamiltonian*:

$$
X \circ Y = \{\{X, \Theta\}, Y\}. \tag{19}
$$

We should have probably written "$\circ_\Theta$" for the operation, but let us fix Θ and keep writing simply "$\circ$". Note that the above formula implies immediately

$$
\rho(X)(f) = \{\{X, \Theta\}, f\}. \tag{20}
$$

Here X, Y are functions on $\widetilde{A}$ of degree 1 (i.e. sections of A) and f is of degree 0 (i.e. f is a function on M). Writing the graded algebra of super-functions on $\widetilde{A}$ as $\mathcal{A} = \bigoplus_{k=0}^{\infty} \mathcal{A}^k$, we can identify the algebra of functions on M with $\mathcal{A}^0$ and the $\mathcal{A}^0$-module of sections of A with $\mathcal{A}^1$. The Poisson bracket reduced to $\mathcal{A}^1$ is just the pseudo-Riemannian form $\langle \cdot, \cdot \rangle$. Moreover, the Hamiltonian vector field $\partial_\Theta = \{\Theta, \cdot\}$ is a cohomology operator in $\mathcal{A}$ defining the corresponding cohomology. The pseudo-Riemannian form $\langle \cdot, \cdot \rangle$, thus the Poisson bracket, identifies canonically A with A^* by $X \mapsto \{X, \cdot\}$ (on sections) and any orthogonal $(1,1)$-tensor $N \in \mathrm{Sec}(A \otimes A^*)$ can be clearly identified with an element in $\mathcal{A}^1 \cdot \mathcal{A}^1 \subset \mathcal{A}^2$, denoted, with some abuse of notation, also by N. In this language, $N(X) = \{N, X\}$ for $X \in \mathcal{A}^1$. In an affine Darboux chart (x^i, ξ^a, p_j) on $\widetilde{A}$, corresponding to a chart (x^i) on M and a local basis $\{e_a\}$ of sections of A such that $\langle e_a, e_b \rangle = g_{ab}$, the symplectic form Ω reads

$$
\Omega = \sum_i \mathrm{d}p_i \mathrm{d}x^i + \frac{1}{2} \sum_a \mathrm{d}\xi^a g_{ab} \mathrm{d}\xi^b
$$

and $\Theta \in \mathcal{A}^3$ is of the form

$$
\Theta = \sum_{a,i} \xi^a \rho_a^i(x) p_i - \frac{1}{6} \sum_{a,b,c} \phi_{a,b,c}(x) \xi^a \xi^b \xi^c,
$$

where $\rho_a^i = \rho(e_a)(x^i)$ and $\phi_{a,b,c} = \langle e_a \circ e_b, e_c \rangle$. Any $(1,1)$-tensor $N : A \to A$, $N(e_a) = \sum_b N_a^b(x) e_b$ is orthogonal if and only if $\sum_b \left(N_a^b g_{bc} + N_c^b g_{ab} \right) = 0$ and then it is represented by the element

$$
N = \frac{1}{2} \sum_b N_a^b g_{bc} \xi^c \xi^a \in \mathcal{A}^1 \cdot \mathcal{A}^1.
$$

Note, however, that what we have denoted N^2 before, and which is $\{N, \{N, \cdot\}\}$ in the present notation, is not the square of N in the algebra $\mathcal{A}$. Since we will not use powers in the algebra $\mathcal{A}$, we will keep the old notation.

Proposition 1 *The derived bracket (19) generated by any cubic Hamiltonian Θ always satisfies the compatibility conditions (12) and (13). The Jacobi identity (3) is equivalent to the homological condition $\{\Theta, \Theta\} = 0$.*

Proof.- Since $\{X, \{Y, Y\}\} = 0$ for $X, Y \in \mathcal{A}^1$, we have, due to the graded Jacobi identity,

$$
\begin{aligned}
0 &= \{\Theta, \{X, \{Y, Y\}\}\} = \{\{\Theta, X\}, \{Y, Y\}\}\} - 2\{X, \{\{\Theta, Y\}, Y\}\} \\
&= \rho(X)\langle Y, Y\rangle - 2\langle X, Y \circ Y\rangle,
\end{aligned}
$$

whence (12). On the other hand,

$$
\begin{aligned}
\langle X \circ Y, Y\rangle &= \{\{\{\Theta, X\}, Y\}, Y\} = \{\{\Theta, X\}, \{Y, Y\}\} - \{\{\{\Theta, X\}, Y\}, Y\} \\
&= \rho(X)\langle Y, Y\rangle - \langle X \circ Y, Y\rangle,
\end{aligned}
$$

that proves (13). That the Jacobi identity is equivalent to the homological condition $\{\Theta, \Theta\} = 0$ follows now from [Ro2], Thorem 4.5. $\qquad\square$

