Monografías de la Real Academia de Ciencias de Zaragoza. **29:** 79–92, (2006).

On the geometry of higher-order ordinary differential equations and the Wuenschmann invariant

M. Crampin[1] and D. J. Saunders[2]

[1]Department of Mathematical Physics and Astronomy, Ghent University, Krijgslaan 281, B–9000 Gent, Belgium

[2]Department of Applied Mathematics, The Open University, Walton Hall, Milton Keynes MK7 6AA, UK

For Pepin Cariñena on his 60th birthday

Introduction

The Wuenschmann invariant is a quantity associated with a 3rd-order differential equation

$$y''' = F(x, y, y', y'').$$

Set $p = y'$, $q = y''$, and

$$\Gamma = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + q\frac{\partial}{\partial p} + F\frac{\partial}{\partial q};$$

the Wuenschmann invariant is

$$\frac{1}{6}\Gamma^2(F_q) - \frac{1}{3}F_q\Gamma(F_q) - \frac{1}{3}\Gamma(F_p) + \frac{2}{27}F_q^3 + \frac{1}{3}F_pF_q + F_y.$$

It appears that Wuenschmann discovered a method of associating with certain classes of 3rd-order differential equations a metric of Lorentz signature $(+, -, -)$ on the 3-dimensional solution space of the equation, determined up to conformal equivalence; the equations for which this construction is possible are those whose Wuenschmann invariant vanishes. This result, it is said, appeared in his PhD thesis of 1905. The study of 3rd-order differential equations, and their possible association with 3-dimensional conformal structures, was taken up again in the early 1940s by Chern [4] and Cartan [3]; Cartan mentions Wuenschmann and his result in a footnote in his paper, and this is so far as we are aware the only known reference to Wuenschmann's scientific career.

The subject then remained in comparative obscurity until the beginning of the present century, when Newman and his coworkers, in the course of their programme of formulating general relativity in terms of null surfaces, discovered an association between 4-dimensional conformal Lorentzian structures and certain systems of partial differential equations [9]; they took Wuenschmann's result as a model for the 4-dimensional case. Inspired by this, Nurowski has recently discovered and investigated a number of situations in which there is some relation between differential equations and conformal structures [13]; see also [10, 14]. In addition, there has been renewed interest recently in the study of invariants of differential equations per se: see for example [7, 8].

All of these studies are bedevilled to some degree, or so it seems to us, by the complexity of the expressions for the invariants discovered: the Wuenschmann invariant is in fact one of the simpler ones. There is however a considerable and relevant body of knowledge of geometrical structures associated with systems of 2nd-order ordinary differential equations, built up in the late 1980s and early 1990s by a group in which Pepin Cariñena played a significant role, and comprehensively recorded in [11, 12]. Anyone familiar with the 2nd-order case who looks at the Wuenschmann invariant will recognize some structure in it: in fact it looks as though it is constructed out of Jacobi endomorphisms and their dynamical covariant derivatives.

We shall show in this paper that the Wuenschmann invariant does indeed have a simple expression in such terms. In arriving at this conclusion we had of course to decide how exactly one should define and calculate the higher-order analogues of the Jacobi endomorphism of the 2nd-order case. We should explain first that although the Wuenschmann invariant is an invariant of a single differential equation, in order to understand the relevant differential geometry it is helpful to deal with systems of equations. Rather to our surprise, we discovered that there are at least two ways of approaching the construction of Jacobi endomorphisms for systems of equations of arbitrary order, which coincide in the 2nd-order case but are distinct for higher-order equations; moreover the one which comes from the most obvious way of generalizing the approach of [11, 12] to higher-order equations is not the one best adapted to the problem at hand.

This paper accordingly falls into two parts. The first deals with the matter of the Jacobi endomorphisms, and occupies Section 1. The derivation of the expression for the Wuenschmann invariant in terms of Jacobi endomorphisms will be found in Section 2.

1 The Jacobi endomorphisms

We start with a manifold fibred over the real line, say $\pi : E \to \mathbf{R}$, with local coordinates (x, y^i), $i = 1, 2, \dots, n$. We denote by VE the vertical sub-bundle of TE with respect to π.

Let $J^N\pi$ denote the Nth jet bundle of π, and $(x, y_0^i, y_1^i, \dots, y_N^i)$ its natural jet coordinates. Recall that $J^N\pi$ is fibred over $J^r\pi$ for $r = 0, 1, \dots, N-1$, where $J^0\pi = E$; the corresponding projections are written $\pi_r : J^N\pi \to J^r\pi$. There is a corresponding filtration of $TJ^N\pi$,

$$V_N \subset V_{N-1} \subset \cdots \subset V_1 \subset V_0 \subset TJ^N\pi,$$

where for $N \geq r \geq 1$, V_r is the vector sub-bundle of $TJ^N\pi$ consisting of tangent vectors vertical with respect to π_{r-1}, while V_0 is the vector sub-bundle consisting of vectors vertical with respect to $\pi \circ \pi_0$. Thus the fibre of V_r at any point is spanned by the vectors $\partial/\partial y^i_s$ for $s \geq r$. We can identify V_N, and V_{r-1}/V_r, with $\pi_0^*(VE)$.

We denote by θ^i_r the contact 1-form $dy^i_r - y^i_{r+1}dx$, $0 \leq r \leq N-1$. The type $(1,1)$ tensor field S on $J^N\pi$ given by

$$S = \sum_{r=1}^{N} r \frac{\partial}{\partial y^i_r} \otimes \theta^i_{r-1}$$

is called the vertical endomorphism.

A differential equation field of order $N+1$ is a vector field Γ on $J^N\pi$ of the form

$$\Gamma = \frac{\partial}{\partial x} + y^i_1 \frac{\partial}{\partial y^i_0} + y^i_2 \frac{\partial}{\partial y^i_1} + \cdots + y^i_N \frac{\partial}{\partial y^i_{N-1}} + F^i \frac{\partial}{\partial y^i_N};$$

such a vector field is a geometrical expression for the system of $(N+1)$st-order ordinary differential equations

$$y^i_{N+1} = F^i(x, y^j, y^j_1, \ldots, y^j_N), \qquad y^i_r = \frac{d^r y^i}{dx^r}.$$

Evidently $TJ^N\pi = \langle\Gamma\rangle \oplus V_0$. If furthermore we define $\theta^i_N = dy^i_N - F^i dx$ then $\{dx, \theta^i_r\}$, $r = 0, 1, \ldots, N$, is a local basis of 1-forms, and $\langle\Gamma, \theta^i_r\rangle = 0$, $\langle\Gamma, dx\rangle = 1$.

A differential equation field defines a dynamical covariant derivative: this is a linear operator ∇ on $\operatorname{sect}\pi_0^*(VE)$ such that $\nabla(fX) = f\nabla X + \Gamma(f)X$, and is determined by

$$\nabla\left(\frac{\partial}{\partial y^i}\right) = -\frac{1}{N+1}\frac{\partial F^j}{\partial y^i_N}\frac{\partial}{\partial y^j} = \Gamma^j_i \frac{\partial}{\partial y^j}.$$

The significance of the dynamical covariant derivative can be expressed, albeit a trifle crudely, in terms of coordinate transformations, as follows. Suppose one carries out a coordinate transformation of the type appropriate to the bundle structure of E, namely $\bar{y}^i = \bar{y}^i(x, y^j)$, $\bar{x} = x$. Any section X of $\pi_0^*(VE)$ can be written

$$X = X^i \frac{\partial}{\partial y^i} = \bar{X}^i \frac{\partial}{\partial \bar{y}^i}, \quad \text{where} \quad \bar{X}^i = \frac{\partial \bar{y}^i}{\partial y^j} X^j;$$

we shall say that quantities that transform in this way transform nicely. Then if the X^i transform nicely, and we set $\nabla X^i = \Gamma(X^i) + \Gamma^i_j X^j$, the ∇X^i transform nicely too.

Now $\mathcal{L}_\Gamma \theta^i_r = \theta^i_{r+1}$, $r = 0, 1, \ldots, N-1$; it follows that the type $(1,1)$ tensor field $\mathcal{L}_\Gamma S$, considered as a fibre-linear map on $TJ^N\pi$, maps V_0 to itself. Consider the type $(1,1)$ tensor field

$$P = \frac{1}{N+1}\left(I + \mathcal{L}_\Gamma S\right),$$

restricted to operating on V_0. It can be shown that P is a projection operator whose image is V_N. We denote the kernel of P by $\mathcal{H} \subset V_0$, and call the vector sub-bundle $\langle \Gamma \rangle \oplus \mathcal{H}$ the horizontal distribution of Γ. Then $\langle \Gamma \rangle \oplus \mathcal{H} \oplus V_N = TJ^N\pi$. We have the following explicit expression for P:

$$P = \frac{\partial}{\partial y^i_N} \otimes \left(\theta^i_N - \sum_{r=0}^{N-1} \left(\frac{r+1}{N+1} \right) \frac{\partial F^i}{\partial y^j_{r+1}} \theta^j_r \right).$$

We set

$$\phi^i_N = \theta^i_N - \sum_{r=0}^{N-1} \left(\frac{r+1}{N+1} \right) \frac{\partial F^i}{\partial y^j_{r+1}} \theta^j_r;$$

the ϕ^i_N transform nicely. A vector field ξ on $J^N\pi$ is horizontal if and only if $\langle \xi, \phi^i_N \rangle = 0$ (notice that Γ satisfies these conditions).