Example 2 ([Ro2]) The symplectic $\mathcal{N}$-manifold of degree 2 associated with the canonical Courant algebroid from Example 1 is

$$
\widetilde{A} = \mathrm{T}^*[2]\mathrm{T}[1]M \simeq \mathrm{T}^*[2]\mathrm{T}^*[1]M
$$

with the canonical symplectic form Ω of degree 2. In local affine Darboux coordinates $(x^i, \xi^j, p_k, \vartheta_l)$, where (x^i) are local coordinates (of degree 0) on M, (ξ^j, ϑ_l) are degree-1 coordinates associated with adapted linear functions on the bundle $A = \mathcal{T}M = \mathrm{T}M \oplus \mathrm{T}^*M$ corresponding to $\mathrm{d}x^j$ and ∂_{x^l}, and (p_l) are degree-2 coordinates associated with linear functions on another copy of T^*M – the core of the double vector bundle $\mathrm{T}^*[2]\mathrm{T}[1]M \simeq \mathrm{T}^*[2]\mathrm{T}^*[1]M$. In these coordinates $\Omega = \sum_i (\mathrm{d}x^i\mathrm{d}p_i + \mathrm{d}\xi^i\mathrm{d}\vartheta_i)$ and the corresponding Poisson superbracket reads

$$
\{F, G\} = i_{\mathrm{d}G}i_{\mathrm{d}F} \sum_i \left(\partial_{p_i}\partial_{x^i} + \partial_{\vartheta_i}\partial_{\xi^i}\right).
$$

The canonical cubic Hamiltonian is in this case $\Theta = \sum_i \xi^i p_i$ which is just the Hamiltonian lift of the de Rham vector field $\mathrm{d} = \sum_i \xi^i \partial_{x^i}$ on $\mathrm{T}[1]M$.

Proposition 2 *For $N \in \mathcal{A}^1 \cdot \mathcal{A}^1 \subset \mathcal{A}^2$, the contracted product $\circ_N$ is the derived bracket associated with the cubic Hamiltonian $\{\Theta, N\}$. Moreover, N represents a weak Courant-Nijenhuis tensor if and only if $\{\{\Theta, N\}, N\}$ is a ∂_Θ-cocycle, and N represents a Courant-Nijenhuis tensor if and only if $\{\{\Theta, N\}, N\}$ is the generating Hamiltonian for "$\circ_{N^2}$".*

Proof.- We have

$$
\begin{aligned}
\{\{X, \{\Theta, N\}\}, Y\} &= -\{\{N, \Theta\}, X\}, Y\} = \\
&= -\{N, \{\{\Theta, X\}, Y\}\} + \{\{\Theta, \{N, X\}\}, Y\} + \{\{\Theta, X\}, \{N, Y\}\} = \\
&= -N(X \circ Y) + NX \circ Y + X \circ NY = X \circ_N Y.
\end{aligned}
$$

Thus, the product "$\circ_N$" defines another Courant algebroid structure on $(A, \langle \cdot, \cdot \rangle)$ if and only if $\{\{\Theta, N\}, \{\Theta, N\}\} = 0$. But

$$
\{\{\Theta, N\}, \{\Theta, N\}\} = \{\{\{\Theta, N\}, \Theta\}, N\}\} - \{\Theta, \{\{\Theta, N\}, N\}\} = 0 - \{\Theta, \{\{\Theta, N\}, N\}\}.
$$

Since, as easily seen, $X(\circ_N)_N Y = 2\mathrm{Tor}_N(X, Y) + X \circ_{N^2} Y$, the vanishing of the Nijenhuis torsion is equivalent that the generator of "$(\circ_N)_N$", i.e. $\{\{\Theta, N\}, N\}$ is the generator of "$\circ_{N^2}$". $\qquad\square$

Corollary 3 *A complex (resp., product, tangent) Courant structure on a Courant algebroid A associated with the qubic Hamiltonian Θ is exactly an element $N \in \mathcal{A}^1 \cdot \mathcal{A}^1$ satisfying $\{\{\Theta, N\}, N\} = -\Theta$ (resp, $\{\{\Theta, N\}, N\} = \Theta$, $\{\{\Theta, N\}, N\} = 0$).*

A detailed description of such quadratic Hamiltonians is in general difficult, since in particular it contains all true complex structures (cf. also [Cr]). There are also relations to presymplectic-Nijenhuis and Poisson-Nijenhuis structures, thus bihamiltonian systems (cf. [CGM3, Theorem 10]).

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