It would be desirable to break $\mathcal{H}$ down into pieces which respect the vertical filtration. We can do so as follows. Define

$$\phi^i_r = \frac{1}{r+1} S^*(\phi^i_{r+1}), \quad r = N-1, \ldots, 0;$$

the ϕ^i_r transform nicely. Now $S^*(\theta^i_r) = r\theta^i_{r-1}$, from which it follows that

$$\phi^i_r = \theta^i_r + \sum_{s=0}^{r-1} (C^s_r)^i_j \theta^j_s$$

for some coefficients $(C^s_r)^i_j$. In particular, $\phi^i_0 = \theta^i_0$. It is evident that $\{dx, \phi^i_r\}$ is a local 1-form basis adapted both to the horizontal distribution and the vertical filtration; let $\{\Gamma, \xi^r_i\}$ be the dual basis. Then

- $\xi^N_i = \partial/\partial y^i_N$, $\xi^r_i \sim \partial/\partial y^i_r \bmod V_{r+1}$, $r < N-1$
- ξ^r_i is horizontal for $r < N$
- $\xi^r_i = (1/r)S(\xi^{r-1}_i)$
- if $\mathcal{H}_r = \langle \xi^r_i \rangle$ then $\mathcal{H}_r \subset V_r$
- $\mathcal{H}_r \equiv V_r/V_{r+1} \equiv \pi^*_0(VE)$, $r < N-1$
- $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{N-1}$
- $\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_r$ is a complement to V_{r+1} in V_0.

We therefore obtain a multiconnection on V_0, that is, for each vertical sub-bundle $V_r \subset V_0$, a connection or complementary horizontal distribution H_r, these connections being compatible in the sense that $H_r \subset H_{r+1}$; here $H_r = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{r-1}$.

We now turn to the formulation of the analogue of the Jacobi equation in this context. The Jacobi equation should be thought of as the linearization of the original system, that is to say, it is the equation satisfied by, and determining, a connecting vector between two neighbouring integral curves of Γ; in short, the Jacobi equation is the equation of Lie transport, namely $[\Gamma, \xi] = 0$, and we can take $\xi \in V_0$. We must therefore consider, in the light of the previous analysis, how best to represent $[\Gamma, \xi]$.

First of all, every $X \in \operatorname{sect} \pi_0^*(VE)$ determines a vector field on $J^N\pi$ lying in $\mathcal{H}_r$, which we denote by X^r. We can express $[\Gamma, X^0]$ in terms of its components with respect to the $\mathcal{H}_r$, where for convenience (but disregarding the nomenclature) we set $\mathcal{H}_N = V_N$. We write

$$[\Gamma, X^0] = (\nabla X)^0 + \sum_{r=1}^{N} \Phi_r(X)^r :$$

then ∇ is the dynamical covariant derivative, and each Φ_r is a section of $\pi_0^*(VE \otimes VE^*)$, or loosely speaking a type $(1,1)$ tensor field along π_0. It is in fact enough to know $[\Gamma, X^0]$, since for $1 \leq r < N$

$$[\Gamma, X^r] = -X^{r-1} + (\nabla X)^r + \sum_{s=r+1}^{N} \begin{pmatrix} s \\ r \end{pmatrix} \Phi_{s-r}(X)^s.$$

To see this, notice that

$$r[\Gamma, X^r] = [\Gamma, S(X^{r-1})] = \mathcal{L}_\Gamma S(X^{r-1}) + S([\Gamma, X^{r-1}] = -X^{r-1} + S([\Gamma, X^{r-1}]$$

since X^{r-1} is horizontal; then use induction. Finally

$$[\Gamma, X^N] = -X^{N-1} + (\nabla X)^N.$$

We can return now to consideration of the Jacobi equation. Set

$$\xi = (X_0)^0 + (X_1)^1 + \cdots + (X_N)^N, \quad X_r \in \operatorname{sect} \pi_0^*(VE).$$

Then

$$\begin{array}{rccccccccccc}
[\Gamma, \xi] = & (\nabla X_0)^0 & + & \Phi_1(X_0)^1 & + & \Phi_2(X_0)^2 & + & \cdots & & \cdots & + & \Phi_N(X_0)^N \\
& -(X_1)^0 & + & (\nabla X_1)^1 & + & 2\Phi_1(X_1)^2 & + & \cdots & & \cdots & + & k\Phi_{N-1}(X_1)^N \\
& & - & (X_2)^1 & + & (\nabla X_2)^2 & + & \cdots & & \cdots & + & k'\Phi_{N-2}(X_2)^N \\
& & & & & & & & & & \vdots & \\
& & & & & & & & - & (X_N)^{N-1} & + & (\nabla X_N)^N
\end{array}$$

(where k, k' are numerical coefficients). Thus for $[\Gamma, \xi] = 0$

$$\begin{aligned}
X_1 &= \nabla X_0 \\
X_2 &= \nabla X_1 + \Phi_1(X_0) \\
X_3 &= \nabla X_2 + 2\Phi_1(X_1) + \Phi_2(X_0) \\
\vdots \quad & \quad \vdots \\
X_N &= \nabla X_{N-1} + \cdots \\
\nabla(X_N) &= N\Phi_1(X_{N-1}) + \cdots
\end{aligned}$$

Everything can be expressed in terms of $X_0 = X$ say, which must satisfy an equation of the form

$$\nabla^{N+1}X + \Psi_1(\nabla^{N-1}X) + \Psi_2(\nabla^{N-2}X) + \cdots + \Psi_N(X) = 0$$

where Ψ_r is an expression in the Φ_s and their ∇-derivatives, which is complicated in all but low order cases. This is the Jacobi equation, and the coefficients Ψ_r are therefore justly called the Jacobi endomorphisms; but to obtain them via the obvious vertical projection operator P is not entirely straightforward.

There is however another, better, method of obtaining them.

The dynamical covariant derivative ∇ can be made to act on n-tuples of 1-forms α^i which transform nicely, as follows:

$$\nabla\alpha^i = \mathcal{L}_\Gamma\alpha^i + \Gamma^i_j\alpha^j, \qquad \Gamma^i_j = -\frac{1}{N+1}\frac{\partial F^i}{\partial y^j_N};$$

as before, the fact that the α^i transform nicely and the properties of the Γ^i_j mean that the $\nabla\alpha^i$ transform nicely too. Alternatively, one can note that saying that the α^i transform nicely is equivalent to saying that $\partial/\partial y^i \otimes \alpha^i$ is a section of $\pi_0^*(VE)\otimes T^*J^N\pi$ (a VE-valued 1-form on $J^N\pi$, if you will); we can make ∇ act on $\operatorname{sect}\pi_0^*(VE)\otimes T^*J^N\pi$ by $\nabla(X\otimes\alpha) = (\nabla X)\otimes\alpha + X\otimes\mathcal{L}_\Gamma\alpha$. Moreover, the θ^i_0 transform nicely: it is easy to see that $\bar\theta^i_0 = (\partial\bar y^i/\partial y^j)\theta^j_0$. Alternatively, note that $\partial/\partial y^i\otimes\theta^i_0$ is a well-defined section of $\pi_0^*(VE)\otimes T^*J^N\pi$, specified as a map as follows: it maps a vector field on $J^N\pi$ into $\operatorname{sect}\pi_0^*(TE)$ by projection, then takes the vertical component of the result with respect to the canonical splitting of $\pi_0^*(TE)$ defined by the total derivative.

We may therefore define 1-forms Θ^i_r, $0\le r\le N$ by

$$\Theta^i_{r+1} = \nabla\Theta^i_r, \qquad \Theta^i_0 = \theta^i_0;$$

such 1-forms will transform nicely. Since $\mathcal{L}_\Gamma\theta_r = \theta_{r+1}$,

$$\Theta^i_r = \theta^i_r + \sum_{s=0}^{r-1}(C^s_r)^i_j\theta^j_s$$

for some coefficients $(C^s_r)^i_j$. In fact it is not difficult to see, by induction, that

$$\Theta^i_r = \theta^i_r + r\Gamma^i_j\theta^j_{r-1} + \sum_{s=0}^{r-2}(C^s_r)^i_j\theta^j_s.$$

Thus $\{dx, \Theta^i_r\}$, $0\le r\le N$, is a new adapted basis, $\{\Theta^i_N\}$ defines a new horizontal distribution, and we have a new multiconnection.

The $\nabla\Theta^i_N$ must be linearly dependent on the Θ^i_r, $0\le r\le N$: say

$$\nabla\Theta^i_N + (\Psi_0)^i_j\Theta^j_N + (\Psi_1)^i_j\Theta^j_{N-1} + \cdots + (\Psi_N)^i_j\Theta^j_0 = 0.$$

In fact $\Psi_0 = 0$:

$$\Theta^i_N = \theta^i_N + N\Gamma^i_j\theta^j_{N-1} + \sum_{s=0}^{N-2}(C^s_r)^i_j\theta^j_s,$$

whence $\nabla\Theta^i_N = \mathcal{L}_\Gamma\theta^i_N + (N+1)\Gamma^i_j\theta^j_N + \cdots$; now

$$\mathcal{L}_\Gamma\theta^i_N = dF^i - \Gamma(F^i)dx = \frac{\partial F^i}{\partial y^j_N}\theta^j_N + \cdots;$$

but

$$\frac{\partial F^i}{\partial y^j_N} + (N+1)\Gamma^i_j = 0,$$

and the result follows. Clearly

$$\begin{aligned}
\nabla\Theta^i_N + (\Psi_1)^i_j\Theta^j_{N-1} + \cdots + (\Psi_N)^i_j\Theta^j_0 &= 0\\
\nabla^{N+1}\Theta^i_0 + (\Psi_1)^i_j\nabla^{N-1}\Theta^j_0 + \cdots + (\Psi_N)^i_j\Theta^j_0 &= 0
\end{aligned}$$

is the Jacobi equation in dual form. In fact, suppose that $[\Gamma, \xi] = 0$, where $\xi = \sum_{r=0}^{N}(\Xi_r)^r$. Then $\langle\xi, \Theta^i_r\rangle = \Xi^i_r$, and so

$$\Gamma(\Xi^i_r) = \Gamma\langle\xi, \Theta^i_r\rangle = \langle\xi, \mathcal{L}_\Gamma\Theta^i_r\rangle = \langle\xi, \Theta^i_{r+1} - \Gamma^i_j\Theta^j_r\rangle,$$

whence $\nabla\Xi_r = \Xi_{r+1}$ for $r < N$, and

$$\nabla\Xi_N + \Psi_1(\Xi_{N-1}) + \cdots + \Psi_N(\Xi_0) = 0$$

as required.

To confirm that the two methods really are different we consider the 3rd-order case. For the first method we have the adapted basis

$$\begin{aligned}
\phi^i_2 &= \theta^i_2 - \frac{2}{3}\frac{\partial F^i}{\partial y^j_2}\theta^j_1 - \frac{1}{3}\frac{\partial F^i}{\partial y^j_1}\theta^j_0\\
\phi^i_1 &= \theta^i_1 - \frac{1}{3}\frac{\partial F^i}{\partial y^j_2}\theta^j_0\\
\phi^i_0 &= \theta^i_0,
\end{aligned}$$

and the Jacobi equation $\nabla^3X + 3\Phi_1(\nabla X) + \nabla\Phi_1(X) + \Phi_2(X) = 0$; thus $\Psi_1 = 3\Phi_1$, $\Psi_2 = \Phi_2 + \nabla\Phi_1$.

For the second method the adapted basis is

$$\begin{aligned}
\Theta^i_0 &= \theta^i_0\\
\Theta^i_1 &= \theta^i_1 - \frac{1}{3}\frac{\partial F^i}{\partial y^j_2}\theta^j_0\\
\Theta^i_2 &= \theta^i_2 - \frac{2}{3}\frac{\partial F^i}{\partial y^j_2}\theta^j_1 - \frac{1}{3}\left(\Gamma\left(\frac{\partial F^i}{\partial y^j_2}\right) - \frac{1}{3}\frac{\partial F^i}{\partial y^k_2}\frac{\partial F^k}{\partial y^j_2}\right)\theta^j_0,
\end{aligned}$$

and the Jacobi equation is of course $\nabla^3X + \Psi_1(\nabla X) + \Psi_2(X) = 0$.

The equality of ϕ^i_1 and Θ^i_1 holds for all orders, and we see that for 2nd-order the two methods agree, but not for any higher order.

The account of the first method given here is based in the first place, for the definition and properties of P, on [6], and secondly, for the multiconnection construction, on [15]. Method 2 was given originally for 4th-order equations in [1]; the simple extension to arbitrary order is given here (so far as we know) for the first time. It appears, though this observation is based entirely on the comparison of quoted formulas in the 3rd-order case, that method 2 is closely related in its results to Chern's theory of higher-order path spaces [5]; Chern's approach, however, is completely different, being based on Cartan's method of equivalence (indeed, the paper referred to is dedicated to Cartan on his 70th birthday). We should also mention the contribution of de León and his co-workers [2]: this paper contains a construction of a distribution H_r complementary to V_r for all r, which seems to contain elements of both of our methods; however, this construction does not produce a multiconnection, since the compatibility condition $H_r \subset H_{r+1}$ is not satisfied.

2 The Wuenschmann condition

We now turn to consideration of the relation between 3-dimensional Lorentzian conformal geometry and 3rd-order ordinary differential equations which gives rise to the Wuenschmann condition.

We begin with a simple example: null 2-planes in 3-dimensional Minkowski space. We consider $\mathbf{R}^3$, with coordinates (t^0, t^1, t^2), and with (the conformal class of) the Minkowski metric

$$(dt^0)^2 - (dt^1)^2 - (dt^2)^2.$$

A 2-plane $a_0 t^0 + a_1 t^1 + a_2 t^2 = c$ is null if and only if its normal covector (a_0, a_1, a_2) is null, that is, if and only if $a_0^2 - a_1^2 - a_2^2 = 0$. The set of null 2-planes is 2-dimensional; we can for example parametrize those null 2-planes for which $a_0 \neq 0$ with two parameters x, y by

$$y = t^0 + (\cos x)t^1 + (\sin x)t^2.$$

Then fixing (x, y) gives a null 2-plane in 3-dimensional Minkowski space. Fixing (t^a), on the other hand, leads to a curve in the (x, y) plane. Clearly such curves are solutions of the third-order differential equation $y''' + y' = 0$; and converesly every solution of this equation determines a null 2-plane. So we have an association between the null 2-planes and a third-order differential equation.

The association is in fact rather stronger than is immediately apparent, because the set of 2-planes determines the conformal structure, in the following sense. Consider the components of the contravariant metric $g^{ab} = g^{ab}(t^c)$ as unknown. Then the condition for the covector $(1, \cos x, \sin x)$ to be null is that for all x

$$g^{00} + (\cos x)^2 g^{11} + (\sin x)^2 g^{22} \\ + 2(\cos x) g^{01} + 2(\sin x) g^{02} + 2(\cos x \sin x) g^{12} = 0.$$

By differentiating repeatedly with respect to x one obtains a set of simultaneous equations for the 6 unknowns g^{ab}. The first five of these are satisfied if and only if $g^{11} = g^{22} = -g^{00}$, $g^{ij} = 0$, $i \neq j$; the sixth, and all subsequent ones, are linearly dependent on the first five.

This example illustrates how given a particular 3-dimensional Lorentzian conformal structure one can obtain a 3rd-order ordinary differential equation. The obvious converse question is whether given any 3rd-order ordinary differential equation one can construct an associated 3-dimensional Lorentzian conformal structure. In general one cannot: in order that one can do so the equation must satisfy the Wuenschmann condition. We now address this question.

Suppose given a 3rd-order ordinary differential equation $y''' = F(x, y, y', y'')$. Its general solution depends on 3 constants t^a, and can be written $y = Z(x, t^a)$. On the other hand, on fixing x and y we obtain a 2-surface in the 3-dimensional space with coordinates (t^a). The question is, can we find a Lorentzian conformal structure on this 3-dimensional space for which the surfaces of the 2-parameter family $y = Z(x, t^a)$ are null; in other words, can we find $g^{ab}(t^c)$ such that

$$g^{ab} \frac{\partial Z}{\partial t^a} \frac{\partial Z}{\partial t^b} = 0,$$

where $g^{ba} = g^{ab}$ and the symmetric bilinear form (g^{ab}) is non-singular and of Lorentz signature?

In order that we can use the geometric methods of the first section in this problem we must consider the fibred manifold $\pi : \mathbf{R}^2 \to \mathbf{R}$ and the jet bundle $J^2\pi$. We use coordinates x, y, $p = y'$, $q = y''$; the differential equation is represented by the vector field

$$\Gamma = \frac{\partial}{\partial x} + p \frac{\partial}{\partial y} + q \frac{\partial}{\partial p} + F \frac{\partial}{\partial q}.$$

The space of coordinates (t^a) is the path space $\mathcal{P} = J^2\pi/\Gamma$; we denote by τ the projection $J^2 \to \mathcal{P}$. (Considered globally, the path space need not be a manifold; but all of our considerations are local, so we ignore this difficulty.)

The conformal structure, if it exists, will be defined on $\mathcal{P}$; we would prefer however to work on $J^2\pi$. We can in fact transfer the problem to $J^2\pi$, as follows. Consider a symmmetric contravariant 2-tensor $\hat{g}$ on $J^2\pi$ such that $\mathcal{L}_\Gamma \hat{g} = 0$. Then for any 1-forms ϕ, ψ on $\mathcal{P}$, $\Gamma(\hat{g}(\tau^*\phi, \tau^*\psi)) = 0$, so $g(\phi, \psi) = \hat{g}(\tau^*\phi, \tau^*\psi)$ defines a symmmetric contravariant 2-tensor on $\mathcal{P}$. Moreover, every symmmetric contravariant 2-tensor on $\mathcal{P}$ can be obtained in this way, as we now show. Define new coordinates $(t^0, t^a) = (t^\alpha)$ on $J^2\pi$ $(\alpha = 0, 1, 2, 3)$

by

$$x = t^0, \quad y = Z(t^0, t^a), \quad p = \frac{\partial Z}{\partial t^0}, \quad q = \frac{\partial^2 Z}{\partial (t^0)^2}.$$

Then

$$\begin{aligned} dy - pdx &= \frac{\partial Z}{\partial t^a} dt^a \\ dp - qdx &= \frac{\partial^2 Z}{\partial t^0 \partial t^a} dt^a \\ dq - Fdx &= \frac{\partial^3 Z}{\partial (t^0)^2 \partial t^a} dt^a; \end{aligned}$$

all are semi-basic with respect to τ. It follows that

$$\Gamma = \frac{\partial}{\partial t^0},$$

as indeed one would expect; so

$$\mathcal{L}_\Gamma \hat{g} = 0 \quad \Longleftrightarrow \quad \frac{\partial \hat{g}^{\alpha\beta}}{\partial t^0} = 0.$$

Thus starting with g we may take $\hat{g}^{ab} = g^{ab}$ and (for example) $\hat{g}^{0\alpha} = 0$.

We must express the condition

$$g^{ab} \frac{\partial Z}{\partial t^a} \frac{\partial Z}{\partial t^b} = 0$$

in terms of $\hat{g}$. But

$$\frac{\partial Z}{\partial t^a} dt^a = dy - pdx = \theta_0,$$

so $\hat{g}$ must satisfy $\hat{g}(\theta_0, \theta_0) = 0$.

We shall now assume that there is a symmetric contravariant 2-tensor $\hat{g}$ on $J^2\pi$ such that $\mathcal{L}_\Gamma \hat{g} = 0$ and $\hat{g}(\theta_0, \theta_0) = 0$, and attempt to characterize $\hat{g}$.

We use the 1-forms Θ_r where $\Theta_0 = \theta_0$ and $\Theta_{r+1} = \nabla \Theta_r$, so that $\mathcal{L}_\Gamma \Theta_r = \Theta_{r+1} + \frac{1}{3} F_q \Theta_r$. We also have at our disposal the dual Jacobi equation $\Theta_3 = \nabla \Theta_2 = -(\Psi_1 \Theta_1 + \Psi_2 \Theta_0)$. The following argument is similar in concept to the one we used in the example to show that the null 2-planes determine the conformal Minkowski metric; it is now carried out on $J^2\pi$, however. We take the Lie derivative of the condition $\hat{g}(\Theta_0, \Theta_0) = 0$, to obtain

$$\hat{g}(\Theta_1 + \tfrac{1}{3} F_q \Theta_0, \Theta_0) = 0 \quad \Longrightarrow \quad \hat{g}(\Theta_0, \Theta_1) = 0.$$

Continuing in this way:

$$\begin{aligned} \hat{g}(\Theta_0, \Theta_0) &= 0; \\ \hat{g}(\Theta_0, \Theta_1) &= 0; \\ \hat{g}(\Theta_0, \Theta_2) + \hat{g}(\Theta_1, \Theta_1) &= 0; \\ \hat{g}(\Theta_0, \Theta_3) + 3\hat{g}(\Theta_1, \Theta_2) &= 0 \\ \Longrightarrow \quad \hat{g}(\Theta_1, \Theta_2) &= 0; \\ \hat{g}(\Theta_1, \Theta_3) + \hat{g}(\Theta_2, \Theta_2) &= 0 \\ \Longrightarrow -\Psi_1 \hat{g}(\Theta_1, \Theta_1) + \hat{g}(\Theta_2, \Theta_2) &= 0. \end{aligned}$$

We now have

$$\hat{g}(\Theta_0,\Theta_2) = -\hat{g}(\Theta_1,\Theta_1), \quad \hat{g}(\Theta_2,\Theta_2) = \Psi_1\hat{g}(\Theta_1,\Theta_1), \quad \hat{g}(\Theta_r,\Theta_s) = 0 \text{ otherwise;}$$

thus every component $\hat{g}(\Theta_r,\Theta_s)$ is determined in terms of $\hat{g}(\Theta_1,\Theta_1)$. We still have to consider the components of $\hat{g}$ involving dx. We may take $\hat{g}(dx,dx) = \hat{g}(dx,\Theta_r) = 0$; since $\mathcal{L}_\Gamma dx = 0$, and $\mathcal{L}_\Gamma\Theta_r$ is a linear combination of the Θ_r, these choices are consistent.

There is one consistency condition, coming from the derivative of the equation $\hat{g}(\Theta_2,\Theta_2) = \Psi_1\hat{g}(\Theta_1,\Theta_1)$: it is

$$2\hat{g}(\Theta_2,\Theta_3) = \Gamma(\Psi_1)\hat{g}(\Theta_1,\Theta_1) + 2\Psi_1\hat{g}(\Theta_1,\Theta_2),$$

or equivalently

$$-2\Psi_2\hat{g}(\Theta_2,\Theta_0) = 2\Psi_2\hat{g}(\Theta_1,\Theta_1) = \Gamma(\Psi_1)\hat{g}(\Theta_1,\Theta_1).$$

Thus in order that there should be a non-trivial solution $\hat{g}$ to the problem the condition $\Gamma(\Psi_1) = 2\Psi_2$ must be satisfied. This is the Wuenschmann condition. A type $(1,1)$ tensor along the projection in 1 dimension is a function, so it is permissible, and would be better, to write the condition as

$$\nabla\Psi_1 = 2\Psi_2.$$

Finally, we must choose $\hat{g}(\Theta_1,\Theta_1)$: we must have

$$\Gamma(\hat{g}(\Theta_1,\Theta_1)) = \tfrac{2}{3}F_q\hat{g}(\Theta_1,\Theta_1),$$

so we take $\hat{g}(\Theta_1,\Theta_1) = -z^2$ (the choice of sign here determines the signature of the metric (g^{ab})) where z is any nowhere-vanishing solution of the equation

$$\Gamma(z) = \frac{\partial z}{\partial t^0} = \tfrac{1}{3}F_q z.$$

We can write this equation as $\nabla z = 0$.

We draw the following conclusions from this analysis.

Theorem Given a 3rd-order differential equation field Γ we can define a symmetric contravariant 2-tensor $\hat{g}$ on $J^2\pi$ by taking any nowhere-vanishing solution z of the partial differential equation $\nabla z = 0$ and setting

$$\hat{g}(\Theta_1,\Theta_1) = -\hat{g}(\Theta_0,\Theta_2) = -z^2, \quad \hat{g}(\Theta_2,\Theta_2) = -\Psi_1 z^2,$$

with all other components zero. In terms of the vector field basis $\{\Gamma, U^r\}$ dual to the 1-form basis $\{dx,\Theta_r\}$

$$\hat{g} = z^2\left((2U^0 - \Psi_1 U^2)\odot U^2 - U^1\odot U^1\right).$$

Then if the Wuenschmann condition $\nabla\Psi_1 = 2\Psi_2$ holds, $\hat{g}$ will be Γ-invariant, and will define a metric of Lorentz signature on the path space; and the 2-surfaces $y = Z(x, t^a)$, x, y constant, will be null with respect to this metric. Since the solutions of the partial differential equation for z are determined only up to multiplication by a non-vanishing function constant along the flow of Γ, the metrics on the path space constructed in this manner are conformal.

Conversely, given a conformal class of Lorentz metrics on a 3-manifold and a suitable 2-parameter family of null 2-surfaces one can construct a differential equation field and a symmmetric covariant 2-tensor $\hat{g}$ on $J^2\pi$ as above; then $\hat{g}$ will be Γ-invariant, and so the Wuenschmann condition must hold. ∎

We have asserted that the Wuenschmann invariant may be expressed in terms of the Jacobi endomorphisms of the differential equation as $\nabla\Psi_1 - 2\Psi_2$. It may be checked (with a little effort) that when expanded out in terms of F this is indeed the expression given in the introduction. Moreover, it is now clear that the Wuenschmann invariant is indeed an invariant, so far as coordinate transformations of the (rather restricted) form $\bar{y} = \bar{y}(x, y)$, $\bar{x} = x$ are concerned. It is also clear, however, that there are numerous opportunities for varying the association between differential equations and conformal structures: for example one can change the coordinates in the path space, change the parametrization of the null surfaces, and indeed choose quite a different family of null surfaces to characterize the conformal structure. The overall effect of all these possibilities is that the association is with contact classes of differential equations, and the Wuenschmann condition is a condition on such classes, or in other words the Wuenschmann invariant is a relative invariant of contact transformations. We mention this point (which is discussed more fully in [13]) for completeness' sake.

The argument leading to the theorem above is based on one given by Newman et al. in [9]; however, our use of the invariant methods of Section 1 gives an improved result, namely a simple and illuminating expression for the Wuenschmann condition which is manifestly invariant under a relevant if restricted class of coordinate transformations.

There is a final twist to our story. Recall that $\Psi_1 = 3\Phi_1$, $\Psi_2 = \Phi_2 + \nabla\Phi_1$; thus

$$\nabla\Psi_1 - 2\Psi_2 = \nabla\Phi_1 - 2\Phi_2,$$

and so the Wuenschmann invariant turns out unexpectedly to have the same expression in terms of the Φ_r and the Ψ_r after all.

Acknowledgements

The first author is a Guest Professor at Ghent University, and a Visiting Senior Research Fellow of King's College, University of London: he is grateful to the Department of Mathematical Physics and Astronomy at Ghent and the Department of Mathematics at King's for their hospitality.

Address for correspondence

M. Crampin, 65 Mount Pleasant, Aspley Guise, Beds MK17 8JX, UK
Crampin@btinternet.com

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